# THE CONNECTION BETWEEN THE SUM OF THE FIRST $n$ INTEGERS AND THE SUM OF THE FIRST $n$ CUBES.

ELIZABETH GOODE, JUSTIN R. HUGHES

Abstract. Mathematical induction is one of the more challenging proof methods for the novice mathematician. Two of the canonical examples prove formulas for the sum of the first $n$ integers and for the sum of the first $n$ perfect cubes. These formulas are related: the sum of the first $n$ cubes is the square of the sum of the first $n$ integers. An intuitive connection between these two examples is given here with induction serving as the vehicle. Thus our little proof might be useful as a problem to be given after the two examples above are treated.

An introduction to the principle of mathematical induction often contains the formulae for computing the sum of the first $n$ integers and computing the sum of the first $n$ cubes. The formula for the sum of the first $n$ cubes is the square of the formula for computing the sum of the first $n$ integers. The proof of the formula for the sum of the first $n$ cubes is an excellent example of a proof by the principle of mathematical induction. However, the usual proof leaves one question: Why is the formula for the sum of the first $n$ cubes the *square* of the formula for computing the sum of the first $n$ integers? This paper will give one approach to illustrate this connection.

We begin by introducing the following matrix:

$$\begin{array}{c} \\ 1 \\ + \\ 2 \\ + \\ 3 \\ + \\ \vdots \\ + \\ n \end{array} \begin{array}{c} 1 \quad + \quad 2 \quad + \quad 3 \quad + \quad \cdots \quad + \quad n \\ \begin{pmatrix} 1\cdot 1 & 1\cdot 2 & 1\cdot 3 & \cdots & 1\cdot n \\ 2\cdot 1 & 2\cdot 2 & 2\cdot 3 & \cdots & 2\cdot n \\ 3\cdot 1 & 3\cdot 2 & 3\cdot 3 & \cdots & 3\cdot n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n\cdot 1 & n\cdot 2 & n\cdot 3 & \cdots & n\cdot n \end{pmatrix} \end{array}$$

Observe that the sum of the entries in the matrix is the sum of the first $n$ integers squared. In fact the matrix is simply being used as a convenient container for the distribution of $(1+2+3+\ldots+n)$ across $(1+2+3+\ldots+n)$.

The cubes are also hidden in this matrix. First the $1,1$ position holds $1^2$ which is the same as $1^3$. Observe, $2^3$ is given by the $2^{nd}$ *L-strip*. See the entries highlighted in the following figure.

$$\begin{array}{c} \\ 1 \\ + \\ 2 \\ + \\ 3 \\ + \\ \vdots \\ + \\ n \end{array} \begin{array}{c} 1 \quad + \quad 2 \quad + \quad 3 \quad + \quad \cdots \quad + \quad n \\ \begin{pmatrix} 1\cdot 1 & \boxed{1\cdot 2} & 1\cdot 3 & \cdots & 1\cdot n \\ \boxed{2\cdot 1} & \boxed{2\cdot 2} & 2\cdot 3 & \cdots & 2\cdot n \\ 3\cdot 1 & 3\cdot 2 & 3\cdot 3 & \cdots & 3\cdot n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n\cdot 1 & n\cdot 2 & n\cdot 3 & \cdots & n\cdot n \end{pmatrix} \end{array}$$

$$(1\cdot 2)+(2\cdot 2)+(2\cdot 1)=2^3$$

Similarly $3^3$ is given by the $3^{rd}$ *L-strip*.

$$\begin{array}{c} \\ 1 \\ + \\ 2 \\ + \\ 3 \\ + \\ \vdots \\ + \\ n \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & \cdots & + & n \end{array} \\ \begin{pmatrix} 1\cdot 1 & 1\cdot 2 & 1\cdot 3 & \cdots & 1\cdot n \\ 2\cdot 1 & 2\cdot 2 & 2\cdot 3 & \cdots & 2\cdot n \\ 3\cdot 1 & 3\cdot 2 & 3\cdot 3 & \cdots & 3\cdot n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n\cdot 1 & n\cdot 2 & n\cdot 3 & \cdots & n\cdot n \end{pmatrix} \end{array}$$

$$(1\cdot 3)+(2\cdot 3)+(3\cdot 3)+(3\cdot 2)+(3\cdot 1)=3^3$$

**Definition 1.** *Given an $m \times n$ matrix, we define the $k^{th}$ L-strip as the set of entries that appear in the $k^{th}$ row or the $k^{th}$ column of the matrix, and that are indexed by $i, j$ where $1 \leq i, j \leq k$.*

**Lemma 2.** *The sum of the entries in the $n^{th}$ L-strip is $n^3$.*

$$\begin{array}{c} \\ 1 \\ + \\ 2 \\ + \\ 3 \\ + \\ \vdots \\ + \\ n \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & \cdots & + & n \end{array} \\ \begin{pmatrix} 1\cdot 1 & 1\cdot 2 & 1\cdot 3 & \cdots & 1\cdot n \\ 2\cdot 1 & 2\cdot 2 & 2\cdot 3 & \cdots & 2\cdot n \\ 3\cdot 1 & 3\cdot 2 & 3\cdot 3 & \cdots & 3\cdot n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n\cdot 1 & n\cdot 2 & n\cdot 3 & \cdots & n\cdot n \end{pmatrix} \end{array}$$

*Proof.* Consider the sum of the entries in the marked row:

$$\begin{aligned} (n\cdot 1)+(n\cdot 2)+(n\cdot 3)+\ldots+(n\cdot n) &= n(1+2+3+\ldots+n) \\ &= n\left(\frac{n(n+1)}{2}\right) \end{aligned}$$

Next consider the sum of the entries in marked column without the $n \cdot n$ entry since that has already been added above.

$$\begin{aligned} (1\cdot n)+(2\cdot n)+(3\cdot n)+\ldots+((n-1)\cdot n) &= n(1+2+3+\ldots+(n-1)) \\ &= n\left(\frac{(n-1)(n)}{2}\right) \end{aligned}$$

Therefore the sum of the entries in the $n^{th}$ L-strip is

$$\begin{aligned}(n\cdot 1)+(n\cdot 2)+(n\cdot 3)+\ldots+(n\cdot n)&+(1\cdot n)+(2\cdot n)+(3\cdot n)+\ldots+((n-1)\cdot n)\\ &= n\left(\frac{n(n+1)}{2}\right)+n\left(\frac{(n-1)(n)}{2}\right)\\ &= n\left(\frac{n(n+1)}{2}+\frac{(n-1)(n)}{2}\right)\\ &= n(n)\left(\frac{n+1}{2}+\frac{n-1}{2}\right)\\ &= n(n)\left(\frac{n+1+n-1}{2}\right)\\ &= n(n)\left(\frac{2n}{2}\right)=n^3.\end{aligned}$$

□

In order to emphasize the connection between the two sums, we denote the sum of the entries in the $k \times k$ upper-left submatrix as $S_k$.

**Definition 3.** *Given an $m \times n$ matrix $M$, we let $S_k$ denote the sum of the entries in the $k \times k$ upper-left submatrix. In other words $S_k$ is the sum of the entries of $M$ with indexes $i, j$ where $1 \le i, k \le k$.*

Therefore $S_k$ is the sum of the entries in the matrix

$$\begin{array}{c|c} & \begin{matrix}1 & + & 2 & + & 3 & + & \cdots & + & k\end{matrix}\\ \hline \begin{matrix}1\\+\\2\\+\\3\\+\\ \vdots\\+\\k\end{matrix} & \begin{pmatrix}1\cdot 1 & 1\cdot 2 & 1\cdot 3 & \cdots & 1\cdot k\\ 2\cdot 1 & 2\cdot 2 & 2\cdot 3 & \cdots & 2\cdot k\\ 3\cdot 1 & 3\cdot 2 & 3\cdot 3 & \cdots & 3\cdot k\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ k\cdot 1 & k\cdot 2 & k\cdot 3 & \cdots & k\cdot k\end{pmatrix}\end{array}.$$

Clearly $S_k = (1+2+3+\ldots+k)^2$.

**Theorem 4.** *The sum of the first $n$ cubes equals* $\left(\frac{n(n+1)}{2}\right)^2$.

*Proof.* (By mathematical induction.)

Base Case: $1^3 = 1$ and $\left(\frac{1(1+1)}{2}\right)^2 = 1^2 = 1$. Thus the base case holds.

Inductive Step: Assume that

$$\sum_{i=1}^{k} i^3 = S_k$$

As observed earlier $S_k = (1+2+3+\ldots+k)^2$.

Consider

$$\begin{aligned}\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3\\ &= S_k + (k+1)^3 && \text{by the inductive hypothesis}\\ &= S_k \text{ plus the } (k+1)^{st} \text{ L-strip} && \text{by Lemma 2}\end{aligned}$$

However, $S_{k+1}$ is the sum of $S_k$ plus the entries in the $(k+1)^{st}$ L-strip. Thus

$$\sum_{i=1}^{k+1} i^3 = S_{k+1} = (1+2+3+\ldots+(k+1))^2 .$$

Therefore by the principle of mathematical induction, the sum of the first $n$ cubes is $\left(\frac{n(n+1)}{2}\right)^2$ for all positive integers $n$. $\square$