EXPLICIT BOUNDS FOR LARGE GAPS BETWEEN CUBEFREE INTEGERS

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Abstract. We obtain explicit forms of the current best known asymptotic upper bounds for gaps between cubefree integers. In particular, we show that the interval $(x, x+5x^{1/7}\log x]$ contains a cubefree integer for any $x \geq 2$. The constant 5 can be improved further, if x is assumed to be larger than a very large constant.

1. Introduction

If $k \geq 2$ is a fixed integer, an integer n is called *k-free* if it is not divisible by p^k for any prime p. In the special cases $k = 2$ and $k = 3$, the k-free integers are also known, respectively, as *squarefree* and *cubefree*. The distribution of k-free integers, especially in the squarefree case, has been studied for a long time and by many mathematicians.

An important part of that research concerns the study of the size of gaps between consecutive k-free numbers. Its history originates with Fogels' result [5] that if $\theta > 2/5$ the interval $(x, x + x^{\theta}]$ contains a squarefree integer for all sufficiently large x. In 1951, Halberstam and Roth [6] proved that if $k \geq 2$, the interval $(x, x + x^{\theta}]$ contains a k-free integer for any $\theta > 1/(2k)$ and for all sufficiently large x. Around the same time, Erdős [1] proved that there exist infinitely many intervals $(x, x + h]$, with

$$h \gg \frac{\log x}{\log\log x},$$

which contain no squarefree integers. Together, these results inspired the conjecture that for any fixed $\varepsilon > 0$, the interval $(x, x + x^{\varepsilon}]$ contains a squarefree integer for sufficiently large x. This conjecture appears to be far beyond the reach of present methods, with the sharpest asymptotic bounds known to date going back to the early 1990s. In 1991, Filaseta and Trifonov [4] proved that there exists a constant $c > 0$ such that the interval $(x, x + cx^{1/5}\log x]$ contains a squarefree integer for all sufficiently large x. Shortly after, Trifonov [10] generalized this result and proved that, for each $k \geq 3$, there exists a constant $c = c(k) > 0$ such that the interval $(x, x + cx^{1/(2k+1)}\log x]$ contains a k-free integer for all sufficiently large x. The reader interested in the detailed history of the problem can consult the survey [3] by Filaseta, Graham, and Trifonov or the introduction of our earlier work [7].

In recent years, the number-theoretic community has shown an increased interest in numerically explicit results, so the present authors set on a quest to obtain fully explicit versions of the results of Filaseta and Trifonov [4, 10]. In [7], we proved the following explicit theorem about the gaps between squarefree integers.

Theorem 1. *For any $x \geq 2$, the interval $(x, x + 11x^{1/5} \log x]$ contains a squarefree integer.*

In the present paper, we continue our investigations and obtain a similar theorem on the gaps between cubefree integers.

Theorem 2. *For any $x \geq 2$, the interval $(x, x + 5x^{1/7} \log x]$ contains a cubefree integer.*

The focus of the above theorems is on providing explicit intervals that work for all x. The price we pay for this universality are the somewhat elevated values of the constants 11 and 5 in the theorems. The next result is a version of [7, Theorem 2]. It provides a variant of our main result above, which reduces the constant 5 in Theorem 2, assuming that one is willing to accept a result that holds only for sufficiently large x.

Theorem 3. *Every interval*

- $(x, x + 2x^{1/7} \log x]$ *contains a cubefree number for* $x \geq e^{550}$*;*
- $(x, x + x^{1/7} \log x]$ *contains a cubefree number for* $x \geq e^{2300}$*;*
- $(x, x + \frac{1}{2}x^{1/7} \log x]$ *contains a cubefree number for* $x \geq e^{75\,000}$*.*

Mossinghoff, Oliveira e Silva and Trudgian [8] investigated long gaps between squarefree numbers numerically. Their computational work establishes the size of the longest gaps up to 10^{18}, which are all dramatically smaller than the bounds that we get in this paper. The largest gap that they find is a string of 18 consecutive non-squarefree numbers, the first of which is 125 781 000 834 058 568. As a result of their work, we can assume $x \geq 10^{18} > e^{41}$ throughout the rest of this paper.

Our proof of Theorem 2 relies also on several propositions giving results with larger exponents, which are however superior to the theorem for small x. Those results may also be of independent interest. For example, we show that the interval $(x, x + 2x^{1/5}]$ always contains a cubefree integer (Proposition 2) and the intervals $(x, x+10x^{1/6}]$ and $(x, x+8.5x^{1/6}]$ contain a cubefree integer for $x \geq e^{95}$ and $x \geq e^{191}$ respectively (Proposition 3).

We conclude this introduction by remarking that the interested reader can find the `SageMath` code for the computational part of our work at

https://github.com/agreatnate/explicit-k-free-integer-bounds

This should allow a motivated reader not only to check the numerical values that appear below, but also the explore the possibilities for future improvements.

Notation. Throughout the paper, for a real number θ, we use $\lfloor\theta\rfloor$ to denote the greatest integer less than or equal to θ; also, $\{\theta\} = \theta - \lfloor\theta\rfloor$. We write $|A|$ for the size of the set A, and $\pi(x)$ for the prime counting function. Finally, we use $c_1, c_2, \ldots$ to denote constants that appear in the proofs. Those constants tend to depend on various parameters introduced throughout our arguments (such as λ, δ and m); we may indicate such dependencies by labeling our constants as functions of said parameters—see the constant $c_2(m)$ in (3.12), for example.

2. Preliminaries

2.1. **Outline of the method.** Let $N(x,h)$ be the number of integers in $(x, x+h]$ that are not cubefree. To prove any of our theorems, it suffices to show that $N(x,h) < h-1$ for the respective choices of x and h. We first sieve this interval of the cubes of very small primes. Let J to be a parameter to be chosen later. The number of integers in $(x, x+h]$ divisible by the cube of a prime up to J is at most

$$h\left(1 - \prod_{p\le J}\left(1-\frac{1}{p^3}\right)\right) + 2^{\pi(J)} = h\left(1 - \prod_{p\le J}\left(1-\frac{1}{p^3}\right) + \frac{2^{\pi(J)}}{h}\right) =: h\sigma_0'(h,J).$$

We then count separately the integers divisible by p^3 for each prime $p > J$. We find that

$$N(x,h) \le h\sigma_0'(h,J) + \sum_{p>J}\left(\left\lfloor\frac{x+h}{p^3}\right\rfloor - \left\lfloor\frac{x}{p^3}\right\rfloor\right), \tag{2.1}$$

where the sum on the right is over all primes greater than J. To bound the latter sum, we introduce a parameter H, which we will later choose as $H = mh$, with $m \ge 1$ of moderate size, and we use this parameter to split the sum in (2.1) as follows:

$$\left(\sum_{J<p\le H} + \sum_{p>H}\right)\left(\left\lfloor\frac{x+h}{p^3}\right\rfloor - \left\lfloor\frac{x}{p^3}\right\rfloor\right) =: \Sigma_1 + \Sigma_2. \tag{2.2}$$

Beginning with the contribution of the small primes, we find that

$$\begin{aligned}\Sigma_1 &\le \sum_{J<p\le H}\left(\frac{h}{p^3}+1\right) \le h\sum_{p>J}\frac{1}{p^3} + \pi(H)\\ &< h\left(\sigma_1 - \sum_{p\le J}\frac{1}{p^3}\right) + \pi(H),\end{aligned} \tag{2.3}$$

where σ_1, the sum of the reciprocals of the cubes of all primes, satisfies

$$\sigma_1 < 0.1748. \tag{2.4}$$

Hence,

$$N(h,x) \le h\big(\sigma_0(h,J) + \sigma_1\big) + \pi(H) + \Sigma_2, \tag{2.5}$$

where

$$\sigma_0(h, J) = 1 - \prod_{p \le J} \left(1 - \frac{1}{p^3}\right) - \sum_{p \le J} \frac{1}{p^3} + \frac{2^{\pi(J)}}{h}. \tag{2.6}$$

The term $\pi(H)$ in (2.5) can be bounded with the help of the following lemma due to Rosser and Schoenfeld [9, (3.2)].

Lemma 1. *For any $x > 1$, one has*

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1.5}{\log x}\right). \tag{2.7}$$

Applying this lemma, we see that

$$\pi(H) < \sigma_2(h, m)h, \quad \sigma_2(h, m) := \frac{m}{\log(mh)}\left(1 + \frac{1.5}{\log(mh)}\right). \tag{2.8}$$

The estimation of the sum Σ_2 occupies the remainder of the paper. We remark that primes $p > \sqrt[3]{2x}$ do not contribute to that sum. Moreover, if $p > h^{1/3}$, we get

$$0 \le \left\lfloor \frac{x+h}{p^3} \right\rfloor - \left\lfloor \frac{x}{p^3} \right\rfloor \le \frac{h}{p^3} + 1 < 2.$$

Thus, the finite sum Σ_2 counts the primes $p \in \big(H, \sqrt[3]{2x}\big]$ for which there exists an integer m with

$$\frac{x}{p^3} < m \le \frac{x+h}{p^3},$$

or equivalently, such that $\{xp^{-3}\} > 1 - hp^{-3}$ (where $\{\cdot\}$ is the fractional part). Thus,

$$\Sigma_2 \le \big|S(H, \sqrt[3]{2x})\big|, \tag{2.9}$$

where

$$S(M, N) := \left\{u \in \mathbb{Z} : M < u \le N,\ \gcd(u, 2) = 1,\ 1 - \frac{h}{u^3} \le \left\{\frac{x}{u^3}\right\} < 1\right\}. \tag{2.10}$$

Putting together (2.5), (2.8) and (2.9), we reduce the problem of bounding $N(x, h)$ to choosing H so that

$$\big|S(H, \sqrt[3]{2x})\big| \le h\sigma_3(h, m), \tag{2.11}$$

for some bounded function $\sigma_3(h, m)$ such that

$$\sigma_0(h, J) + \sigma_1 + \sigma_2(h, m) + \sigma_3(h, m) < 1 - \frac{1}{h}. \tag{2.12}$$

3. Some lemmas

Our bounds on $|S(M,N)|$ use the simple idea that if the minimum distance between distinct elements of a set of integers $\mathcal{A}$ is at least d, then

$$|\mathcal{A} \cap (M,N]| \leq d^{-1}(N-M)+1. \tag{3.1}$$

In particular, we will be interested in the special case of (3.1) when $\mathcal{A}$ is a set of the form

$$S(M) := S(M, \lambda M),$$

where $\lambda > 1$ is a constant and M is a large parameter, with $H \leq M \leq \sqrt[3]{2x}$. As we pointed out in the introduction, the computational work in [8] allows us to assume that x is large. Thus, we assume in the remainder of the paper that

$$x \geq e^{41}, \quad 1000 \leq h \leq H \leq 2x^{1/3}. \tag{3.2}$$

.

We will the next lemma to sum the ensuing estimates for $|S(M)|$ to obtain bounds of the form (2.11). For the proof of this result see [2, Lemma 1] and the comments in [7, Lemma 2].

Lemma 2. *Suppose that* A_1, A_2, A_3, b_1, b_2 *are positive reals and* u, v, λ *are real numbers with* $0 < u < v < 1 < \lambda$. *Assume that for all* $M \in [x^u, x^v]$ *the estimate*

$$|S(M)| \leq A_1 M^{b_1} + A_2 M^{-b_2} + A_3$$

holds. Then

$$|S(x^u, x^v)| \leq A_1' x^{b_1 v} + A_2' x^{-b_2 u} + A_3' \log x + A_3,$$

where

$$A_1' = \frac{A_1}{1-\lambda^{-b_1}}, \quad A_2' = \frac{A_2}{1-\lambda^{-b_2}}, \quad A_3' = A_3 \cdot \frac{v-u}{\log \lambda}.$$

Lemma 3. *Suppose that* $H \leq M$. *If* u *and* $u+a$ *are distinct elements of* $S(M)$, *then*

$$a > 0.3333x^{-1}M^4. \tag{3.3}$$

Proof. This is a variant of [7, Lemma 4], and the proof is similar to the proof of that result, so we will be brief. Let $f(u) = xu^{-3}$. If $u, u+a \in S(M)$, we have

$$f(u) = n_1 - \theta_1, \quad f(u+a) = n_2 - \theta_2, \tag{3.4}$$

with $n_1, n_2 \in \mathbb{Z}$, $0 < \theta_1, \theta_2 < hM^{-3} < 10^{-5}$. Thus, the mean-value theorem yields a number $\xi \in (u, u+a)$ such that

$$|f(u+a) - f(u)| = a|f'(\xi)| = \frac{3ax}{\xi^4} > \frac{3x}{(\lambda M)^4}.$$

If $n_1 = n_2$, we deduce that

$$\frac{3x}{(\lambda M)^4} < |\theta_2 - \theta_1| < \frac{h}{M^3} <\leq \frac{1}{M^2},$$

which contradicts (3.2); thus, $|n_1 - n_2| \geq 1$. So, we have

$$0.9999 \leq 1 - |\theta_2 - \theta_1| \leq |f(u+a) - f(u)| = 3ax\xi^{-4} < 3axM^{-4},$$

and the lemma follows. □

Applying (3.1) to the result of the last lemma, we obtain the following bound on the size of $S(M)$.

Corollary 1. *Under the hypotheses of Lemma 3, we have*

$$|S(M)| \leq 0.3333^{-1}(\lambda - 1)xM^{-3} + 1.$$

We now use the above lemma to prove the following alternative bound, which is stronger than (3.3) for $M \leq 1.5x^{2/7}$. This is the first of several results that make use of polynomial identities, similar to (3.6) below, which are inspired by the theory of Padé approximations. Such identities play a central role in our forthcoming work on gaps between k-free integers for general k.

Lemma 4. *Let $\lambda \leq 1.2$, and suppose that $3\lambda^5 h \leq H \leq M$. If u and $u + a$ are distinct elements of $S(M)$, then*

$$a > 0.7934x^{-1/3}M^{5/3}. \tag{3.5}$$

Proof. We recall the algebraic identity

$$\frac{a^3(2u+a)}{u^3(u+a)^3} = \frac{u+2a}{(u+a)^3} - \frac{u-a}{u^3}. \tag{3.6}$$

From this (and defining $n_1, n_2, \theta_1, \theta_2$ as in (3.4)), we deduce that

$$\begin{aligned} \frac{a^3(2u+a)x}{u^3(u+a)^3} &= f(u+a)(u+2a) - f(u)(u-a) \\ &= (n_2 - \theta_2)(u+2a) - (n_1 - \theta_1)(u-a) = n + \theta, \end{aligned} \tag{3.7}$$

where $n \in \mathbb{Z}$ and $\theta = (u-a)\theta_1 - (u+2a)\theta_2$. In particular, using that $0 < \theta_i < hM^{-3}$ and $u, u+a \in (M, 1.2M]$, we get

$$|\theta| \leq a(2\theta_1 + \theta_2) + u|\theta_1 - \theta_2| < (u+3a)hM^{-3} \leq 1.6hM^{-2}. \tag{3.8}$$

Next, we will prove that $n \neq 0$. We have

$$\frac{a^3(2u+a)x}{u^3(u+a)^3} > \frac{2a^3ux}{u^3(u+a)^3} \geq \frac{2a^3x}{(\lambda M)^5} > 0.6666\lambda^{-5}M^{-1}, \tag{3.9}$$

on recalling (3.3) and the trivial bound $a^2 \geq 1$. On the other hand, if $n = 0$, the right side of (3.9) equals θ (which must be positive), and (3.8) and (3.9) together yield

$$0.6666\lambda^{-5} < 1.6hM^{-1} \leq 1.6hH^{-1},$$

which contradicts the hypothesis on H. Thus, we must have $|n| \geq 1$.

Similarly to (3.9), we find that

$$\frac{a^3(2u+a)x}{u^3(u+a)^3} \le \frac{2a^3(u+a)x}{u^3(u+a)^3} = \frac{2a^3x}{u^3(u+a)^2} \le \frac{2a^3x}{M^5}. \tag{3.10}$$

On the other hand, the hypotheses of the lemma and (3.2) yield

$$|\theta| < 1.6hH^{-2} < 10^{-3},$$

so

$$\frac{a^3(2u+a)x}{u^3(u+a)^3} \ge |n| - |\theta| > 0.999.$$

Combining the last inequality with (3.10), we deduce

$$a^3 > 0.4995x^{-1}M^5, \tag{3.11}$$

and the conclusion of the lemma follows. □

We can use this to obtain the following.

Corollary 2. *Under the hypotheses of Lemma 4, we have*

$$|S(M)| < 1.2604(\lambda-1)x^{1/3}M^{-2/3} + 1.$$

Next, we consider any three distinct elements $u, u+a, u+b$ of $S(M)$, with $0 < a < b$, and obtain lower bounds on b.

Lemma 5. *Let $\lambda \le 1.04$, $m \ge 3$, and suppose that $mh \le H \le M$. If $0 < a < b$ and $u, u+a, u+b$ are elements of $S(M)$, then*

$$ab^4 \ge c_2(m)x^{-1}M^6, \tag{3.12}$$

where $c_2(m) = 0.4 - 0.05m^{-1}$.

Proof. We begin with the identity

$$\frac{a^5}{u^3(u+a)^3} = \frac{6u^2-3au+a^2}{u^3} - \frac{6u^2+15ua+10a^2}{(u+a)^3}. \tag{3.13}$$

By substitution, this yields also the two companion identities

$$\frac{b^5}{u^3(u+b)^3} = \frac{6u^2-3bu+b^2}{u^3} - \frac{6u^2+15ub+10b^2}{(u+b)^3}; \tag{3.14}$$

$$\begin{aligned}\frac{(b-a)^5}{(u+a)^3(u+b)^3} &= \frac{6(u+a)^2-3(b-a)(u+a)+(b-a)^2}{(u+a)^3} \\ &\quad - \frac{6(u+a)^2+15(u+a)(b-a)+10(b-a)^2}{(u+b)^3}.\end{aligned} \tag{3.15}$$

In order to cancel out the higher order terms of u, we subtract (3.13) and (3.15) from (3.14). This gives

$$\begin{aligned} \frac{(b-a)(a+b-3u)}{u^3} &+ \frac{b(5a-b+3u)}{(u+a)^3} + \frac{a(a-5b-3u)}{(u+b)^3} \\ &= \frac{b^5}{u^3(u+b)^3} - \frac{a^5}{u^3(u+a)^3} - \frac{(b-a)^5}{(u+a)^3(u+b)^3}. \end{aligned} \tag{3.16}$$

For $u_1 = u$, $u_2 = u + a$, and $u_3 = u + b$, let

$$f(u_i) = n_i - \theta_i, \quad 0 < \theta_i < hM^{-3} \qquad (i = 1, 2, 3).$$

Multiplying both sides of (3.16) by x, we have that

$$(b-a)(a+b-3u)f(u_1) + b(5a-b+3u)f(u_2) + a(a-5b-3u)f(u_3) =: n - \theta,$$

where

$$n = (b-a)(a+b-3u)n_1 + b(5a-b+3u)n_2 + a(a-5b-3u)n_3,$$

(note that n must be even, as a and b are even) and

$$\begin{aligned} |\theta| &= |b(3u-b)(\theta_2-\theta_1) + 5ab(\theta_2-\theta_3) + a(3u-a)(\theta_1-\theta_3)| \\ &< hM^{-3}(3bu + 3au + 3ab - (a-b)^2) < 3bhM^{-3}(2u+a) \\ &< 6bhM^{-3}(u+b) < 6\lambda bhM^{-2}. \end{aligned} \tag{3.17}$$

Next, we show that $n \neq 0$. Suppose not. A direct check reveals that

$$\frac{b^5}{u^3(u+b)^3} - \frac{a^5}{u^3(u+a)^3} - \frac{(b-a)^5}{(u+a)^3(u+b)^3} = \frac{P(u;a,b)}{u^3(u+a)^3(u+b)^3},$$

where

$$\begin{aligned} P(u) &= 5ab(b-a)(b^2-ab+a^2)u^3 + 3ab(b^4-a^4)u^2 \\ &\quad + 3a^2b^2(b^3-a^3)u + a^3b^3(b^2-a^2) \\ &> 5ab(b-a)(b^2-ab+a^2)u^3 \geq 5a^2b^2(b-a)u^3 > 10a^3bu^3. \end{aligned} \tag{3.18}$$

In particular, the two sides of (3.16) are positive. When $n = 0$, this entails that $\theta < 0$, and

$$-\theta = \frac{P(u;a,b)x}{u^3(u+a)^3(u+b)^3} > \frac{10a^3bu^3x}{u^3(u+a)^3(u+b)^3} \geq \frac{10b(0.4995M^5)}{(\lambda M)^6},$$

after appeals to (3.18) and (3.11). Hence, comparing the bound above to (3.17),

$$4.995\lambda^{-6}bM^{-1} < 6\lambda bhM^{-2} \leq 6\lambda bh(HM)^{-1},$$

or

$$m = \frac{H}{h} < \frac{6\lambda^5}{4.995}$$

which contradicts the hypotheses of the lemma. As such, we must have that $|n| \geq 2$.

On the other hand, bounding the numerator $P(u;a,b)$ from above, we find that

$$\begin{aligned} P(u) &\le 5ab(b-a)(b^2-ab+a^2)u^3 + 3ab^5u^2 + 3a^2b^5u + a^3b^5 \\ &\le 5ab^3(b-a)u^3 + 3ab^5u^2 + 3ab^6u + ab^7 \\ &< ab^4(5u^3 + 3bu^2 + 3b^2u + b^3) < 5ab^4(u+b)^3. \end{aligned}$$

Thus,

$$n - \theta = \frac{P(u;a,b)x}{u^3(u+a)^3(u+b)^3} \le \frac{5ab^4x}{u^3(u+a)^3} < 5ab^4xM^{-6}. \tag{3.19}$$

Further, recalling that $b < (\lambda-1)M$, we deduce that

$$|\theta| \le 6\lambda bhM^{-2} < 0.25hM^{-1} \le 0.25hH^{-1}. \tag{3.20}$$

From (3.19) and (3.20), we deduce that

$$5ab^4xM^{-6} \ge \frac{P(u;a,b)x}{u^3(u+a)^3(u+b)^3} \ge 2 - |\theta| > 2 - 0.25hH^{-1},$$

which implies the conclusion of the lemma. □

Corollary 3. *Under the hypotheses of Lemma 5, we have*

$$|S(M)| \le 2(\lambda-1)c_2(m)^{-1/5}x^{1/5}M^{-1/5} + 2.$$

4. Main bounds on $|S(M)|$

In this section, we first study a special family of quadruples u, $u+a$, $u+b$, $u+a+b$ of elements of $S(M)$. The special form of the spacing between the four numbers allows us to use several algebraic identities to obtain a series of lower bounds on products of the form a^ib^j. These yield bounds on b that are stronger than those for general quadruples in $S(M)$. Later in the section, we will average these bounds over b.

Lemma 6. *Let $\lambda \le 1.04$, $m \ge 3$, and suppose that $mh \le H \le M$. If $0 < a \le b$ and $u, u+a, u+b, u+a+b$ are elements of $S(M)$, then*

$$a^3b > 0.0999x^{-1}M^6. \tag{4.1}$$

Proof. Note that under the hypotheses of the lemma, we have

$$a \le \frac{1}{2}(a+b) < 0.5(\lambda-1)M. \tag{4.2}$$

Recall that, by (3.7) and (3.8), we have

$$\frac{a^3(2u+a)x}{u^3(u+a)^3} = n_1 + \theta_1$$

with $n_1 \in \mathbb{Z}$ and $|\theta_1| < 1.6hM^{-2}$. Indeed, using (4.2) we can strengthen this to $|\theta_1| < 1.06hM^{-2}$. Similarly,

$$\frac{a^3(2u+2b+a)x}{(u+b)^3(u+a+b)^3} = n_2 + \theta_2$$

with $n_2 \in \mathbb{Z}$ and $|\theta_2| < 1.06hM^{-2}$. Combining these identities, we find that

$$\frac{a^3(2u+a)x}{u^3(u+a)^3} - \frac{a^3(2u+2b+a)x}{(u+b)^3(u+a+b)^3} = n + \theta$$

with $|\theta| \le |\theta_1| + |\theta_2| < 2.12hM^{-2}$. Now, we note that when the right side of the above identity is combined into a single fraction, the numerator can be written as

$$a^3x[4bu^3(u+a)^3 + 3abu^2(u+a)^3 + 3b(2u+a)u^3(u+a)^2 + \text{other positive terms}].$$

Since

$$3abu^2(u+a)^3 + 3b(2u+a)u^3(u+a)^2 > 6bu^3(u+a)^3,$$

we can use (3.11) to obtain the lower bound

$$\begin{aligned} n+\theta &\ge \frac{10a^3bu^3(u+a)^3x}{u^3(u+a)^3(u+b)^3(u+a+b)^3} \\ &= \frac{10a^3bx}{(u+b)^3(u+a+b)^3} \ge \frac{20a^3x}{(\lambda M)^6} \ge \frac{9.99}{\lambda^6 M}. \end{aligned}$$

In particular, if $n = 0$, we see that

$$9.99\lambda^{-6} \le M\theta < 2.12hM^{-1} < 2.12m^{-1},$$

which contradicts the hypotheses. Thus, we must have $|n| \ge 1$.

We now observe that (note the first equality can be checked using a computer algebra system)

$$\begin{aligned} &(2u+a)(u+b)^3(u+a+b)^3 - (2u+2b+a)u^3(u+a)^3 \\ &\quad = 10bu^3(u+a+b)^3 + b(3(a+b)^2 + 7b^2 + 9ab)u^2(u+a+b)^2 \\ &\qquad + b^2(3(a+b)^3 + ab^2 + 2a^2b - b^3)u(u+a+b) + ab^3(a+b)^3 \\ &\quad < 10bu^3(u+a+b)^3 + 10b(a+b)^2u^2(u+a+b)^2 \\ &\qquad + 3b^2(a+b)^3u(u+a+b) + ab^3(a+b)^3 \\ &\quad < 10bu^3(u+a+b)^3 + b^2u^2(u+a+b)^3 + b^3u(u+a+b)^3 + b^4(a+b)^3 \\ &\quad < b(u+a+b)^3(10u^3 + bu^2 + b^2u + b^3) < 10b(u+b)^3(u+a+b)^3. \end{aligned}$$

Thus,

$$1 - |\theta| \le n + \theta \le \frac{10a^3bx}{u^3(u+a)^3} \le 10a^3bxM^{-6}.$$

Since $|\theta| \le 2.12hH^{-2} < 0.001$, we conclude that

$$0.999 < 10a^3bxM^{-6},$$

and the lemma follows. □

Lemma 7. *Let $\lambda \le 1.04$, $m \ge 8$, and suppose that $mh \le H \le M$. If $0 < a \le b$ and $u, u+a, u+b, u+a+b$ are elements of $S(M)$, then*

$$a^3b^3 > 0.0664x^{-1}M^7. \tag{4.3}$$

Proof. First, we observe that if $b \ge 0.01M$, Lemma 6 gives

$$a^3b^3 \ge 0.0999b^2x^{-1}M^6 \ge 0.000009x^{-1}M^8 > 0.07x^{-1}M^7,$$

since $M \ge mh \ge 8000$. Thus, we may assume for the remainder of the proof that $0 < a \le b < 0.01M$. Let $u_1 = u$, $u_2 = u + a$, $u_3 = u + b$, and $u_4 = u + a + b$, and recall that by the definition of the set $S(M)$, there exist integers $n_1, \dots, n_4$ and reals $\theta_1, \dots, \theta_4$ such that

$$f(u_i) = n_i - \theta_i, \quad 0 < \theta_i < hM^{-3} \qquad (1 \le i \le 4). \tag{4.4}$$

We begin by constructing a rational function of the form

$$R(u; a, b) = \frac{P_1(u; a, b)}{u^3} + \frac{P_2(u; a, b)}{(u+a)^3} + \frac{P_3(u; a, b)}{(u+b)^3} + \frac{P_4(u; a, b)}{(u+a+b)^3}, \tag{4.5}$$

where $P_i(u; a, b)$ are homogeneous quadratic polynomials, which are at most linear in u. Clearly, any such rational function can be rewritten as

$$R(u; a, b) = \frac{C(u; a, b)}{u^3(u+a)^3(u+b)^3(u+a+b)^3}, \tag{4.6}$$

for some homogeneous polynomial $C(u; a, b)$ of total degree 11, which has at most degree 10 in u. We choose the polynomials $P_i(u; a, b)$ as to minimize the degree of C with respect to u. The identities

$$\frac{u+2a}{(u+a)^3} - \frac{u-a}{u^3} = \frac{a^3(2u+a)}{u^3(u+a)^3}, \qquad \frac{1}{u^3} - \frac{1}{(u+a)^3} = \frac{a(3u^2+3ua+a^2)}{u^3(u+a)^3},$$

imply that any choice of the form

$$P_1(u, a, b) = \alpha(-u+a) - \beta, \qquad P_2(u, a, b) = \alpha(u+2a) + \beta,$$
$$P_3(u, a, b) = P_1(u+b, a, b) + 2\beta, \qquad P_4(u, a, b) = P_2(u+b, a, b) - 2\beta,$$

where α, β depend only on a, b, reduces the u-degree of C to at most 7. The choice $\alpha = 3b$ and $\beta = a^2$ then ensures that the coefficients of u^7 and u^6 in $C(u; a, b)$ also cancel out. Thus, we choose

$$P_1(u, a, b) = -3bu + 3ab - a^2, \qquad P_2(u, a, b) = 3bu + 6ab + a^2,$$
$$P_3(u, a, b) = -3bu - 3b^2 + 3ab + a^2, \qquad P_4(u, a, b) = 3bu + 3b^2 + 6ab - a^2.$$

With the above choice, a straightforward (but tedious) direct calculation reveals that

$$C(u;a,b) = 6a^3b(5b^2-a^2)v^5 - 15a^3b(a+b)(5b^2-a^2)v^4 + D(v;a,b), \tag{4.7}$$

where $v = u + a + b$ and $D(v;a,b)$ is a homogeneous polynomial of total degree 11, which is cubic in v. In particular, the coefficient of v^3 in $D(v;a,b)$ is

$$72a^3b^5 + 150a^4b^4 + 52a^5b^3 - 30a^6b^2 - 12a^7b < 74a^3b^3(a+b)^2,$$

on recalling that $0 < a \le b$. Note that we have also

$$72a^3b^5 + 150a^4b^4 + 52a^5b^3 - 30a^6b^2 - 12a^7b > 58a^3b^3(a+b)^2.$$

Similarly, the coefficient of v^2 is bounded above and below as

$$-36a^3b^3(a+b)^3 < 3a^8b + 18a^7b^2 - 3a^6b^3 - 93a^5b^4 - 108a^4b^5 - 33a^3b^6 < -27a^3b^3(a+b)^3;$$

the coefficient of v is bounded above and below as

$$6a^3b^4(a+b)^3 < -3a^8b^2 - 6a^7b^3 + 18a^6b^4 + 48a^5b^5 + 33a^4b^6 + 6a^3b^7 < 14a^3b^4(a+b)^3;$$

and the constant (in v) terms are bounded as

$$-4.25a^4b^5(a+b)^2 < a^8b^3 - 6a^6b^5 - 8a^5b^6 - 3a^4b^7 < -3a^4b^5(a+b)^2.$$

Moreover, since $M < u < v \le \lambda M$ and $0 < a, b < 0.01M$, we find that

$$a^4b^5(a+b)^2 < 0.01a^3b^4(a+b)^3v, \quad a^3b^4(a+b)^3v < 0.01a^3b^3(a+b)^3v^2,$$
$$a^3b^3(a+b)^3v^2 < 0.02a^3b^3(a+b)^2v^3.$$

From these and the earlier bounds on the coefficients of $D(v;a,b)$, we deduce that

$$0 < D(v;a,b) < 74a^3b^3(a+b)^2v^3. \tag{4.8}$$

Inserting this upper bound into (4.7) we now have

$$\begin{aligned} C(u;a,b) &= 6a^3b(5b^2-a^2)v^5 - 15a^3b(a+b)(5b^2-a^2)v^4 + D(v;a,b) \\ &\le a^3bv^3\big(6(5b^2-a^2)v^2 - 15(a+b)(5b^2-a^2)v + 74b^2(a+b)^2\big). \end{aligned}$$

We expand $v = u + a + b$ (and use $a + b < 0.02u$) to bound the term in parentheses above as

$$\begin{aligned} &6(5b^2-a^2)v^2 - 15(a+b)(5b^2-a^2)v + 74b^2(a+b)^2 \\ &\quad = 6(5b^2-a^2)u^2 - 3(a+b)(5b^2-a^2)u - 9(a+b)^2(5b^2-a^2) + 74b^2(a+b)^2 \\ &\quad \le 6(5b^2-a^2)u^2 - 3(a+b)(5b^2-a^2)u + 38b^2(a+b)^2 \\ &\quad < 6(5b^2-a^2)u^2 - 12(a+b)b^2u + (a+b)b^2u < 30b^2u^2. \end{aligned}$$

Thus $C(u;a,b) < 30a^3b^3u^2v^3$. We can also use (4.8) to bound C from below:

$$\begin{aligned} C(u;a,b) &= 3a^3b(5b^2-a^2)v^4(2v-5(a+b)) + D(v;a,b) \\ &> 12a^3b^3v^4(2u-3(a+b)) \\ &> 12a^3b^3v^2\big(u^2+2(a+b)u\big)(2u-3(a+b)) \\ &> 12a^3b^3uv^2\big(2u^2+0.98(a+b)u\big) > 24a^3b^3u^3v^2. \end{aligned}$$

From these bounds and (4.6), we deduce that

$$18.238a^3b^3M^{-7} < \frac{24a^3b^3}{(\lambda M)^7} \le R(u;a,b) \le 30a^3b^3M^{-7}. \tag{4.9}$$

On the other hand, multiplying both sides of (4.5) by x, we get from (4.4) that

$$xR(u;a,b) = n - \theta$$

where $n \in \mathbb{Z}$ is even (since all of the coefficients in each of the P_i are even) and

$$\theta = \sum_{i=1}^{4} P_i(u;a,b)\theta_i.$$

We can bound $|\theta|$ as follows:

$$\begin{aligned} |\theta| &\le 3bu|\theta_1-\theta_2| + 3b(u+b)|\theta_3-\theta_4| \\ &\qquad + 3ab(\theta_1+2\theta_2+\theta_3+2\theta_4) + a^2|\theta_1-\theta_2-\theta_3+\theta_4| \\ &< \big(3b(u+b)+3bu+18ab+2a^2\big)hM^{-3} \\ &< (6b(u+a+b)+11ab)hM^{-3} < (6\lambda+0.11)bhM^{-2}, \end{aligned}$$

upon recalling that $u+a+b<\lambda M$ and $a < 0.01M$.

Next, we will show that $n \neq 0$. Suppose that $n = 0$. Then we have

$$18.238a^3b^3xM^{-7} \le |Cx| = |\theta| \le (6\lambda+0.11)bhM^{-2} \le 6.35bhM^{-2}.$$

Combining this inequality with (4.1), we obtain

$$1.822b^2M^{-1} < 18.238a^3b^3xM^{-7} \le 6.35b(mM)^{-1},$$

which is a contradiction under the hypotheses of the lemma. Thus $n \ge 2$, and we get

$$Cx = n - \theta > 2 - 6.35bhM^{-2} > 2 - 0.0635m^{-1} > 1.992.$$

Combining this with the upper bound in (4.9), we get that

$$1.992 < Cx < 30a^3b^3xM^{-7},$$

and the desired conclusion follows. □

Our last lemma differs from the spacing lemmas established hitherto. In this lemma, instead of proving that the distance b between the two pairs exceeds some lower bound in terms of x, M, and possibly, a, we establish a kind of a dichotomy for b: either $b \geq B_1$ for some lower bound B_1, or $b \leq B_2$, with B_2 significantly smaller than B_1. Note, however, that this dichotomy is not complete, and conditions (4.10) and (4.11) below are not always mutually exclusive—when M is close to H and $m \leq 15$, the two inequalities may hold simultaneously.

Lemma 8. *Let $\lambda \leq 1.04$, $m \geq 8$, and suppose that $mh \leq H \leq M$. If $0 < a \leq b$ and $u, u+a, u+b, u+a+b$ are elements of $S(M)$, then at least one of the conditions*

$$a^5 b < 2\lambda^9 h x^{-1} M^6, \tag{4.10}$$

or

$$a^5 b > (\tfrac{1}{3} - 2\lambda^2 m^{-1}) x^{-1} M^7, \tag{4.11}$$

must hold.

Proof. As in the proof of the last lemma, let $u_1 = u$, $u_2 = u + a$, $u_3 = u + b$, and $u_4 = u + a + b$, and recall (4.4). We rely on the identity

$$\begin{aligned} \frac{a^5 x}{u^3(u+a)^3} &= \frac{(6u^2 - 3au + a^2)x}{u^3} - \frac{(6u^2 + 15ua + 10a^2)x}{(u+a)^3} \\ &= (6u^2 - 3ua + a^2) f(u) - (6u^2 + 15ua + 10a^2) f(u+a) = n' - \theta', \end{aligned}$$

with $n' \in \mathbb{Z}$, even and

$$\theta' = (6u^2 - 3ua + a^2)\theta_1 - (6u^2 + 15ua + 10a^2)\theta_2.$$

Using this relation and the analogous one for the pairs $u+b, u+a+b$, we find that

$$\frac{xa^5}{u^3(u+a)^3} - \frac{xa^5}{(u+b)^3(u+a+b)^3} = n + \theta, \tag{4.12}$$

with $n \in \mathbb{Z}$ even and

$$\begin{aligned} |\theta| &< (6u^2 + 6(u+b)^2 + 12au + 15ab + 11a^2) h M^{-3} \\ &< (6(u+a+b)^2 + 6(u+a)^2) h M^{-3} < 12\lambda^2 h M^{-1}. \end{aligned}$$

By the mean-value theorem,

$$(u+b)^3(u+a+b)^3 - u^3(u+a)^3 = 3b\xi^2(\xi+a)^2(2\xi+a)$$

for some $\xi \in (u, u+b)$, so

$$6bu^3(u+a)^2 < (u+b)^3(u+a+b)^3 - u^3(u+a)^3 < 6b(u+b)^2(u+a+b)^3.$$

We use this to bound the left side of (4.12) and get that

$$n + \theta \leq \frac{6a^5 bx}{u^3(u+a)^3(u+b)} < 6a^5 bx M^{-7}.$$

Similarly, we have

$$n+\theta \geq \frac{6a^5bx}{(u+a)(u+b)^3(u+a+b)^3} > 6\lambda^{-7}a^5bxM^{-7}.$$

Since $M \geq H$, we have that $|\theta| < 12\lambda^2 m^{-1}$. So, it follows from the upper bound on $n+\theta$ that if $b \leq (\frac{1}{3} - 2\lambda^2 m^{-1})a^{-5}x^{-1}M^7$, we have

$$n+\theta < 6a^5bxM^{-7} < 2 - |\theta|;$$

hence, $n < 2$. On the other hand, if $b \geq 2\lambda^9 a^{-5}hx^{-1}M^6$, then using the lower bound for $n+\theta$, we have that

$$n+\theta > 6\lambda^{-7}a^5bxM^{-7} \geq 12\lambda^2 hH^{-1} > |\theta|,$$

implying that $n > 0$. Since $n > 0$ and $n < 2$ cannot occur simultaneously when n is even, the lemma follows. □

Let

$$A = (0.4 - 0.05m^{-1})^{1/5}x^{-1/5}M^{6/5}. \tag{4.13}$$

In Lemma 5, we proved that $b \geq A$ whenever $u, u+a, u+b$ are distinct elements of $S(M)$. Therefore, if $u_0, u_1, \ldots, u_s$ are the elements of $S(M)$, listed in increasing order, the set $S'(M) = \{u_0, u_2, u_4, \ldots\}$ has no gaps $< A$ and satisfies

$$|S(M)| \leq 2|S'(M)|. \tag{4.14}$$

We will use (4.14) and Lemmas 7 and 8 to prove the following result.

Proposition 1. *Suppose $h = 5x^{1/7}\log x$, let $\lambda = 1.04$ and $x \geq e^{200}$, and suppose that $11h \leq M \leq x^{2/7}$. Then*

$$|S(M)| \leq h(\sigma_3(M) + \sigma_4(M)), \tag{4.15}$$

where

$$\sigma_3(M) = \big(0.4212x^{1/7} + 0.1900x^{-1/7}M\big)h^{-1} + 0.0374x^{1/21}M^{-1/3}, \tag{4.16}$$

and

$$\sigma_4(M) = \begin{cases} 0.9519x^{-2/3}M^{11/3} & \text{if } M \leq 18x^{1/6}, \\ 5.1698x^{-1/15}M^{1/15} & \text{if } M > 18x^{1/6}. \end{cases} \tag{4.17}$$

Remark 1. Notice that when x is relatively small, the condition $M \leq 18x^{1/6}$ is impossible, and so only the second condition will be used in the range of "small" values of x.

The proof of these proposition uses the set

$$T(M;a) = \{u : u, u+a \text{ are consecutive elements of } S'(M)\}$$

to bound $|S'(M)|$. The starting point is the elementary identity

$$|S'(M)| = 1 + \sum_{a=1}^{\infty} |T(M;a)| = 1 + \sum_{a \geq A} |T(M;a)|, \tag{4.18}$$

which is a direct consequence of the definition of $T(M;a)$. Further, for any $B \geq A$, we have

$$\sum_{a \geq B} a|T(M;a)| \leq \sum_{a \geq A} a|T(M;a)| \leq (\lambda - 1)M + 1,$$

so

$$\sum_{a \geq B} |T(M;a)| \leq (\lambda - 1)MB^{-1} + B^{-1}.$$

Applying this inequality to the right side of (4.18), we find that, for any parameter $B \geq 2$,

$$|S'(M)| \leq 1.5 + (\lambda - 1)MB^{-1} + \sum_{A \leq a < B} |T(M;a)|. \tag{4.19}$$

Proof of Proposition 1. We select

$$B = \delta x^{-1/7} M, \quad \delta = 0.19, \tag{4.20}$$

in the application of (4.19). Fix an integer a, with $A \leq a \leq B$. By Lemma 8, if we consider an interval I of length

$$|I| \leq (\tfrac{1}{3} - 2\lambda^2 m^{-1})x^{-1}M^7,$$

we must have $b < 2\lambda^9 ha^{-5}x^{-1}M^6$ for any elements $u, u + b \in T(M;a) \cap I$. Then we can use (4.3) to get

$$\begin{aligned} |T(M;a) \cap I| &\leq \frac{2\lambda^9 ha^{-5}x^{-1}M^6}{(0.0664)^{1/3}a^{-1}x^{-1/3}M^{7/3}} + 1 \\ &< 7.0298a^{-4}hx^{-2/3}M^{11/3} + 1. \end{aligned}$$

Since we need at most

$$\frac{(\lambda - 1)M}{(\frac{1}{3} - 2\lambda^2 m^{-1})x^{-1}M^7 a^{-5}x^{-1}M^7} + 1 < 0.2929a^5 xM^{-6} + 1$$

intervals of length $|I|$ to cover $(M, \lambda M]$, we conclude that

$$\begin{aligned} |T(M;a)| &\leq (0.2929a^5 xM^{-6} + 1)(7.0298a^{-4}hx^{-2/3}M^{11/3} + 1) \\ &\leq 2.0591ahx^{1/3}M^{-7/3} + 0.2929a^5 xM^{-6} + 7.0298a^{-4}hx^{-2/3}M^{11/3} + 1. \end{aligned} \tag{4.21}$$

Next, we use this to bound the right side of (4.19). With our choice of parameters, (4.19) yields

$$|S'(M)| \leq 1.5 + 0.04\delta^{-1}x^{1/7} + \sum_{\substack{A \leq a < B \\ a \text{ even}}} |T(M;a)|. \tag{4.22}$$

Next, we sum each of the four terms on the right side of (4.21) over $a \in [A, B)$. Recall the inequality

$$\sum_{k \leq K} k^s < \frac{(K+1)^{s+1}}{s+1} \qquad (s > 0),$$

and note that $B = \delta M x^{-1/7} \geq 11h\delta x^{-1/7} > 10.45 \log x > 2090$. Thus,

$$\sum_{\substack{2 \leq a \leq B \\ a \text{ even}}} a^s < \frac{(B+2)^{s+1}}{2(s+1)} < \frac{(1.001B)^{s+1}}{2(s+1)},$$

and we deduce

$$\begin{aligned} 2.0591hx^{1/3}M^{-7/3} \sum_{\substack{A \leq a < B \\ a \text{ even}}} a &< \frac{2.0591(1.001B)^2}{4} x^{1/3} M^{-7/3} \\ &< 0.0187hx^{1/21}M^{-1/3}, \end{aligned} \tag{4.23}$$

$$0.2929xM^{-6} \sum_{\substack{A \leq a < B \\ a \text{ even}}} a^5 < \frac{0.2929 \cdot (1.001B)^6}{12} xM^{-6} < 0.00001x^{1/7}. \tag{4.24}$$

Combining (4.20)–(4.24), we conclude that

$$|S'(M)| \leq h\sigma_3'(M) + 7.0298hx^{-2/3}M^{11/3} \sum_{\substack{A \leq a < B \\ a \text{ even}}} a^{-4}, \tag{4.25}$$

where

$$\sigma_3'(M) = \left(0.2106x^{1/7} + 0.095x^{-1/7}M\right) h^{-1} + 0.0187x^{1/21}M^{-1/3}. \tag{4.26}$$

We estimate the sum on the right side of (4.25) in different ways, depending on the size of M. When $M \leq 18x^{1/6}$, we use that

$$\sum_{\substack{A \leq a < B \\ a \text{ even}}} a^{-4} < \frac{\zeta(4)}{16} = \frac{\pi^4}{1440} < 0.0677. \tag{4.27}$$

On the other hand, when $M > 18x^{1/6}$, we have $A > 0.8306 \cdot 18^{6/5} > 26.6513$, so

$$\sum_{\substack{A \leq a < B \\ a \text{ even}}} a^{-4} < \frac{1}{3 \cdot 2^4} \left(\frac{A}{2} - 1\right)^{-3} < 0.2107A^{-3} < 0.3677x^{3/5}M^{-18/5}. \tag{4.28}$$

The proposition follows from (4.14) and (4.25)–(4.28). □

5. Proof of Theorem 2

As stated in the introduction, the theorem can be checked by brute force for $x \leq e^{41}$. Thus, we focus on proving it for $x \geq e^{41}$.

5.1. **Large** x**.** Let $x \ge e^{200}$ and set $H = 11h$ in (2.2) and (2.11). We will use Proposition 1 and Lemma 2 to bound $|S(H, x^{2/7})|$.

Suppose first that $H \le 18x^{1/6}$, (in this case we can assume $x \ge e^{284}$) we again split $S(H, x^{2/7})$ according to the two cases in (4.17). When we apply Lemma 2 to the bound (4.15) for $M \in [H, 18x^{1/6}]$, we find that

$$\begin{aligned} |S(H, 18x^{1/6})| &< h\left(\frac{0.9519 \cdot 18^{11/3}x^{-1/18}}{1 - 1.04^{-11/3}} + \frac{0.0374x^{1/21}H^{-1/3}}{1 - 1.04^{-1/3}}\right) + \frac{3.8x^{1/42}}{1 - 1.04^{-1}} \\ &\quad + 0.4212x^{1/7}\left(\frac{\log x}{42\log(1.04)} - \frac{\log(\frac{55}{18}\log x)}{\log(1.04)} + 1\right) \\ &< 0.1552h + 0.2557x^{1/7}\log x - 72.2396x^{1/7} < 0.2064h. \end{aligned}$$

Similarly, when we apply Lemma 2 to (4.15) for $M \in [18x^{1/6}, x^{2/7}]$, we get

$$\begin{aligned} |S(18x^{1/6}, x^{2/7})| &< h\left(\frac{5.1698x^{-1/21}}{1 - 1.04^{-1/15}} + \frac{0.0374 \cdot 18^{-1/3}x^{-1/126}}{1 - 1.04^{-1/3}}\right) + \frac{0.19x^{1/7}}{1 - 1.04^{-1}} \\ &\quad + 0.4212x^{1/7}\left(\frac{5\log x}{42\log(1.04)} - \frac{\log 18}{\log(1.04)} + 1\right) \\ &< 0.1180h + 1.2785x^{1/7}\log x - 25.6791x^{1/7} < 0.3737h. \end{aligned}$$

Hence,

$$|S(H, x^{2/7})| < 0.2064h + 0.3737h = 0.5801h. \tag{5.1}$$

Next, we consider the case $H > 18x^{1/6}$, noting that we must have $x \le e^{285}$. We use the latter case of Proposition 1 for M in the full range $(H, x^{2/7}]$, and we find that

$$\begin{aligned} |S(H, x^{2/7})| &< h\left(\frac{5.1698x^{-1/21}}{1 - 1.04^{-1/15}} + \frac{0.0374x^{1/21}H^{-1/3}}{1 - 1.04^{-1/3}}\right) + \frac{0.19x^{1/7}}{1 - 1.04^{-1}} \\ &\quad + 0.4212x^{1/7}\left(\frac{\log x}{7\log(1.04)} - \frac{\log(55\log x)}{\log(1.04)} + 1\right) \\ &< 0.2742h + 1.5342x^{1/7}\log x - 94.5742x^{1/7} < 0.5148h, \end{aligned} \tag{5.2}$$

on noting that $94.5742x^{1/7} > 0.0663h$ when $x \le e^{285}$.

To complete the estimation of $|S(H, \sqrt[3]{2x})|$, we apply Lemma 2 to the bound in Corollary 1 for $M \in [x^{2/7}, \sqrt[3]{2x}]$. This yields

$$|S(x^{2/7}, \sqrt[3]{2x})| \le \frac{0.1202x^{1/7}}{1 - 1.04^{-3}} + \frac{\log x}{21\log(1.04)} + \frac{\log 2}{3\log(1.04)} + 1 < 0.0011h. \tag{5.3}$$

Combining (5.1)–(5.3), we obtain (2.11) with

$$\sigma_3 = \begin{cases} 0.5812 & \text{if } H \le 18x^{1/6}, \\ 0.5159 & \text{if } H > 18x^{1/6}, \end{cases}$$

for all $x \geq e^{200}$. Taking $J = 100$ in (2.6), we have $\sigma_0(h, 100) \leq -0.0066$ in the same range. Furthermore, we have

$$\sigma_2(h, 11) < \begin{cases} 0.2256 & \text{if } x \geq e^{284}, \\ 0.3020 & \text{if } x \geq e^{200}. \end{cases}$$

Thus,

$$\sigma_0(h, 100) + \sigma_1 + \sigma_2(h, 11) + \sigma_3(h, 11) < \begin{cases} 0.9750 & \text{if } H \leq 18x^{1/6}, \\ 0.9861 & \text{if } H > 18x^{1/6}, \end{cases}$$

which establishes (2.12), and therefore the theorem, for $x \geq e^{200}$.

5.2. **Intermediate x.** Suppose that $x \geq e^{41}$. We take $h = 2x^{1/5}$, $H = 2.7h$, and $\lambda = 1.06$, and apply Lemma 2 to the result of Corollary 2. We obtain

$$\begin{aligned} |S(H, \sqrt[3]{2x})| &\leq \frac{1.2604(0.06)(xH^{-2})^{1/3}}{1 - 1.06^{-2/3}} + \frac{\log \sqrt[3]{2x} - \log H}{\log(1.06)} + 1 \\ &< 0.3225h + \frac{2 \log x}{15 \log(1.06)} - \frac{\log(5.4) - \frac{1}{3} \log 2}{\log(1.06)} + 1 < 0.3354h. \end{aligned}$$

That is, (2.11) holds with $\sigma_3(h, 2.7) = 0.3354$. As $\sigma_2(h, 2.7) \leq 0.3146$, and $\sigma_0(h, 6) < -0.0047$ we get

$$\sigma_0(h, 6) + \sigma_1 + \sigma_2(h, 2.7) + \sigma_3(h, 2.7) < 0.8201.$$

This establishes that the interval $(x, x + 2x^{1/5}]$ contains a cubefree integer for all $x \geq e^{41}$. Moreover, recalling the results of [8], we obtain the following proposition.

Proposition 2. *For any $x \geq 2$, the interval $(x, x + 2x^{1/5}]$ contains a cubefree integer.*

Since $2x^{1/5} \leq 5x^{1/7} \log x$ for $x \leq e^{95.8}$ the main result follows from Proposition 2 for $x \leq e^{95}$. Next, we prove that $h = 10x^{1/6}$ is admissible when $x \geq e^{95}$. With this choice of h, we let $H = 4h$ and $\lambda = 1.03$. An application of Lemma 2 to the bound of Corollary 3 yields

$$|S(H, \sqrt[3]{2x})| \leq \frac{0.0726(xH^{-1})^{1/5}}{1 - 1.03^{-1/5}} + \frac{2 \log x}{6 \log(1.03)} < 0.5891h,$$

or $\sigma_3(h, 4) = 0.5891$ in (2.11). Since $\sigma_0(h, 20) < -0.0066$ and $\sigma_2(h, 4) < 0.2207$ for $x \geq e^{95}$, we deduce that

$$\sigma_0(h, 20) + \sigma_1 + \sigma_2(h, 4) + \sigma_3(h, 4) < 0.9780.$$

Since $10x^{1/6} \leq 5x^{1/7} \log x$ in the range $x \leq e^{191.6}$, this establishes the theorem for $x \leq e^{191}$.

By the same method we show that $h = 8.5x^{1/6}$ is admissible when $x \geq e^{191}$. With this choice of h, we let $H = 5h$ and $\lambda = 1.01$. In this case, applying Lemma 2 to the bound of Corollary 3 yields

$$|S(H, \sqrt[3]{2x})| \leq \frac{0.0242(xH^{-1})^{1/5}}{1 - 1.01^{-1/5}} + \frac{2\log x}{6\log(1.01)} < 0.6767h,$$

or $\sigma_3(h, 5) = 0.6767$ in (2.11). Since $\sigma_0(h, 20) < -0.0066$ and $\sigma_2(h, 5) < 0.1465$ for $x \geq e^{191}$, we deduce that

$$\sigma_0(h, 20) + \sigma_1 + \sigma_2(h, 5) + \sigma_3(h, 5) < 0.9914.$$

Since $8.5x^{1/6} \leq 5x^{1/7}\log x$ in the range $x \leq e^{200.3}$, this completes the proof of Theorem 2.

As we close this section, we take a moment to record the following result, which we just proved.

Proposition 3. *For any $x \geq e^{95}$, the interval $(x, x + 10x^{1/6}]$ contains a cubefree integer, and for any $x \geq e^{191}$, the interval $(x, x+8.5x^{1/6}]$ contains a cubefree integer.*

6. Asymptotic results

We conclude by noting a few of the explicit bounds that can be obtained by these methods if one no longer requires the bounds to be admissible for all values of $x \geq 2$, allowing instead results valid for sufficiently large values of x.

Some of the possible results that can be obtained by tweaking the parameters used in the proof of Theorem 2 are given in the statement of Theorem 3. To prove any of those results, we reset the parameters m, J, λ, δ that appear in the proofs of Proposition 1 and Theorem 2 and then update the various constants. (When x is as large as in Theorem 3, the inequality $H \leq 18x^{1/6}$ always holds, so only the first case in the proof of Theorem 2 can occur.) To establish the claims of Theorem 3, we always select $J = 20$, $\lambda = 1.002$, and $m = \sqrt{\log x_0}$, where x_0 is the lower bound on x in each result; we only vary the choice of δ. For example, when $h = 2x^{1/7}\log x$, $x \geq e^{550}$ (hence, $m = 23.4520\cdots$), and $\delta = 0.38$, we have

$$\sigma_0(h, 20) + \sigma_1 + \sigma_2(h, m) + \sigma_3(h, m) < 0.9914.$$

For $h = x^{1/7}\log x$ and $x \geq e^{2300}$, the choice $\delta = 0.66$ yields an upper bound of 0.9919; and for $h = \frac{1}{2}x^{1/7}\log x$ and $x \geq e^{75\,000}$, $\delta = 0.9$ gives a bound of 0.9977.

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References

[1] P. Erdős, *Some problems and results in elementary number theory*, Publ. Math. Debrecen **2** (1951), 103–109. MR45759

[2] M. Filaseta, *Short interval results for squarefree numbers*, J. Number Theory **35** (1990), no. 2, 128–149. MR1057318

[3] M. Filaseta, S. Graham, and O. Trifonov, *Starting with gaps between k-free numbers*, Int. J. Number Theory **11** (2015), no. 5, 1411–1435. MR3376218

[4] M. Filaseta and O. Trifonov, *On gaps between squarefree numbers. II*, J. London Math. Soc. (2) **45** (1992), no. 2, 215–221. MR1171549

[5] E. Fogels, *On average values of arithmetic functions*, Proc. Cambridge Philos. Soc. **37** (1941), 358–372. MR4843

[6] H. Halberstam and K. F. Roth, *On the gaps between consecutive k-free integers*, J. London Math. Soc. **26** (1951), 268–273. MR43120

[7] A. Kumchev, W. McCormick, N. McNew, A. Park, R. Scherr, S. Ziehr, *Explicit bounds for large gaps between squarefree integers*, J. Number Theory, to appear.

[8] Michael J. Mossinghoff, Tomás Oliveira e Silva, and Timothy S. Trudgian, *The distribution of k-free numbers*, Math. Comp. **90** (2021), no. 328, 907–929. MR4194167

[9] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94. MR137689

[10] O. Trifonov, *On gaps between k-free numbers*, J. Number Theory **55** (1995), no. 1, 46–59. MR1361558

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