Squarefree Integers in Short Intervals

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1 Introduction

The problem of finding an $h = h(x)$ such that for x sufficiently large $(x, x+h]$ contains a squarefree integer is an old one with a number of authors' contributions. The best result is due to M. Filaseta and O. Trifonov [1] who prove that $h = cx^{\frac{1}{5}} \log x$ is admissible. Here $c > 0$ is an unspecified absolute constant. In this note we give concrete meaning to both the constant c and the words "sufficiently large" in the Theorem from [1]. We establish the following

Theorem. *For every $x \geq 2$ the interval $(x, x+1000x^{\frac{1}{5}} \log x]$ contains a squarefree integer.*

We follow closely the approach from [1], but at the same time pay much more attention to the exact constants in our estimates. Also, for "small" values of x we make use of the asymptotically weaker results $h = 2x^{\frac{1}{2}}$, $h = 2x^{\frac{1}{3}}$ and $h = 9x^{\frac{1}{4}}$. These are stated as lemmas which are proved in Sections 3 and 4.

2 Preliminaries

Let $N(x,h)$ be the number of the integers in $(x, x+h]$ that are not squarefree. As in [1] we establish that $(x, x+h]$ contains a squarefree number by showing that $N(x,h) < h - 1$. First we note that

$$
\begin{aligned}
(1) \qquad N(x,h) &\leq \sum_p \left(\left[\frac{x+h}{p^2} \right] - \left[\frac{x}{p^2} \right] \right) \\
&\leq \sum_{p \leq H} \left(\frac{h}{p^2} + 1 \right) + \sum_{p > H} \left(\left[\frac{x+h}{p^2} \right] - \left[\frac{x}{p^2} \right] \right) \\
&=: \textstyle\sum_1 + \sum_2 .
\end{aligned}
$$

For $\sum_1$ we obtain

$$
(2) \qquad \textstyle\sum_1 \leq h \displaystyle\sum_p \frac{1}{p^2} + \pi(H) < 0.454h + \pi(H) .
$$

On the other hand, if $p > \sqrt{2x}$ and $h \leq x$, we have

$$
0 < \frac{x}{p^2} < \frac{x+h}{p^2} < \frac{x+h}{2x} \leq 1 ,
$$

so that all terms in $\sum_2$ with $p > \sqrt{2x}$ are zero and

$$\sum\nolimits_2 = \sum_{H<p\le\sqrt{2x}} \left(\left[\frac{x+h}{p^2}\right] - \left[\frac{x}{p^2}\right] \right) .$$

Also, if $p > \sqrt{h}$

$$0 \le \left[\frac{x+h}{p^2}\right] - \left[\frac{x}{p^2}\right] \le \frac{h}{p^2} + 1 < 2$$

with

$$\left[\frac{x+h}{p^2}\right] - \left[\frac{x}{p^2}\right] = 1 \quad \Longleftrightarrow \quad \exists n \in \mathbb{Z}: \ \frac{x}{p^2} < n \le \frac{x+h}{p^2} .$$

Hence, if we define

$$S(A,B) = \# \left\{ n \in \mathbb{Z}: \ A < n \le B, \ \{x/n^2\} \in (1 - hn^{-2}, 1) \right\} ,$$

we will have

$$\sum\nolimits_2 \le S(H, \sqrt{2x}) .$$

Therefore, in view of (1) and (2), in order to prove that $(x, x+h]$ contains a squarefree integer it suffices to show that

$$\pi(H) + S(H, \sqrt{2x}) < 0.546h - 1 \tag{3}$$

for some $H > \sqrt{h}$.

We end this section with two technical lemmas. Lemma 1 is an effective version of a well known result (e.g. Lemma 1 of [1]). Lemma 2 contains (3.1) and (3.6) of [2].

Lemma 1. *Suppose that A, a, b, u and v are real numbers with $A > 0$ and $0 < u < v < 1$, and assume that for all $M \in [x^u, x^v]$ the estimate*

$$S(M, 2M) \le Ax^a M^b$$

holds. Then

(a) if $b > 0$,

$$S(x^u, x^v) \le \frac{A}{1 - 2^{-b}} \cdot x^{a+bv} ;$$

(b) if $b < 0$,

$$S(x^u, 2x^v) \le \frac{A}{1 - 2^{b}} \cdot x^{a+bu} ;$$

(c) if $b = 0$,

$$S(x^u, 2x^v) \le Ax^a \left(\frac{v-u}{\log 2} \cdot \log x + 2 \right) .$$

Lemma 2. *For any $x > 1$ we have*

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{3}{2\log x}\right)$$

and

$$\pi(x) < 1.256x/\log x \; .$$

3 Small Values of x

In this section we prove that $h = 2x^{\frac{1}{2}}$ and $h = 2x^{\frac{1}{3}}$ are admissible in the gap problem for the squarefree numbers. We establish the following lemmas

Lemma 3. *If $x \geq 200$, the interval $(x, x + 2x^{\frac{1}{2}}]$ contains a squarefree number.*

Lemma 4. *If $x \geq e^{20}$, the interval $(x, x + 2x^{\frac{1}{3}}]$ contains a squarefree number.*

Initially, we show that the Theorem is true for $x \leq e^{79}$ assuming that Lemmas 3 and 4 hold. Note that if $x \in (e^{20}, e^{79}]$ then

$$2x^{\frac{1}{3}} \leq 1000x^{\frac{1}{5}} \log x \; ,$$

and for $x \in [200, e^{20}]$

$$2x^{\frac{1}{2}} \leq 1000x^{\frac{1}{5}} \log x \; .$$

Hence, in the first case the Theorem follows from Lemma 4, and in the second from Lemma 3. Finally, if $2 \leq x \leq 200$, the interval $(x, x{+}1000x^{\frac{1}{5}} \log x]$ contains the integer 201 that is squarefree.

Now we prove Lemmas 3 and 4.

Proof of Lemma 3. We need to prove (3) with $h = 2x^{\frac{1}{2}}$. Choosing $H = \sqrt{2x}$, we have $S(H, \sqrt{2x}) = 0$, so (3) is equivalent to

$$\pi(h/\sqrt{2}) < 0.546h - 1 \; .$$

By the second part of Lemma 2,

$$\pi(h/\sqrt{2}) < 1.256 \times \frac{h/\sqrt{2}}{\log(h/\sqrt{2})} < 0.3h$$

if $x \geq 200$. Hence, the lemma follows. □

For the rest of this note we let $f(u) = xu^{-2}$ and

$$T_\varphi = \left\{n \in \mathbb{Z} : \; x^\varphi < n \leq 2x^\varphi, \; \{f(n)\} \in (1 - hn^{-2}, 1)\right\} \; .$$

Proof of Lemma 4. We prove (3) with $h = 2x^{\frac{1}{3}}$ and $H = 2h$. Let u and $u + a$ be two elements of T_φ, $\frac{1}{3} \le \varphi \le \frac{1}{2}$. We have

$$f(u) = n_1 - \theta_1 \qquad \text{and} \qquad f(u+a) = n_2 - \theta_2$$

where n_i are integers and $0 < \theta_i < hx^{-2\varphi}$. So,

$$f(u+a) - f(u) = (n_2 - n_1) + (\theta_1 - \theta_2) =: n + \theta$$

with $n \in \mathbb{Z}$ and $|\theta| < hx^{-2\varphi}$. First we show that $n \neq 0$. Suppose n is zero. Then we get

$$|f(u+a) - f(u)| < hx^{-2\varphi}.$$

On the other hand, by the Mean Value Theorem,

$$|f(u+a) - f(u)| = a|f'(\xi)| = \frac{2ax}{\xi^3} \ge \frac{1}{4}x^{1-3\varphi}.$$

Combining these inequalities, we obtain

$$x < 4hx^{\varphi} \le 4hx^{\frac{1}{2}} = 8x^{\frac{5}{6}} ,$$

which is impossible for $x \ge e^{20}$. Hence, n cannot be zero and we must have $|n| \ge 1$. Also, for $x^\varphi \ge H$ we have

$$|\theta| < hx^{-2\varphi} \le hH^{-2} = \tfrac{1}{8}x^{-\frac{1}{3}} \le \tfrac{1}{8}e^{-\frac{20}{3}} =: \delta ,$$

so that

$$|f(u+a) - f(u)| = |n - \theta| \ge 1 - \delta .$$

This inequality and another application of the Mean Value Theorem give the estimate

$$a \ge 0.5(1-\delta)x^{3\varphi-1}, \tag{4}$$

and hence,

$$S(x^\varphi, 2x^\varphi) \le 2(1-\delta)^{-1}x^{1-2\varphi} + 1 . \tag{5}$$

Now (5) and Lemma 1 give

$$\begin{aligned} S(H, \sqrt{2x}) &\le \frac{8}{3(1-\delta)} \cdot xH^{-2} + \frac{\log x}{6\log 2} + 2 \\ &\le h\left(\frac{1}{12(1-\delta)} + \frac{\log x}{12x^{\frac{1}{3}}\log 2} + x^{-\frac{1}{3}}\right) < 0.089h - 1 \end{aligned} \tag{6}$$

if $x \ge e^{20}$. Also, by the first estimate of Lemma 2, we have

$$\pi(2h) < 0.295h ,$$

which in combination with (6) proves the lemma. □

4 Intermediate Values of x

In this section we prove that $h = 9x^{\frac{1}{4}}$ is admissible. The result is

Lemma 5. *If $x \geq e^{79}$, the interval $(x, x + 9x^{\frac{1}{4}}]$ contains a squarefree number.*

Note that since for $x \in (e^{79}, e^{200}]$

$$9x^{\frac{1}{4}} \leq 1000x^{\frac{1}{5}} \log x ,$$

this lemma and the results from Section 3 establish the Theorem for all $x \leq e^{200}$.

Proof of Lemma 5. We prove (3) with $H = h = 9x^{\frac{1}{4}}$. Let $u_1 = u$, $u_2 = u + a$, $u_3 = u + a + b$ be 3 consecutive elements of T_φ, $\frac{1}{4} \leq \varphi \leq \frac{1}{2}$, and let

$$f(u_i) = n_i - \theta_i \qquad ; n_i \in \mathbb{Z},\ 0 < \theta_i < hx^{-2\varphi}.$$

Consider the second divided difference

$$f[u_1, u_2, u_3] = \frac{f(u_1)(u_3 - u_2) - f(u_2)(u_3 - u_1) + f(u_3)(u_2 - u_1)}{(u_2 - u_1)(u_3 - u_1)(u_3 - u_2)} =: \frac{A}{V} .$$

We have $A = n + \theta$ where

$$n = bn_1 - (a + b)n_2 + an_3 \qquad \text{and} \qquad \theta = b\theta_1 - (a + b)\theta_2 + a\theta_3 ,$$

and hence, $|\theta| < (a + b)hx^{-2\varphi}$. Assuming that $n = 0$, we obtain

$$|f[u_1, u_2, u_3]| = \frac{|\theta|}{V} < \frac{(a + b)hx^{-2\varphi}}{ab(a + b)} = \frac{hx^{-2\varphi}}{ab} ,$$

and in the same time

$$|f[u_1, u_2, u_3]| = \left| \frac{f''(\xi)}{2!} \right| = \frac{3x}{\xi^4} \geq \frac{3}{16} x^{1-4\varphi} .$$

These inequalities imply

$$ab < \frac{16}{3} hx^{2\varphi - 1} = 48x^{2\varphi - \frac{3}{4}} .$$

Since a and b are positive integers, we have $ab \geq 1$, and hence we must have $x^\varphi \geq \frac{1}{4\sqrt{3}} x^{\frac{3}{8}}$. On the other hand, by (4), $ab \geq 0.5(1 - \delta)x^{3\varphi - 1}$ and we get

$$x^\varphi < \frac{32h}{3(1 - \delta)} = \frac{96x^{\frac{1}{4}}}{1 - \delta} .$$

For $x \geq e^{79}$ this is a contradiction, whence $|n| \geq 1$.

Now, we proceed to show that

(7) $$a + b \geq \sqrt[3]{1.32} x^{(4\varphi - 1)/3} .$$

If $a+b \geq 0.01x^{\varphi}$, this is true by virtue of the condition $x \geq e^{79}$, so assume that $a+b < 0.01x^{\varphi}$. In this case

$$|\theta| < (a+b)hx^{-2\varphi} < 0.01\,.$$

Hence, we have

$$|f[u_1,u_2,u_3]| = \frac{|A|}{V} \geq \frac{|n|-|\theta|}{V} \geq \frac{0.99}{ab(a+b)}\,,$$

and

$$|f[u_1,u_2,u_3]| = \frac{3x}{\xi^4} \leq 3x^{1-4\varphi}\,.$$

Since $4ab \leq (a+b)^2$, we can derive (7) from these inequlities. Now, using (7), we find

$$S(x^{\varphi},2x^{\varphi}) \leq \frac{2}{\sqrt[3]{1.32}}x^{(1-\varphi)/3}+2\,, \tag{8}$$

and putting this in Lemma 1

$$\begin{aligned} S(H,\sqrt{2x}) &\leq \frac{2}{\sqrt[3]{1.32}(1-2^{-\frac{1}{3}})}x^{\frac{1}{3}}h^{-\frac{1}{3}} + \frac{\log x}{2\log 2}+2 \\ &\leq h\left(\frac{2}{9\sqrt[3]{9}\sqrt[3]{1.32}(1-2^{-\frac{1}{3}})} + \frac{\log x}{18x^{\frac{1}{4}}\log 2} + \frac{2}{9}x^{-\frac{1}{4}}\right) \\ &< 0.473h-1 \end{aligned} \tag{9}$$

if $x \geq e^{79}$. Also, by the first estimate in Lemma 2, we obtain that for $x \geq e^{79}$

$$\pi(h) \leq 0.049h\,.$$

The last estimate and (9) complete the proof of the lemma. □

5 Large Values of x

First note that Lemma 1 and (5) imply that if $x \geq e^{200}$, then

$$S(x^{0.4},\sqrt{2x}) \leq \frac{8x^{\frac{1}{5}}}{3(1-\delta)} + \frac{\log x}{10\log 2}+2 < 0.001h\,.$$

Hence, by the discussion in the previous sections, it suffices to show that if $x \geq e^{200}$,

$$\pi(H) + S(H,x^{0.4}) \leq 0.545h-1$$

with $h = 1000x^{\frac{1}{5}}\log x$ and a suitable H. We choose $H = 12h$. Then by Lemma 2 we have

$$\pi(H) \leq 0.226h\,,$$

and it remains to show that

$$S(H, x^{0.4}) \le 0.319h - 1 \,. \tag{10}$$

Now, we define the set

$$T(a) = \{u : \; u, \, u + a \text{ are consecutive elements of } T_\varphi\}$$

and denote its cardinality by $t(a)$. We also set

$$A = \sqrt[3]{1.32} x^{(4\varphi-1)/3} \qquad \text{and} \qquad B = 0.005x^{\varphi - \frac{1}{5}} \,.$$

Then

$$S(x^\varphi, 2x^\varphi) = 1 + \sum_{a=1}^{\infty} t(a) \le 2 + 2\sum_{a \ge A} t(a) \,,$$

by virtue of the estimate (7). Also, since for any $B \ge 1$

$$x^\varphi \ge \sum_{a=1}^{\infty} at(a) \ge \sum_{a \ge B} at(a) \ge B \sum_{a \ge B} t(a) \,,$$

we obtain

$$S(x^\varphi, 2x^\varphi) \le 2 + 2x^\varphi B^{-1} + 2 \sum_{A \le a < B} t(a) \,. \tag{11}$$

At this point it is convenient to replace the set $T(a)$ by its subset $T_1(a)$, obtained by dropping the last element of $T(a)$ and every other element $u \in T(a)$ such that $u - a$ is also in $T(a)$. Set $t_1(a) = |T_1(a)|$. Then $t(a) \le 2t_1(a) + 1$, and if u and $u+b$ are two elements of $T_1(a)$, $u_1 = u$, $u_2 = u+a$, $u_3 = u+b$, $u_4 = u+a+b$ are distinct elements of T_φ. Also, for any $u \in T_1(a)$, $u + 2a \le 2x^\varphi$. Now, consider

$$f[u_1, \ldots, u_4] = -\frac{f(u_1)}{ab(a+b)} + \frac{f(u_2)}{ab(b-a)} - \frac{f(u_3)}{ab(b-a)} + \frac{f(u_4)}{ab(a+b)} =: -\frac{U}{V} \,.$$

We proceed to show that $U \ge 0.5$. Since $u_i \in T_\varphi$,

$$f(u_i) = n_i - \theta_i \qquad ; n_i \in \mathbb{Z}, \; 0 < \theta_i < hx^{-2\varphi}.$$

So, if $0 < U < 0.5$, we have

$$\begin{aligned} U = \|U\| &= \| -\theta_1(b-a) + \theta_2(b+a) - \theta_3(b+a) + \theta_4(b-a) \| \\ &\le hx^{-2\varphi}((b+a) + (b-a)) = 2bhx^{-2\varphi} \,, \end{aligned}$$

and hence

$$|f[u_1, \ldots, u_4]| \le \frac{2bhx^{-2\varphi}}{ab(a+b)(b-a)} \,.$$

On the other hand,

$$|f[u_1,\dots,u_4]| = \frac{4x}{\xi^5} \geq \frac{1}{8}x^{1-5\varphi} .$$

Combining these estimates, we get

$$\frac{2bhx^{-2\varphi}}{ab(a+b)(b-a)} \geq \frac{1}{8}x^{1-5\varphi} ,$$

and this implies

$$a(b-a)(b+a) \leq 16hx^{3\varphi-1} .$$

Since $a \geq A$ and $b \geq 2A$, we also have

$$a(b-a)(b+a) \geq 3A^3 = 3.96x^{4\varphi-1} .$$

So, if $x^\varphi \geq H$, we obtain $12h \leq x^\varphi \leq 5h$. Hence, we must have $U \geq 0.5$. We now find that

$$4x^{1-5\varphi} \geq \frac{4x}{\xi^5} = |f[u_1,\dots,u_4]| \geq \frac{1}{2ab(a+b)(b-a)} \geq \frac{1}{3ab^3} .$$

Finally we obtain

$$ab^3 \geq \frac{1}{12}x^{5\varphi-1} . \tag{12}$$

Now, we use the polynomial identity

$$\frac{2u-a}{u^2} - \frac{2u+3a}{(u+a)^2} = -\frac{a^3}{u^2(u+a)^2} .$$

If $u \in T_1(a)$, this implies

$$-\frac{a^3x}{u^2(u+a)^2} = (2u-a)f(u) - (2u+3a)f(u+a) = n + \theta$$

where n is an integer and

$$|\theta| = |-\theta_1(2u-a) + \theta_2(2u+3a)| < 4hx^{-\varphi} .$$

Applying this to u and $u+b$, we obtain

$$\frac{-xa^3}{(u+b)^2(u+a+b)^2} + \frac{xa^3}{u^2(u+a)^2} = n + \theta$$

with $n \in \mathbb{Z}$ and $|\theta| < 8hx^{-\varphi}$, i.e.

$$\frac{xa^3b(2u+a+b)[(u+b)(u+a+b)+u(u+a)]}{u^2(u+a)^2(u+b)^2(u+a+b)^2} = n + \theta .$$

The left hand side is both

$$\leq \frac{4xa^3b}{u^2(u+a)^2(u+b)} \leq 4a^3bx^{1-5\varphi} ,$$

and

$$\geq \frac{4xa^3b}{(u+a)(u+b)^2(u+a+b)^2} \geq \frac{1}{8}a^3bx^{1-5\varphi} .$$

Since the condition $x^\varphi \geq H$ implies $|\theta| < \frac{2}{3}$, we obtain from the upper estimate that if $b \leq \frac{1}{12}a^{-3}x^{5\varphi-1}$, we must have $n < 1$. On the other hand, the lower estimate implies that if $b > 64a^{-3}hx^{4\varphi-1}$, n is a positive integer. Therefore these two conditions cannot hold simultaneously, i.e. either

$$b > \frac{1}{12}a^{-3}x^{5\varphi-1} ,$$

or

$$b \leq 64a^{-3}hx^{4\varphi-1} . \tag{13}$$

Consequently, if I is an interval of length $|I| \leq \frac{1}{12}a^{-3}x^{5\varphi-1}$, and $u, u+b \in T_1(a) \cap I$, the condition (13) must hold. When combined with (12) this implies

$$|I \cap T_1(a)| \leq \frac{64a^{-3}hx^{4\varphi-1}}{\sqrt[3]{\frac{1}{12}}a^{-\frac{1}{3}}x^{(5\varphi-1)/3}} + 1 = 64\sqrt[3]{12}a^{-\frac{8}{3}}hx^{(7\varphi-2)/3} + 1 .$$

Also, since

$$12a^3x^{1-4\varphi} \geq 12A^3x^{1-4\varphi} > 15 ,$$

the maximum possible number of intervals I with the above properties is

$$\leq \frac{x^\varphi}{\frac{1}{12}a^{-3}x^{5\varphi-1}} + 1 = 12a^3x^{1-4\varphi} + 1 < 12.8a^3x^{1-4\varphi} .$$

Putting the pieces together, we find

$$\begin{aligned} t_1(a) &\leq 12.8a^3x^{1-4\varphi} \times (64\sqrt[3]{12}a^{-\frac{8}{3}}hx^{(7\varphi-2)/3} + 1) \\ &= 819.2\sqrt[3]{12}a^{\frac{1}{3}}hx^{\frac{1}{3}-5\varphi/3} + 12.8a^3x^{1-4\varphi} . \end{aligned}$$

From (11) and the last estimate we obtain

$$\begin{aligned} S(x^\varphi, 2x^\varphi) &\leq 2 + 2x^\varphi B^{-1} + 2B + 7502hx^{(1-5\varphi)/3}\sum_{a\leq B} a^{\frac{1}{3}} + 51.2x^{1-4\varphi}\sum_{a\leq B} a^4 \\ &\leq 2 + 400.01x^{\frac{1}{5}} + 5626.5hx^{(1-5\varphi)/3}(B+1)^{\frac{4}{3}} + 12.8x^{1-4\varphi}(B+1)^4 \\ &\leq 400.011x^{\frac{1}{5}} + 4.816hx^{\frac{1}{15}-\varphi/3} . \end{aligned}$$

Now, applying Lemma 1, we get

$$\begin{aligned} S(H, x^{0.4}) &\leq 400.011 x^{\frac{1}{5}} \left(\frac{\log x}{5 \log 2} + 2 \right) + \frac{4.816}{1 - 2^{-\frac{1}{3}}} h x^{\frac{1}{15} - \varphi/3} H^{-\frac{1}{3}} \\ &\leq 120.004 x^{\frac{1}{5}} \log x + \frac{1.0197h}{(\log x)^{\frac{1}{3}}} \\ &< 0.295h < 0.3h - 1 \ . \end{aligned}$$

Thus, (10) holds and the proof is complete. □

References

[1] M. Filaseta, O. Trifonov, *On gaps between squarefree numbers II*, J. London Math. Soc. (2) **45** (1992), 215–221.

[2] J. Rosser, L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–89.