# NOTES ON GROUPS, MATH 369.101

[First we should discuss the Prep about Exercise 1 from before. To what is $(\mathbb{Z}_{16})^\times/\langle 9\rangle$ isomorphic? How about $(\mathbb{Z}_{16})^\times/\langle 7\rangle$]

## 2-dimensional Complexes

Before we introduce complexes, we need to introduce *vertices* (plural of *vertex*), *edges* and *triangular faces* which, for our purposes, will be defined as follows.

**Definition 1.** A **vertex (or 0-simplex)** is a point in $\mathbb{R}^n$ for some $n$, $n \geq 2$. If $v_0$ and $v_1$ are two vertices, an **edge (or 1-simplex)** corresponding to $v_0$ and $v_1$ is simply the straight line segment between the points. If $v_0, v_1$ and $v_2$ are three vertices, a **triangular face (or 2-simplex)** corresponding to the three vertices is the triangular region in the plane containing $v_0, v_1, v_2$ which has corners at the vertices.

**Remark 1:** If we think of the vertices as vectors in $\mathbb{R}^n$, then the points on edges and triangular faces are linear combinations of the vectors. For example, say $v_0 = (1,2)$ and $v_1 = (3,0)$. Then the edge between $v_0, v_1$ is the set of all endpoints of vectors written as

$$t_0v_0 + t_1v_1, \text{ where } t_0, t_1 \geq 0 \text{ and } t_0 + t_1 = 1. \tag{1}$$

(For example, the midpoint of the edge would be where $t_0 = 1/2 = t_1$. Indeed, $\frac{1}{2}(1,2) + \frac{1}{2}(3,0) = (2,1)$, which is the midpoint.)

Similarly, the triangular face corresponding to $v_0, v_1, v_2$ is the (endpoints of vectors in the) set of all linear combinations of the form

$$t_0v_0 + t_1v_1 + t_2v_2, \text{ where } t_0, t_1, t_2 \geq 0 \text{ and } t_0 + t_1 + t_2 = 1. \tag{2}$$

*Date*: Dec. 5 – Dec. 12.

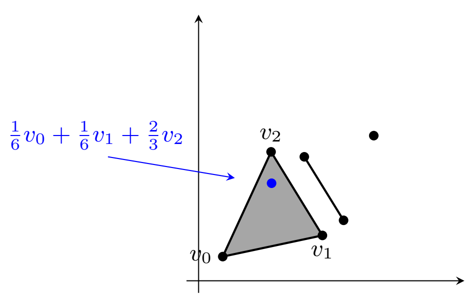


Figure 1. Some simplices in $\mathbb{R}^2$

(In the above, note that the corners correspond to $t_0 = 1; t_1 = 0; t_2 = 0$, $t_0 = 0; t_1 = 1; t_2 = 0$, $t_0 = 0; t_1 = 0; t_2 = 1$. Also note, it is not generally the case that $t_0 = t_1 = t_2 = \frac{1}{3}$ is the point that is equi-distant from the three corners (this depends on the shape of the triangular region).

**Remark 2:** We stopped after considering triples $v_0, v_1, v_2$ of vertices, and because of this Definition 1 makes sense (in most cases) for vertices in $\mathbb{R}^2$ (or $\mathbb{R}^n$, for $n \geq 2$).

The reason to specify *most cases*: What if $v_0, v_1$ were the same point, what does the edge between them mean? Similarly, what if $v_0, v_1, v_2$ were on the same line; what does the triangular face mean? We typically think of the more generic situation where Definition 1 works, but by using the linear combination description in Remark 1 we have a definition even in these degenerate cases. So, if $v_0, v_1, v_2$ are collinear, then the triangular face for them is the set of linear combinations in (2).

Using this linear combination description we could also continue with tuples of vertices $v_0, v_1, \ldots, v_k$ where $k \geq 3$. We would get a higher-dimensional version of the triangular face (for $k = 3$ we get a *tetrahedron*). In general, the object is called the **$k$-simplex** for the vertices $v_0, v_1, \ldots, v_k$. We will stick to $k \leq 2$.

If an edge corresponds to $v_i, v_j$, then we say that $v_i$ is a **border** of the edge, and $v_j$ is also a border.

If a triangular face has corners $v_i, v_j, v_k$, then we say that the three edges (one for $v_i, v_j$, one for $v_i, v_k$ and one for $v_j, v_k$) are each a **border** of this triangular face.

**Definition 2.** A **(simplicial) complex** $X$ (with dimension $\leq 2$) is a collection of vertices, a collection of edges, and a collection of triangular faces where if a simplex is in $X$ then its borders are in $X$. That is,

(1) if an edge is in $X$ then its two borders are in $X$;
(2) if a triangular face is in $X$ then its three borders are in $X$.

When we start to talk about a complex $X$ made up of simplices, we need to make a mental division between the points in $\mathbb{R}^n$ that make up a simplex and the simplex itself. That is, $X$ is a collection of simplices, not a subset of $\mathbb{R}^n$. So a 2-simplex in $X$ is one, indivisible, thing in $X$, and each of its border edges (which will be three 1-simplices) is something else on its own as an element of $X$.

In some of our pictures we will try to represent this separation as done in Figure 2. Despite how it is depicted, the 1-simplices (edges) in Figure 2 *come from* the same points in $\mathbb{R}^2$ that make up the boundary line segments of the triangle that the pictured 2-simplex *comes from*.

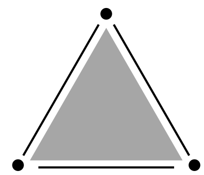

FIGURE 2. A complex consisting of one 2-simplex, its three 1-simplex borders and the three 0-simplices (corners).

## 1. The chain group, $\partial$ and homology

**1.1. The chain group.** Given a complex $X$, we can form an abelian group, the *chain group* $C_*(X)$, as follows. In fact, $C_*(X)$ will be a vector space over the field $\mathbb{Z}_2$.

$C_0(X)$: Think of each vertex $v_i$ as a vector with $\mathbb{Z}_2$ scalars. So there is $0v_i = 0$ and $1v_i = v_i$. We get the property that $v_i + v_i = 0$. (Despite this, $v_i \neq 0$. We are thinking of "working over $\mathbb{Z}_2$" where 2 is 0, so you cannot divide by 2.)

If $i \neq j$ then $v_i + v_j \neq 0$ is a new non-zero vector. But $(v_0+v_1)+(v_1+v_2) = v_0 + v_2$.

$C_1(X)$: Also think of each edge $e_{i,j}$ (the edge with borders $v_i$ and $v_j$) as a vector with $\mathbb{Z}_2$ scalars.

$C_2(X)$: Also think of each triangular face $t_{i,j,k}$ (the triangle with borders $e_{i,j}, e_{i,k}$ and $e_{j,k}$) as a vector with $\mathbb{Z}_2$ scalars.

**Exercise:** If $X$ has $m$ vertices, how many vectors are in $C_0(X)$? (Each element is a vector $\sum_{i=0}^{m-1} c_i v_i$ where $c_i$ is 0 or 1.)

Does the set of vertices form a basis of $C_0(X)$?

What is a basis for $C_1(X)$ and for $C_2(X)$?

**Definition 3.** Define $C_*(X) = C_0(X) \oplus C_1(X) \oplus C_2(X)$.

$C_*(X)$ is a group under vector addition. The identity is 0, and the inverse of every vector is itself.

**Exercise:** If there are $m$ simplices in $X$ (0-simplices,1-simplices and 2-simplices), then show that $C_*(X)$ is isomorphic to $\underbrace{\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \ldots \oplus \mathbb{Z}_2}_{m \text{ times}}$

So far, the group is just encoding *formal sums* of vertices, edges and triangular faces.

Since we used $\mathbb{Z}_2$ for our coefficients, each element of $C_*(X)$ is just an algebraic way to record a union of vertices, edges and triangular faces in $X$. Note that the element $(v_0 + v_1,\ e_{0,1},\ 0)$ is the union of $v_0, v_1$ and the edge between $v_0, v_1$ (so it is a subcomplex of $X$). But $(0,\ e_{0,1},\ 0)$ is also in $C_*(X)$ and, pointedly, does not contain $v_0$ or $v_1$, so is not a subcomplex.

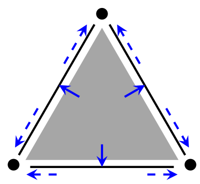

FIGURE 3. A visualization of the boundary map $\partial$.

1.2. **The "boundary" homomorphism $\partial$.** We want to introduce a homomorphism $\partial : C_*(X) \to C_*(X)$. First, we define it on each vertex, edge or triangular face $x$ by:

$$\partial(x) \text{ is the sum representing the union of borders of } x$$

(a vertex is understood to have no border, so the sum would equal zero in that case).

Now force $\partial$ to be a homomorphism by saying that $\partial(x+y) = \partial(x)+\partial(y)$ for any $x$ and any $y$ in $C_*(X)$ (think about how this is well-defined, since the set of simplices make a basis).

We can split $\partial : C_*(X) \to C_*(X)$ into dimensions by $\partial((v, e, f)) = (\partial_1(e), \partial_2(f), 0)$.

So $\partial_1 : C_1(X) \to C_0(X)$ and $\partial_2 : C_2(X) \to C_1(X)$.

**Exercise:** For the complex shown in Figure 4 (each simplex shown is in the complex), determine $\partial_1(e_{0,1} + e_{1,3})$, $\partial_1(e_{1,3} + e_{2,3} + e_{1,2})$ and $\partial_2(t_{0,1,2} + t_{1,3,2})$.

**Exercise:** If $x$ is in $C_*(X)$, then $\partial(x)$ is the sum representing the boundary of the union of simplices represented by $x$.

**The kernel of $\partial$.** Note that if $x \in C_i(X)$ for $i = 1, 2$, then $\partial(x) = 0$ means, in a certain sense, that the union of edges (if $i = 1$) or triangular faces (if $i = 2$) corresponding to $x$ doesn't have a boundary.

Because of the border condition on the complex $X$, if $x = \partial_2(y)$, then $x$ has no boundary (the boundary of a union of triangles doesn't itself have any boundary vertices – vertices that are "at an end by themselves," and so $\partial_1(\partial_2(y)) = 0$.

**Proposition 1.** *For any $x \in C_*(X)$, $\partial(\partial(x)) = 0$.*

**The image of $\partial$.** If $\partial_i(x) = y$ for some $y$ then $y$ represents a union of edges when $i = 2$ (or union of vertices when $i = 1$), that are the boundary of a union of triangular faces (or of edges).

**Exercise:** Show that if $y = \sum_{i=1}^{m} c_i v_i \in C_0(X)$ and $y = \partial_1(x)$ for some $x \in C_1(X)$, then the number of coefficients $c_i$ that are non-zero must be even.

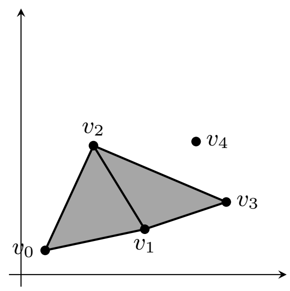


FIGURE 4. Example complex in $\mathbb{R}^2$

For the case $i = 1$, say that $x$ is one sequence of edges connected by endpoints. Then $y = \partial_1(x)$ is the pair of vertices at the extremes of this sequence, and is in the image of $\partial$.

**Exercise:** Name all the elements of $C_0(X)$ which are in the image of $\partial_1 : C_1(X) \to C_0(X)$, where $X$ is the complex in Figure 4.

Proposition 1 means that the image of $\partial$ is contained in its kernel. Since $\partial$ is a homomorphism, $\partial(C_*(X)) \subset C_*(X)$ is a subgroup of $\ker \partial$. It is a normal subgroup since $C_*(X)$ is an abelian group.

**Definition 4.** We write $H_*(X)$ for the factor group $\ker \partial / \partial(C_*(X))$. For a fixed $i = 0, 1, 2$, the factor group $\ker \partial_i / \partial_{i+1}(C_{i+1}(X))$ is denoted $H_i(X)$.

**Exercise:** Using the complex $X$ in Figure 4, write out the cosets in $H_0(X)$. What group is isomorphic to $H_0(X)$?

Solution: Let $N_1$ be equal to the normal subgroup $\partial_1(C_1(X))$.

$H_0(X)$ is defined to be $\ker \partial_0 / N_1$.

$\ker \partial_0 = C_0(X)$: $\partial((v, 0, 0)) = (0, 0, 0)$; that is, every vertex is sent to zero by $\partial$, so each element in $C_0(X) \cong \mathbb{Z}_2^5$ will be in a coset in $H_0(X)$. Note that if $i \neq j$, then $v_i$ and $v_j$ are independent in $C_0(X)$: the only way for $c_i v_i + c_j v_j = 0$ is if $c_i = 0 = c_j$.

If $i = 1, 2, 3$, then $v_0 + v_i$ is in the $N_1 = \partial_1(C_1(X))$ since there is a union of edges connecting $v_i$ to $v_0$: e.g. $\partial_1(e_{0,1} + e_{1,3}) = v_0 + v_3$.

Since $v_i$ is its own inverse, this means that $v_0$ and $v_i$ are in the same coset (for $i = 1, 2, 3$). Note that $v_0 \notin \partial_1(C_1(X))$ (an exercise above was that the number of non-zero coefficients had to be even), so the coset $\overline{v_0} = N_1 + v_0$ is not the identity in $H_0(X)$.

Thus, for $i = 1, 2, 3$, $N_1 + v_i = \overline{v_i} = \overline{v_0}$ (not only are these now linearly dependent in $H_0(X)$, they are equal).

$v_4$ is not in the same coset as $v_0$: in order for $v_0 + v_4$ to be in $N_1$, we would need an element of $C_1(X)$ (a union of edges) so that $v_0 + v_4$ is its boundary. This doesn't happen.

So none of $\overline{v_0}, \overline{v_4}, \overline{v_0 + v_4}$ are the identity in $H_0(X)$, but they each have order 2. The coset of anything else is either equal to one of these, or to the identity: for example,

$$\overline{v_1} + \overline{v_4} = \overline{v_0 + v_4}$$
$$\overline{v_0 + v_4} + \overline{v_3} = \overline{v_0 + v_0 + v_4} = \overline{v_4}$$

So $\{\overline{v_0}, \overline{v_4}\}$ is a basis of cosets for $H_0(X)$, and $H_0(X) = \langle \overline{v_0} \rangle \oplus \langle \overline{v_4} \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

**Exercise:** For the same complex:

(1) $e_{0,1} + e_{1,2} + e_{0,2}$ is in $\ker \partial_1$. Find a different element in $\ker \partial_1$ and convince yourself that these two elements are a basis for $\ker \partial_1$.

(2) Let $N_2 = \partial_2(C_2(X))$. There are only two 2-simplices (triangular faces). What is their image under $\partial_2$? What is a basis for $N_2$?

(3) What is $H_1(X)$?

**Exercise:** A basis of $H_0(X)$ represents the connected components of $X$. A basis of $H_1(X)$ represents a set of loops in $X$ which are not "filled" by triangular faces.

**Exercise (extra):** Figure out $H_1(X)$ for the $X$ in Figure 5.

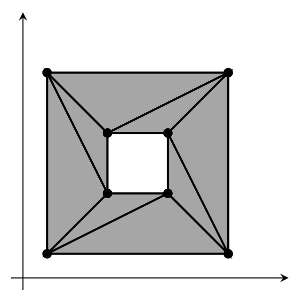

Figure 5. Another complex in $\mathbb{R}^2$