Math 265: Elementary Linear Algebra

LECTURE NOTES BY ANGEL V. KUMCHEV

1. Linear Systems

1.1. **Introduction.** In this and the next few lectures, we will use the familiar problem of solving systems of linear equations as a motivation for some of the basic concepts of linear algebra. The main goal of the present lecture is to review the methods for solving of linear systems and to illustrate how one can use matrices to express the solution of this problem more efficiently.

1.2. **Terminology.** A *linear equation* in the variables $x_1, x_2, \dots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \dots, a_n$ and b are given real (or complex) numbers. For example, the equations

$$4x_1 - x_2 - 3 = x_3 \qquad \text{and} \qquad \sqrt{3}x_1 - 2x_3 = 1.78x_3$$

are linear equations in x_1, x_2, x_3, since they can be written as

$$4x_1 - x_2 - x_3 = 3 \qquad \text{and} \qquad \sqrt{3}x_1 + 0x_2 - 3.78x_3 = 0,$$

respectively. On the other hand, the equations

$$5x_1x_2 + x_4 = 7, \qquad x_1^2 + x_2^{1/3} = 9, \qquad \text{and} \qquad \ln x_1 + 3x_2 = 8$$

are nonlinear. The first equation is nonlinear, because x_1 appears multiplied by x_2. The second equation is nonlinear, because x_1 and x_2 appear raised to powers different from 1. The last equation is nonlinear, because x_1 appears inside the logarithm, which is a nonlinear function.

A *system of linear equations* (or a *linear system*) is a collection of one or more linear equations involving the same variables. For example,

$$\begin{cases} x_1 + x_2 - 3x_3 = 4 \\ x_1 \qquad\;\; - 2x_3 = 2 \end{cases} \tag{1.1}$$

is a linear system of two equations in the variables x_1, x_2, x_3. A "generic" linear system of m equations in n variables is usually written as

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & \vdots & & & & \vdots & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases} \tag{1.2}$$

What exactly does it mean to solve a system of linear equations? A *solution* of a system in $x_1, x_2, \dots, x_n$ is any n-tuple of numbers $(s_1, s_2, \dots, s_n)$ that makes every equation of the system a true statement when the values $s_1, s_2, \dots, s_n$ are substituted for the variables $x_1, x_2, \dots, x_n$, respectively. For example, the triple $(2, 2, 0)$ is a solution of the system (1.1): if we substitute the values $x_1 = 2$, $x_2 = 2$, and $x_3 = 0$, we obtain the true statements $4 = 4$ and $2 = 2$. The set of all possible solutions is called the *solution set* of the system. To solve a linear system means to find its solution set, that is, to find all the values of the variables such that all the equations are satisfied.

Recall the three possibilities for the solution set of a linear system of two equations in two variables: such a system can have infinitely many solutions, one solution, or no solution at all. The same is true for any linear system.

Fact 1.2.1. *A system of linear equations has either no solution, or a unique solution, or infinitely many solutions.*

If a linear system has at least one solution, we say that it is *consistent*; otherwise, we say that the system is *inconsistent*.

Two systems of linear equations are *equivalent* if they have the same solution sets. For example, the systems

$$\begin{cases} 3x + y = 5 \\ 5x - y = 3 \end{cases} \quad \text{and} \quad \begin{cases} x = 1 \\ y = 2 \end{cases}$$

are equivalent: the only solution of both systems is $(1, 2)$.

1.3. **Matrix notation.** A *matrix* is a rectangular array of numbers. If a matrix has m rows and n columns, we say that it is an $m \times n$ *matrix*. The standard notation for a "generic" $m \times n$ matrix A is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}. \tag{1.3}$$

Notice that in (1.3) the first index of an entry a_{ij} indicates the number of the row which the entry belongs to, and the second index indicates the number of the column which the entry belongs to.

Matrices have many applications throughout mathematics and the sciences. In this lecture and the next, we will focus on one such application: to solving linear systems. Given any linear system, we introduce two matrices associated to that system: the coefficient matrix and the augmented matrix of the system. The *coefficient matrix* of the generic linear system (1.2) is the matrix (1.3). For example, the coefficient matrix of the system (1.1) is

$$\begin{bmatrix} 1 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix}.$$

The *augmented matrix* of the system (1.2) is the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} & b_i \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} & b_m \end{bmatrix}.$$

In particular, the augmented matrix of (1.1) is

$$\begin{bmatrix} 1 & 1 & -3 & 4 \\ 1 & 0 & -2 & 2 \end{bmatrix}.$$

It is not hard to see that the augmented matrix "encodes" all the relevant information about a system of linear equations (except for the labeling of the unknowns). Indeed, given a matrix, such as

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -1 & -3 & 6 \\ 1 & 0 & -2 & 4 \end{bmatrix},$$

we can easily "recover" the system of linear equations having the given matrix as its augmented matrix:

$$\begin{cases} x_1 - x_2 - x_3 = 2 \\ 2x_1 - x_2 - 3x_3 = 6 \\ x_1 - 2x_3 = 4 \end{cases}$$

1.4. **Solving linear systems.** The truth is that linear systems are so ubiquitous in mathematics (and in science in general) that you are probably quite familiar with them by now. Let us review some of the things you most likely already know. First, recall the basic operations we use to simplify a system of equations:

- we can multiply all the terms in an equation by the same nonzero number;
- we can replace an equation by the sum of that equation and a multiple of another equation;
- we can change the order of the equations.

The last operation is so trivial that you probably never used it consciously until now, but it is actually quite important when we deal with systems on a more abstract level. The important common property of these three basic operations is that each of them transforms any linear system into an equivalent system.

Example 1.4.1. *Solve the linear system*

$$\begin{cases} x_1 + 3x_2 - 6x_3 = 5 \\ 2x_1 - x_2 + 2x_3 = -4 \\ 3x_1 + 2x_2 = 1 \end{cases}$$

Solution. Since we want to use this solution as a motivation for things to come, we will describe every step in full detail. Also, each time we replace a system by an equivalent system, we will write the augmented matrix of the new system. In particular, we observe that the augmented matrix of the given system is

$$\begin{bmatrix} 1 & 3 & -6 & 5 \\ 2 & -1 & 2 & -4 \\ 3 & 2 & 0 & 1 \end{bmatrix}.$$

First, we use the term x_1 in the first equation to eliminate the variable x_1 from the second and third equations. We can replace the second equation by

$$[\text{eq. } 2] + (-2)[\text{eq. } 1]: \quad -7x_2 + 14x_3 = -14.$$

The given system is then replaced by the equivalent system

$$\begin{cases} x_1 + 3x_2 - 6x_3 = 5 \\ - 7x_2 + 14x_3 = -14 \\ 3x_1 + 2x_2 = 1 \end{cases} \qquad \begin{bmatrix} 1 & 3 & -6 & 5 \\ 0 & -7 & 14 & -14 \\ 3 & 2 & 0 & 1 \end{bmatrix}. \tag{1.4}$$

Replacing the third equation by

$$[\text{eq. } 3] + (-3)[\text{eq. } 1]: \quad -7x_2 + 18x_3 = -14,$$

we see that (1.4) is equivalent to

$$(1.5) \qquad \begin{cases} x_1 + 3x_2 - 6x_3 = 5 \\ - 7x_2 + 14x_3 = -14 \\ - 7x_2 + 18x_3 = -14 \end{cases} \qquad \begin{bmatrix} 1 & 3 & -6 & 5 \\ 0 & -7 & 14 & -14 \\ 0 & -7 & 18 & -14 \end{bmatrix}.$$

At this point, we have replaced the given system by an equivalent system of a very special kind: its first equation contains x_1, but its second and third equations don't. Thus, the second and third equations form a "linear subsystem" in x_2 and x_3 only. We now focus on those equations.

Let us divide the second equation of (1.5) by -7. We get

$$(1.6) \qquad \begin{cases} x_1 + 3x_2 - 6x_3 = 5 \\ x_2 - 2x_3 = 2 \\ - 7x_2 + 18x_3 = -14 \end{cases} \qquad \begin{bmatrix} 1 & 3 & -6 & 5 \\ 0 & 1 & -2 & 2 \\ 0 & -7 & 18 & -14 \end{bmatrix}.$$

We can now eliminate x_2 from the last equation:

$$[\text{eq. } 3] + 7[\text{eq. } 2] : \quad 4x_3 = 0.$$

Hence, (1.6) is equivalent to

$$\begin{cases} x_1 + 3x_2 - 6x_3 = 5 \\ x_2 - 2x_3 = 2 \\ 4x_3 = 0 \end{cases} \qquad \begin{bmatrix} 1 & 3 & -6 & 5 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 4 & 0 \end{bmatrix}.$$

At this stage, we can easily find the value of x_3 from the last equation: we divide that equation by 4 and find that $x_3 = 0$:

$$(1.7) \qquad \begin{cases} x_1 + 3x_2 - 6x_3 = 5 \\ x_2 - 2x_3 = 2 \\ x_3 = 0 \end{cases} \qquad \begin{bmatrix} 1 & 3 & -6 & 5 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We can now substitute the value of x_3 into the first two equations, we can then solve the new second equation for x_2, and finally we can use the result to find x_1 from the first equation. (This approach is known as *backward substitution*.) However, we will use a somewhat longer but more structured mathod.

Let us use the third equation in (1.7) to eliminate x_3 from the remaining two equations. Replacing the first equation by

$$[\text{eq. } 1] + 6[\text{eq. } 3] : \quad x_1 + 3x_2 = 5$$

and the second equation by

$$[\text{eq. } 2] + 2[\text{eq. } 3] : \quad x_2 = 2,$$

we find that (1.7) is equivalent to

$$\begin{cases} x_1 + 3x_2 = 5 \\ x_2 = 2 \\ x_3 = 0 \end{cases} \qquad \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Finally, replacing the first equation of the last system by

$$[\text{eq. } 1] - 3[\text{eq. } 2] : \quad x_1 = -1,$$

we find that the original system is equivalent to

$$\begin{cases} x_1 = -1 \\ x_2 = 2 \\ x_3 = 0 \end{cases} \qquad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

It is now clear that the original system has one solution: the triple $(-1, 2, 0)$. □

Remark 1.4.2. Let us check our answer. To verify that $(-1, 2, 0)$ is really a solution of the original system, we substitute it into the system:

$$\begin{cases} (-1) + 3(2) - 6(0) = 5 & \checkmark \\ 2(-1) - (2) + 2(0) = -4 & \checkmark \\ 3(-1) + 2(2) = 1 & \checkmark \end{cases}$$

That is, the triple $(-1, 2, 0)$ is a solution indeed.

Let us examine how the augmented matrices evolved through the solution. We see that each time we performed one of the three basic operations, the augmented matrix of the system changed according to a similar operation on its rows. The respective operations on rows of matrices are:

- *Interchange:* interchange two rows.
- *Scaling:* multiply a row by a nonzero number.
- *Replacement:* add a multiple of one row to another row.

These three operations on matrices are called *elementary row operations*. We say that two matrices A and B are *row equivalent*, and write $A \sim B$, if there exists a finite sequence of elementary row operations that transforms one of the matrices into the other. Thus, we can restate our observation in the following way.

Fact 1.4.3. *If the augmented matrices of two linear systems are row equivalent, then the two systems are equivalent.*

This fact allows us to use elementary row operations on augmented matrices to express the solution of linear system more efficiently. For example, a "matrix solution" of Example 1.4.1 looks as follows.

Solution 2. The augmented matrix of the given system is

$$\begin{bmatrix} 1 & 3 & -6 & 5 \\ 2 & -1 & 2 & -4 \\ 3 & 2 & 0 & 1 \end{bmatrix}.$$

We have

$$\begin{bmatrix}1&3&-6&5\\2&-1&2&-4\\3&2&0&1\end{bmatrix} \sim \begin{bmatrix}1&3&-6&5\\0&-7&14&-14\\3&2&0&1\end{bmatrix} \qquad R_2 \leftarrow R_2 + (-2)R_1$$

$$\sim \begin{bmatrix}1&3&-6&5\\0&-7&14&-14\\0&-7&18&-14\end{bmatrix} \qquad R_3 \leftarrow R_3 + (-3)R_1$$

$$\sim \begin{bmatrix}1&3&-6&5\\0&1&-2&2\\0&-7&18&-14\end{bmatrix} \qquad R_2 \leftarrow R_2 \div (-7)$$

$$\sim \begin{bmatrix}1&3&-6&5\\0&1&-2&2\\0&0&4&0\end{bmatrix} \qquad R_3 \leftarrow R_3 + 7R_2$$

$$\sim \begin{bmatrix}1&3&-6&5\\0&1&-2&2\\0&0&1&0\end{bmatrix} \qquad R_3 \leftarrow R_3 \div 4$$

$$\sim \begin{bmatrix}1&3&0&5\\0&1&0&2\\0&0&1&0\end{bmatrix} \qquad \begin{array}{l}R_1 \leftarrow R_1 + 6R_3\\ R_2 \leftarrow R_2 + 2R_3\end{array}$$

$$\sim \begin{bmatrix}1&0&0&-1\\0&1&0&2\\0&0&1&0\end{bmatrix} \qquad R_1 \leftarrow R_1 + (-3)R_2$$

Hence, the solution set of the original system is $(-1, 2, 0)$. □

Remark 1.4.4. It becomes clear from this second solution that the use of matrix notation simplifies greatly the bookkeeping. However, matrix notation also helps us keep an eye on the goal and decide which row operation to perform next. Indeed, each of the row operations we performed makes perfect sense if we think of the above solution as sequence of steps that transforms the augmented matrix as follows:

$$\begin{bmatrix}1&*&*&*\\ *&*&*&*\\ *&*&*&*\end{bmatrix} \sim \begin{bmatrix}1&*&*&*\\0&*&*&*\\0&*&*&*\end{bmatrix} \sim \begin{bmatrix}1&*&*&*\\0&1&*&*\\0&*&*&*\end{bmatrix} \sim \begin{bmatrix}1&*&*&*\\0&1&*&*\\0&0&*&*\end{bmatrix}$$

$$\sim \begin{bmatrix}1&*&*&*\\0&1&*&*\\0&0&1&*\end{bmatrix} \sim \begin{bmatrix}1&*&0&*\\0&1&0&*\\0&0&1&*\end{bmatrix} \sim \begin{bmatrix}1&0&0&*\\0&1&0&*\\0&0&1&*\end{bmatrix}.$$

Here a $*$ indicates an entry which can have any value.

Example 1.4.5. *Solve the system*

$$\begin{cases} x_1 - x_2 - x_3 = 2\\ 2x_1 - x_2 - 3x_3 = 6\\ x_1 - 2x_2 = 4\end{cases}$$

Solution. The augmented matrix is

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -1 & -3 & 6 \\ 1 & -2 & 0 & 4 \end{bmatrix}.$$

Using row replacements, we obtain

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -1 & -3 & 6 \\ 1 & -2 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Since the last matrix corresponds to the system

$$\begin{cases} x_1 - x_2 - x_3 = 2 \\ \qquad x_2 - x_3 = 2 \\ \qquad\qquad 0 = 4 \end{cases}$$

the original system is inconsistent. □

Example 1.4.6. *Determine the values of the parameter h for which the system is consistent:*

$$\begin{cases} x_1 - x_2 - x_3 = 2 \\ 2x_1 - x_2 - 3x_3 = 6 \\ x_1 - 2x_2 = h \end{cases}$$

Solution. The augmented matrix is

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -1 & -3 & 6 \\ 1 & -2 & 0 & h \end{bmatrix}.$$

Using row replacements, we obtain

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -1 & -3 & 6 \\ 1 & -2 & 0 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & h-2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & h \end{bmatrix}.$$

The last matrix corresponds to the system

$$\begin{cases} x_1 - x_2 - x_3 = 2 \\ \qquad x_2 - x_3 = 2 \\ \qquad\qquad 0 = h \end{cases}$$

which is clearly inconsistent when $h \neq 0$. When $h = 0$, the third equation turns into $0 = 0$ and can be disregarded, that is, the original system is equivalent to the system

$$\begin{cases} x_1 - x_2 - x_3 = 2 \\ \qquad x_2 - x_3 = 2 \end{cases}$$

Clearly, this system has exactly one solution for every fixed value of x_3. For example, if $x_3 = 1$, we get

$$x_2 = 2 + (1) = 3, \qquad x_1 = 2 + (3) + (1) = 6.$$

So, $(6, 3, 1)$ is a solution when $h = 0$. The given system is consistent if and only if $h = 0$. □

2. Row Reduction

In this lecture, we formalize our observations from §1.4. The result will be the so-called *row reduction algorithm*, which can be used to solve *any* linear system. This algorithm applies to any matrix (whether or not it is viewed as an augmented matrix of a linear system) and has many uses beyond solving linear systems.

2.1. **Echelon forms.**

In the following definition, we say that a row (or column) in a matrix is *nonzero* if not all entries in that row (or column) are zeros, that is, if it contains at least one nonzero entry; a *leading entry* of a row refers to the leftmost nonzero entry (in a nonzero row). For example, the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has three nonzero rows (rows 1, 2, and 3) and four nonzero columns (columns 1, 2, 4, and 5). The leading entry in row 1 is 1, and the leading entry in row 3 is 6.

Definition 2.1.1. A matrix is in *(row) echelon form* if it has the following three properties:

1. All nonzero rows occur above all zero rows.
2. The leading entry of each nonzero row (after the first) occurs to the right of the leading entry of the previous row.
3. All the entries below a leading entry are 0s.

A matrix in echelon form is in *(row) reduced echelon form* if it has the following two additional properties:

4. All leading entries are 1s.
5. All the entries above a leading entry are 0s.

Example 2.1.2. *Determine which of the following matrices are in echelon form*

$$\begin{bmatrix} \bullet & * & * & * & * & * \\ 0 & 0 & \bullet & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & \bullet \end{bmatrix} \quad \begin{bmatrix} \bullet & * & * & * & * & * \\ 0 & 0 & \bullet & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \bullet & * & * & * & * & * \\ 0 & 0 & \bullet & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & \bullet & * & * \end{bmatrix}$$

$$\begin{bmatrix} \bullet & * & * & * & * & * \\ 0 & 0 & \bullet & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & * & * & * \end{bmatrix} \quad \begin{bmatrix} 0 & \bullet & * & * & * & * \\ 0 & 0 & \bullet & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & \bullet & * \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bullet & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & \bullet & * \end{bmatrix}$$

Here, a bullet ($\bullet$) *denotes a nonzero entry and a star* ($*$) *denotes an entry that may be either zero or nonzero.*

Answer. The first and middle matrices on the first line and the middle matrix on the second line are in echelon form. The last matrices on both lines are not in echelon form. It's not possible to say whether the first matrix on the second line is in echelon form. □

Example 2.1.3. *Determine which of the following matrices are in reduced echelon form*

$$\begin{bmatrix}1&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{bmatrix}\quad\begin{bmatrix}1&0&\bullet&*&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&1\end{bmatrix}\quad\begin{bmatrix}1&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&*&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{bmatrix}\quad\begin{bmatrix}1&*&0&0&0&0\\0&0&1&1&0&0\\0&0&0&1&1&1\\0&0&0&0&0&0\end{bmatrix}\quad\begin{bmatrix}1&*&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&1&1\\0&0&0&0&0&0\end{bmatrix}$$

Answer. All but the two middle matrices. □

Example 2.1.4. *Using a bullet (•) to indicate a nonzero entry, a star (∗) to indicate an entry which could be nonzero or zero, list out all possible* 2×2 *matrices in echelon form.*

Answer. $\begin{bmatrix}0&0\\0&0\end{bmatrix},\quad\begin{bmatrix}\bullet&*\\0&0\end{bmatrix},\quad\begin{bmatrix}0&\bullet\\0&0\end{bmatrix},\quad\begin{bmatrix}\bullet&*\\0&\bullet\end{bmatrix}.$ □

Echelon forms are important to us because of the following theorem.

Theorem 1 (Uniqueness of the reduced echelon form)**.** *Each matrix is row equivalent to one and only one reduced echelon matrix.*

Remark 2.1.5. The theorem is false if we omit the requirement that the echelon matrix be reduced. In other words, a matrix is row equivalent to many echelon matrices, but to a single reduced echelon matrix.

2.2. **Pivot positions.**

Definition 2.2.1. A *pivot position* in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A *pivot column (row)* is a column (row) of A that contains a pivot position.

Example 2.2.2. *Find the pivot positions in the matrix*

$$\begin{bmatrix}1&-2&-1&3&0\\-2&4&5&-5&3\\3&-6&-6&8&2\end{bmatrix}.$$

Solution. We have

$$\begin{bmatrix}1&-2&-1&3&0\\-2&4&5&-5&3\\3&-6&-6&8&2\end{bmatrix}\sim\begin{bmatrix}1&-2&-1&3&0\\0&0&3&1&3\\0&0&-3&-1&2\end{bmatrix}\sim\begin{bmatrix}1&-2&-1&3&0\\0&0&3&1&3\\0&0&0&0&5\end{bmatrix}\sim\begin{bmatrix}1&-2&-1&3&0\\0&0&3&1&3\\0&0&0&0&1\end{bmatrix}$$

$$\sim\begin{bmatrix}1&-2&-1&3&0\\0&0&3&1&0\\0&0&0&0&1\end{bmatrix}\sim\begin{bmatrix}1&-2&-1&3&0\\0&0&1&\frac{1}{3}&0\\0&0&0&0&1\end{bmatrix}\sim\begin{bmatrix}1&-2&0&3\frac{1}{3}&0\\0&0&1&\frac{1}{3}&0\\0&0&0&0&1\end{bmatrix}.$$

Thus, the pivot positions are $(1,1)$, $(2,3)$, and $(3,5)$. □

Remark 2.2.3. Notice that in the above example all the echelon forms of the matrix that we went through had the shape

$$\begin{bmatrix} \bullet & * & * & * & * \\ 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & \bullet \end{bmatrix}.$$

That is, the leading entries in all the echelon forms were at the same positions—the pivot positions. This is true in general. In other words, we don't have to go all the way to the reduced echelon form to identify the pivot positions. We can just look at an echelon form: the pivot positions correspond to the leading entries in any echelon form for a given matrix.

2.3. **Row reduction algorithm.** The row reduction algorithm uses repeatedly the following six steps to replace a given matrix by a row equivalent matrix in reduced echelon form (which we know is unique).

1. Begin with the leftmost nonzero column. It is a pivot column; the pivot position is at the top.
2. Using row interchanges (if necessary), move a nonzero entry into the pivot position. If convenient, scale the pivot row so that the leading entry is 1.
3. Adding multiples of the pivot row to subsequent rows, create 0s below the pivot position.
4. Cover the pivot row and all previous rows. If the resulting matrix is nonzero, repeat Steps 1–3 on it; if the resulting matrix is all zeros, move on to Step 5.
5. Cover any possible zero rows. The last row of the remaining matrix is nonzero and its leading entry is the rightmost pivot position.
6. Scale the last row so that its leading entry is 1. By adding multiples of that row to previous rows, create zeros above the rightmost pivot position. Then cover the last row and repeat the step to the remaining matrix (or halt).

Steps 1–4 are known as the *forward phase* of the row reduction algorithm; Steps 5 and 6 form the *backward phase*.

Example 2.3.1. *Apply the row reduction algorithm to the matrix*

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}.$$

Solution. The first pivot position is $(1, 1)$. We have

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}.$$

Cover the first row. The next pivot position is $(2,2)$. We have

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$
$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

Cover the first two rows. The next pivot position is $(3,5)$. Since it is in the last row, this finishes the forward phase of the row reduction.

Start with the last row; the leading entry is already 1. Using row replacements, we get

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

Cover the last row. The leading entry is already 1. By a row replacement,

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

Cover the second row. Finally, we scale the first row and then stop:

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

□

2.4. **Solutions of linear systems.** Now that we know the row reduction algorithm, we can row reduce the augmented matrix of any linear system to its reduced echelon form. In other words, we can replace any linear system by an equivalent system whose augmented matrix is in reduced echelon form. Thus, from now on, we need only worry about linear systems whose augmented matrices are in reduced echelon form. Let us consider some examples.

Example 2.4.1. *Solve the system whose augmented matrix has been reduced to*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}.$$

Solution. The given matrix is 4×5, so the corresponding linear system has four equations and four unknowns. That linear system is

$$\begin{cases} x_1 = 3 \\ x_2 = 0 \\ x_3 = -2 \\ x_4 = -5 \end{cases}$$

and clearly has a unique solution: $(3, 0, -2, -5)$. □

Example 2.4.2. *Solve the system whose augmented matrix has been reduced to*

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Solution. We saw a similar situation in Examples 1.4.5 and 1.4.6. The last row of this matrix represents the equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 2,$$

which has no solution. Thus, the system is inconsistent. □

The next example is a little more challenging.

Example 2.4.3. *Solve the system whose augmented matrix has been reduced to*

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution. This is the augmented matrix of the system

$$\begin{cases} x_1 + x_2 \quad + \; x_4 \qquad = 3 \\ \qquad\quad x_3 + 4x_4 \qquad = -2 \\ \qquad\qquad\qquad\quad x_5 = 1 \end{cases}$$

If we solve the first equation for x_1 and the second for x_3, we get

$$\begin{cases} x_1 = 3 - x_2 - x_4 \\ x_3 = -2 - 4x_4 \\ x_5 = 1 \end{cases} \tag{2.1}$$

For any choice of the variables x_2 and x_4, these formulas determine values of x_1, x_3, x_5 which together with the chosen values of x_2 and x_4 form a solution of the system. For example: when $x_2 = 1$ and $x_4 = 0$, we obtain the solution $(2, 1, -2, 0, 1)$; when $x_2 = 0$ and $x_4 = 1$, we obtain $(2, 0, -6, 1, 1)$; when $x_2 = 0$ and $x_4 = 0$, we obtain $(3, 0, -2, 0, 1)$; etc. The standard way to express this is to write the solution as

$$\begin{cases} x_1 = 3 - x_2 - x_4 \\ x_3 = -2 - 4x_4 \\ x_5 = 1 \\ x_2, x_4 \text{ are free} \end{cases} \tag{2.2}$$

□

The above examples pretty much cover the bases. Example 2.4.1 is typical of the case of linear systems having a unique solution: this can only occur if the reduced echelon form of the augmented matrix is of the shape

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & * \\ 0 & 1 & \cdots & 0 & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \end{bmatrix}.$$

In other words, every column but the last is a pivot column. Example 2.4.2 is typical of the case of inconsistent linear systems: this case occurs if and only if the last column of the augmented matrix is a pivot column. Note that in this case we don't need to go all the way to the reduced echelon form of the augmented matrix to observe that the system has no solution. Any echelon form will do. (In fact, the matrix in Example 2.4.2 is in echelon form but not in reduced echelon form.) Finally, Example 2.4.3 is typical of the case of linear systems having infinitely many solutions. This occurs when the pivot columns of the augmented matrix do not include the last column and at least one of the remaining columns.

In the latter case, we call the variables corresponding to pivot columns in the augmented matrix *basic variables*. The other variables are called *free variables*. In the above example, the basic variables are x_1, x_3, x_5 and the free variables are x_2, x_4. When the augmented matrix is in reduced echelon form, every basic variable appears in exactly one equation. Solving each equation of such a system for its basic variable, we obtain a set of expressions for the basic variables in terms of the free variables (just like (2.1)). Those expressions together with a list of the free variables are called the *general solution* of the system. For example, (2.2) is the general solution of the system whose augmented matrix is considered in Example 2.4.3.

We now summarize all these observations in a formal procedure for solving linear systems:

1. Write the augmented matrix of the system.
2. Use row reduction to find an echelon form of the augmented matrix. We then know which columns of the augmented matrix are pivot columns.
3. If the last column is pivot, the system is inconsistent. Stop.
4. If the last column is not pivot, continue with row reduction until you obtain the reduced echelon form of the augmented matrix.
5. Write the system corresponding to the reduced echelon form of the augmented matrix. It is equivalent to the original system.
6. Solve each equation of the system from Step 5 for its one basic variable. If there are no free variables, this step is trivial and yields the unique solution of the original system. If there is at least one free variable, the original system has infinitely many solutions and this step produces the general solution.

Example 2.4.4. *Solve the system*

$$\begin{cases} \phantom{3x_1 - {}} 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15 \end{cases}$$

Solution. The augmented matrix is

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}.$$

We found its reduced echelon form in Example 2.3.1:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

It follows that the original system is consistent, with basic variables x_1, x_2, x_5 and free variables x_3 and x_4. It is equivalent to the system

$$\begin{cases} x_1 \quad - 2x_3 + 3x_4 \quad = -24 \\ \quad x_2 - 2x_3 + 2x_4 \quad = -7 \\ \qquad\qquad\qquad\quad x_5 = 4 \end{cases}$$

so the general solution is

$$\begin{cases} x_1 = 2x_3 - 3x_4 - 24 \\ x_2 = 2x_3 - 2x_4 - 7 \\ x_5 = 4 \\ x_3, x_4 \text{ free} \end{cases}$$

□

3. Vectors in $\mathbb{R}^n$

3.1. **Definition of and algebraic operations with vectors in $\mathbb{R}^n$.** An *n-dimensional vector* is an n-tuple of numbers: $\mathbf{x} = (x_1, x_2, \dots, x_n)$. It is common in linear algebra to write vectors as columns:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

With this convention, an n-dimensional vector is an $n \times 1$ matrix. The set of all n-dimensional vectors is denoted $\mathbb{R}^n$.

For any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, their *sum*, $\mathbf{x} + \mathbf{y}$, is the vector whose entries are the sums of the respective entries of $\mathbf{x}$ and $\mathbf{y}$:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

Likewise, if $\mathbf{x} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, the *scalar multiple* of $\mathbf{x}$ by c, $c\mathbf{x}$, is the vector whose entries are obtained from the respective entries of $\mathbf{x}$ by multiplication by c:

$$c \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}.$$

The basic properties of these two operations on vectors are summarized in the following proposition.

Proposition 3.1.1 (Algebraic properties of $\mathbb{R}^n$)**.** *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ *and* $c, d \in \mathbb{R}$. *Then:*

1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
3. $\mathbf{x} + \mathbf{0} = \mathbf{x}$, *where* $\mathbf{0}$ *denotes the* ***zero vector****, whose entries are all zero*
4. $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$, *where* $-\mathbf{x} = (-1)\mathbf{x}$
5. $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$
6. $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$
7. $c(d\mathbf{x}) = (cd)\mathbf{x}$
8. $1\mathbf{x} = \mathbf{x}$

Example 3.1.2. *Find* $3\mathbf{x} + (-2)\mathbf{y}$, *if*

$$\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \quad \textit{and} \quad \mathbf{y} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}.$$

Solution. We have

$$3\mathbf{x} = \begin{bmatrix} 9 \\ 0 \\ 6 \end{bmatrix}, \quad (-2)\mathbf{y} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \quad 3\mathbf{x} + (-2)\mathbf{y} = \begin{bmatrix} 11 \\ -2 \\ 10 \end{bmatrix}.$$

□

3.2. **Geometric interpretation.** Geometrically, a vector is a point in n-dimensional space. For example, the two-dimensional vector

$$\mathbf{x} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

corresponds to the point in the plane with Cartesian (rectangular) coordinates $(1, -5)$. Likewise, the vector

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

corresponds to the origin in three-dimensional space. Although a vector represents a point, it can sometimes be drawn as an arrow from the origin to the point represented by the vector. This graphical representation proves especially useful when we try to visualize the sum of two vectors and the scalar product of a vector with a number c. For sums we have the *parallelogram rule*: if $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^2$ represented by points in the plane, then $\mathbf{x} + \mathbf{y}$ is represented by the fourth vertex of the parallelogram having the other three of its vertices at $\mathbf{0}, \mathbf{x}$, and $\mathbf{y}$. If $\mathbf{x}$ is a vector in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and $c \neq 0$, $c\mathbf{x}$ is represented by an arrow lying on the same line as the arrow representing $\mathbf{x}$ and is $|c|$ times as long. The direction of the arrow representing $c\mathbf{x}$ is the same as the direction of the arrow representing $\mathbf{x}$ when $c > 0$ and opposite to it when $c < 0$.

3.3. **Linear combinations.** The *linear combination* of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with *weights* (or *coefficients*) $c_1, \ldots, c_p$ is the vector

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p.$$

Example 3.3.1. *Given the vectors*

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \textit{and} \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix},$$

compute their linear combination with weights $1, -2$. *Then compute three more linear combinations of these two vectors.*

Solution. The linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ with weights 1 and -2 is

$$\mathbf{v}_1 + (-2)\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}.$$

Three (among many) other linear combinations are

$$\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_1 - \mathbf{v}_2 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}, \quad 3\mathbf{v}_1 = \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix}.$$

□

Definition 3.3.2. If $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$, denoted $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, is called the *subset of* $\mathbb{R}^n$ *spanned* (or *generated*) *by* $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is, $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the set of all vectors in $\mathbb{R}^n$ that can be written in the form

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

for some choice of the scalars $c_1, \ldots, c_p$.

Example 3.3.3. *Describe the following sets:*

$$\operatorname{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\},\quad \operatorname{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\},\quad \operatorname{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\},\quad \operatorname{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}.$$

Answers. All of $\mathbb{R}^2$; all of $\mathbb{R}^2$; the xy-plane in $\mathbb{R}^3$; the line in $\mathbb{R}^3$ through the origin and the point $(1, 2, 3)$. □

4. Vector and Matrix Equations

In this lecture, we will provide two different approaches towards linear systems.

4.1. Vector equations.

4.1. Vector equations. First, we will look at linear systems from within the realm of linear combinations of vectors. In order to state the result more elegantly, we will use the notation

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p\end{bmatrix}$$

for the $n \times p$ matrix whose columns are the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \in \mathbb{R}^n$. For example, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are the vectors from Example 3.3.1, we have

$$[\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} 2 & -2 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

Fact 4.1.1. *A vector equation*

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{4.1}$$

has the same solution set as the linear system whose augmented matrix is

$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n \ \mathbf{b}]. \tag{4.2}$$

In particular, $\mathbf{b}$ *belongs to* $\operatorname{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ *if and only if the linear system corresponding to* (4.2) *is consistent.*

We illustrate the meaning of this statement by an example.

Example 4.1.2. *Determine whether the vector*

$$\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

can be written as a linear combination of the vectors $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *in Example 3.3.1.*

Solution. The vector $\mathbf{b}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ if there exist numbers c_1 and c_2 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{b}$, that is, if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{b}$$

has a solution. Using the properties of scalar multiplication and vector addition, we can write this vector equation in the form

$$\begin{bmatrix} 2x_1 - 2x_2 \\ x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

These two vectors are equal if and only if their respective entries match, that is, if the linear system

$$\begin{cases} 2x_1 - 2x_2 = 3 \\ x_2 = 2 \\ x_1 - x_2 = 1 \end{cases} \tag{4.3}$$

has a solution. To find out whether that is the case, we use row reduction:

$$\begin{bmatrix} 2 & -2 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Because the last column of the latter matrix is a pivot column, it follows that (4.3) is inconsistent, and hence, $\mathbf{b}$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2$. □

The solution of this example up to (4.3) illustrates how we pass from a vector equation of the form (4.1) to the linear system corresponding to (4.2). The transition from a linear system to a vector equation simply reverses the steps.

Example 4.1.3. *Describe geometrically* $\mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$*, where*

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}.$$

Solution. The set spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$ consists of all vectors $\mathbf{b} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for which the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$$

has a solution. In other words, we want to describe the set of triples (x, y, z) such that

$$\begin{bmatrix} 1 & -1 & x \\ 0 & -1 & y \\ 2 & 3 & z \end{bmatrix}$$

is the augmented matrix of a consistent linear system. Using row reduction, we find that

$$\begin{bmatrix} 1 & -1 & x \\ 0 & -1 & y \\ 2 & 3 & z \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & x \\ 0 & -1 & y \\ 0 & 5 & z-2x \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & x \\ 0 & -1 & y \\ 0 & 0 & z-2x+5y \end{bmatrix}.$$

Since the last matrix is the augmented matrix of a consistent system if and only if $z - 2x + 5y = 0$, we conclude that $\mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ consists of the vectors $\mathbf{b}$ whose entries satisfy the last equation. Geometrically, these vectors are represented by the points in space that satisfy the equation $-2x + 5y + z = 0$. You may recognize this as the equation of a plane in space. □

4.2. **Matrix equations.** Next, we want to give yet another interpretation of linear systems, one that casts a linear system of the form (1.2) as a generalization of the linear equation $ax = b$. However, before we get to that, we need to define the product of a matrix and a vector.

Definition 4.2.1. Suppose that A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ (in $\mathbb{R}^m$). Given an n-dimensional vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

the *product of A and* $\mathbf{x}$, denoted $A\mathbf{x}$, is the linear combination of the columns of A with weights determined by the entries of $\mathbf{x}$, that is,

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

In particular, $A\mathbf{x} \in \mathbb{R}^m$.

Remark 4.2.2. Note that the number of columns of A must agree with the dimension of $\mathbf{x}$. That is, if A is $m \times n$, then $\mathbf{x}$ must be n-dimensional (i.e., an $n \times 1$ matrix). The answer will be an m-dimensional vector (or an $m \times 1$ matrix). For example, a 3×7 matrix can be multiplied by a 7-dimensional vector to produce a 3-dimensional vector, but a 7×3 matrix cannot be multiplied by a 7-dimensional vector.

Remark 4.2.3. Some of you may have learned a different definition of the product of a matrix A with a vector $\mathbf{x}$. Namely, you may have learned that the ith entry of the answer is the dot product of the ith row of A and $\mathbf{x}$. The book refers to this method (at least until Chapter 6) as the *row-vector rule*. In fact, this is the same definition. However, for our immediate purposes the above definition is more convenient, so we will stick to it.

Example 4.2.4. *Compute the product*

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -5 \\ 2 & -4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Solution.

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -5 \\ 2 & -4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} + z \begin{bmatrix} 4 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} x + 2y + 4z \\ y - 5z \\ 2x - 4y - 3z \end{bmatrix}.$$

□

From the last example, we see that the system

$$\begin{cases} x + 2y + 4z = -2 \\ y - 5z = -1 \\ 2x - 4y - 3z = 3 \end{cases}$$

has the same solution set as the vector equation

$$x \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} + z \begin{bmatrix} 4 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix},$$

which has the same solution set as the *matrix equation*

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -5 \\ 2 & -4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}.$$

This is the reasoning behind the following fact.

Proposition 4.2.5. *If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ (in $\mathbb{R}^m$), and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation*

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b},$$

which has the same solution set as the linear system whose augmented matrix is

$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n \ \mathbf{b}].$$

Example 4.2.6. *Let*

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -6 & 3 & 1 \\ -2 & 1 & 3 \end{bmatrix}.$$

Does the matrix equation $A\mathbf{x} = \mathbf{b}$ have a solution for all $\mathbf{b} \in \mathbb{R}^3$? In other words, do the columns of A span $\mathbb{R}^3$?

Solution. To answer the question, we must check whether the augmented matrix

$$\begin{bmatrix} 2 & -1 & 0 & b_1 \\ -6 & 3 & 1 & b_2 \\ -2 & 1 & 3 & b_3 \end{bmatrix}$$

can have a pivot position in the last column for any choice of b_1, b_2, and b_3. If such a choice exists, then for the respective vector $\mathbf{b} \in \mathbb{R}^3$ the matrix equation $A\mathbf{x} = \mathbf{b}$ will be inconsistent. Since

$$\begin{bmatrix} 2 & -1 & 0 & b_1 \\ -6 & 3 & 1 & b_2 \\ -2 & 1 & 3 & b_3 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 & b_1 \\ 0 & 0 & 1 & b_2 + 3b_1 \\ 0 & 0 & 3 & b_3 + b_1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 & b_1 \\ 0 & 0 & 1 & b_2 + 3b_1 \\ 0 & 0 & 0 & b_3 - 3b_2 - 8b_1 \end{bmatrix},$$

we see that the equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $8b_1 + 3b_2 - b_3 = 0$. Since not all vectors satisfy this condition, the columns of A do not span $\mathbb{R}^3$. □

5. Solution Sets of Linear Systems

In this lecture, we want to reach a better understanding of the solution sets of linear systems. First, let us summarize some of the facts we already know. The algorithm for solving linear systems that we described in §2.4 leads to the following conclusions regarding the number of solutions of a given system.

Theorem 2 (Existence and uniqueness theorem)**.** *A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, that is, if and only if an echelon form of the augmented matrix has no row of the form*

$$\begin{bmatrix} 0 \; 0 \; \cdots \; 0 \; b \end{bmatrix},$$

with $b \neq 0$. If a linear system is consistent, then the solution set contains: either (i) a unique solution, when there are no free variables; or (ii) infinitely many solutions, when there is at least one free variable.

Furthermore, the discussion in the previous lecture leads to the following result.

Theorem 3. *Let A be an $m \times n$ matrix. Then the following are equivalent:*

1. *For each $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.*
2. *Each $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.*
3. *The columns of A span $\mathbb{R}^m$.*
4. *A has a pivot position in every row.*

The equivalence of 1), 2), and 3) follows from the definitions. Also, it isn't difficult to see that if A has a pivot in every row (i.e., 4) holds), then 1) is true. Indeed, the augmented matrix of $A\mathbf{x} = \mathbf{b}$ has no more pivot positions than rows. Since the coefficient matrix A already contains that many pivot positions (because of 4)), it follows that there can be no pivot position in the last column of the augmented matrix, regardless of the choice of $\mathbf{b}$. The reasoning why 1)–3) cannot hold if A does not have a pivot position in every row is more intricate, so we will be content with the explanation that it involves a glorified version of the solution of Example 4.2.6.

5.1. Homogeneous linear systems.

A system of linear equations is *homogeneous* if its right-hand side is the zero vector, that is, the corresponding matrix equation must be of the form $A\mathbf{x} = \mathbf{0}$, where $\mathbf{0}$ is the zero vector. For example, the following linear system is homogeneous:

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ 2x_1 + 4x_2 + 8x_3 + 10x_4 = 0 \\ 3x_1 + 7x_2 + 11x_3 + 14x_4 = 0 \end{cases} \tag{5.1}$$

The importance of the homogeneous linear systems lies in a simple observation: they are always consistent! This is because the zero vector $\mathbf{0}$ (of the right dimension) is always a solution of a homogeneous system (e.g., $(0, 0, 0, 0)$ is obviously a solution of the above system). This solution is known as the *trivial solution* of the given homogeneous system. Thus, for homogeneous systems, the real question is whether they have a unique solution (the trivial solution $x_1 = x_2 = \cdots = 0$) or infinitely many solution. In the latter case, the nonzero solutions are known as *nontrivial solutions*.

Example 5.1.1. *Determine whether* (5.1) *has a nontrivial solution.*

Solution. Yes. The system has four unknowns, but its augmented matrix can have at most three pivot positions. Therefore, there will be at least one free variable. □

Example 5.1.2. *Describe the solution set of* (5.1).

Solution. By row reduction,

$$\begin{bmatrix}1&2&3&4&0\\2&4&8&10&0\\3&7&11&14&0\end{bmatrix}\sim\begin{bmatrix}1&2&3&4&0\\0&0&2&2&0\\0&1&2&2&0\end{bmatrix}\sim\begin{bmatrix}1&2&3&4&0\\0&1&2&2&0\\0&0&2&2&0\end{bmatrix}$$
$$\sim\begin{bmatrix}1&2&3&4&0\\0&1&2&2&0\\0&0&1&1&0\end{bmatrix}\sim\begin{bmatrix}1&2&0&1&0\\0&1&0&0&0\\0&0&1&1&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&1&0\\0&1&0&0&0\\0&0&1&1&0\end{bmatrix}.$$

Thus, the solution is

$$\begin{cases}x_1=-x_4\\x_2=0\\x_3=-x_4\\x_4 \text{ free}\end{cases}$$

Another way to describe the solution is in terms of vectors. Namely, a vector $\mathbf{x}\in\mathbb{R}^4$ is a solution if and only if its four entries satisfy the last system, that is, if it is of the form

(5.2)
$$\mathbf{x}=\begin{bmatrix}-x_4\\0\\-x_4\\x_4\end{bmatrix}=x_4\begin{bmatrix}-1\\0\\-1\\1\end{bmatrix},$$

where x_4 is any number. In more sophisticated language, the solution set of the homogeneous linear system equals

$$\operatorname{Span}\left\{\begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}\right\}.$$

□

Fact 5.1.3. *It is true in general that the solution set of a homogeneous linear system is always the span of certain vectors—as many vectors as there are free variables.*

When the solution of a homogeneous system is expressed as a linear combination of fixed vectors with variable coefficients, such as in (5.2), we say that the solution is written in *parametric vector form*. Let us see another example.

Example 5.1.4. *Describe the solution set of the homogeneous linear system*

$$\begin{cases}-3x_1+5x_2-7x_3+2x_4=0\\-6x_1+7x_2+x_3+4x_4=0\end{cases}$$

Solution. By row reduction,

$$\begin{bmatrix}-3&5&-7&2&0\\-6&7&1&4&0\end{bmatrix}\sim\begin{bmatrix}-3&5&-7&2&0\\0&-3&15&0&0\end{bmatrix}\sim\begin{bmatrix}-3&5&-7&2&0\\0&1&-5&0&0\end{bmatrix}$$
$$\sim\begin{bmatrix}-3&0&18&2&0\\0&1&-5&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&-6&-2/3&0\\0&1&-5&0&0\end{bmatrix}.$$

Thus, the solution is

$$\begin{cases} x_1 = 6x_3 + (2/3)x_4 \\ x_2 = 5x_3 \\ x_3, x_4 \text{ free} \end{cases}$$

or in vector form:

$$\mathbf{x} = \begin{bmatrix} 6x_3 + (2/3)x_4 \\ 5x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 6 \\ 5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2/3 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5.3}$$

□

5.2. **Nonhomogeneous linear systems.** A linear system that is not homogeneous is called *nonhomogeneous*. As we know from experience, nonhomogeneous systems can be inconsistent, in which case its solution set is, of course, empty. What about the solution sets of a consistent nonhomogeneous system? It turns out that it is closely related to the solution set of the corresponding homogeneous system.

Theorem 4. *Suppose that $A\mathbf{x} = \mathbf{b}$ is a consistent nonhomogeneous equation, and let $\mathbf{p}$ be a solution of this equation. Then any other solution $\mathbf{v}_\mathrm{n}$ of this equation is of the form $\mathbf{v}_\mathrm{n} = \mathbf{v}_\mathrm{h} + \mathbf{p}$, where $\mathbf{v}_\mathrm{h}$ is a solution of the corresponding homogeneous equation $A\mathbf{x} = \mathbf{0}$. In other words, the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of vectors*

$$\big\{\mathbf{w} : \mathbf{w} = \mathbf{p} + \mathbf{v}_\mathrm{h},\ \mathbf{v}_\mathrm{h} \textit{ solution of } A\mathbf{x} = \mathbf{0}\big\}.$$

The following two examples illustrate the idea behind the above theorem.

Example 5.2.1. *Use Example 5.1.4 and the fact that $(0, 1, 0, -1)$ is a solution of the nonhomogeneous linear system*

$$\begin{cases} -3x_1 + 5x_2 - 7x_3 + 2x_4 = 3 \\ -6x_1 + 7x_2 + x_3 + 4x_4 = 3 \end{cases}$$

to describe its general solution.

Solution. Using that $(0, 1, 0, -1)$ is a solution, we can rewrite the system as follows:

$$\begin{cases} -3x_1 + 5x_2 - 7x_3 + 2x_4 = -3(0) + 5(1) - 7(0) + 2(-1) \\ -6x_1 + 7x_2 + x_3 + 4x_4 = -6(0) + 7(1) + (0) + 4(-1) \end{cases}$$

and then further as

$$\begin{cases} -3(x_1 - 0) + 5(x_2 - 1) - 7(x_3 - 0) + 2(x_4 + 1) = 0 \\ -6(x_1 - 0) + 7(x_2 - 1) + (x_3 - 0) + 4(x_4 + 1) = 0 \end{cases}$$

It follows that (x_1, x_2, x_3, x_4) is a solution of the original system if and only if $(x_1, x_2 - 1, x_3, x_4 + 1)$ is a solution of the homogeneous system. We know from Example 5.1.4 that the solution set of the homogeneous system is described by (5.3), so the solution set of the given system consists of the vectors $\mathbf{x} \in \mathbb{R}^4$ such that

$$\begin{bmatrix} x_1 \\ x_2 - 1 \\ x_3 \\ x_4 + 1 \end{bmatrix} = s \begin{bmatrix} 6 \\ 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2/3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{5.4}$$

for some $s, t \in \mathbb{R}$ (note that we have to use different letters in place of x_3, x_4 in (5.3)). Since

$$\begin{bmatrix} x_1 \\ x_2 - 1 \\ x_3 \\ x_4 + 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix},$$

we can write (5.4) as

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + s \begin{bmatrix} 6 \\ 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2/3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad (s, t \in \mathbb{R}).$$

□

The last line of the solution of Example 5.2.1 represents the *parametric vector form* of the solution. Observe that it matches exactly the form we claimed in Theorem 4: it is the sum of a particular solution (the given one) and the parametric form of general solution of the homogeneous system (taken from Example 5.1.4). In practice, we seldom know a particular solution in advance. Thus, we usually argue as in the following example.

Example 5.2.2. *Solve the nonhomogeneous linear system*

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 1 \\ 2x_1 + 4x_2 + 8x_3 + 10x_4 = 6 \\ 3x_1 + 7x_2 + 11x_3 + 14x_4 = 7 \end{cases}$$

Solution. By row reduction,

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 8 & 10 & 6 \\ 3 & 7 & 11 & 14 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 1 & 2 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & 2 & 2 & 4 \\ 0 & 0 & 2 & 2 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & 2 & 2 & 4 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 & -5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}.$$

Thus, the solution is

$$\mathbf{x} = \begin{bmatrix} -5 - x_4 \\ 0 \\ 2 - x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

□

6. Linear Independence

In this lecture we introduce one of the main concepts in linear algebra.

6.1. **Definition.** Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ be an (indexed) set of vectors in $\mathbb{R}^n$. This set is called *linearly independent* if the only solution to the homogeneous vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

is the trivial solution. Otherwise, the set is said to be *linearly dependent*. If the set is linearly dependent, any identity of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0},$$

where the weights $c_1, \dots, c_p$ are not all zero, is called a *linear dependence relation*.

Example 6.1.1. *Are the vectors*

$$\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \quad \textit{and} \quad \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$$

linearly independent?

Solution. We have to determine whether the vector equation

$$x_1 \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix} + x_3 \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has nontrivial solutions. From Theorem 2, we know that the corresponding homogeneous system (which is consistent) will have more than one solution if and only if it has at least one free variable. Thus, we row reduce the augmented matrix:

$$\begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -6 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}.$$

The matrix on the right is already in echelon form (though not in reduced echelon form), so we can see that the linear system has no free variables. Hence, the vector equation has only the trivial solution and the vectors are linearly independent. □

Example 6.1.2. *Determine whether the vectors*

$$\begin{bmatrix} -1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -1 \\ 0 \\ -2 \end{bmatrix}, \quad \textit{and} \quad \begin{bmatrix} 3 \\ -2 \\ 2 \\ -1 \end{bmatrix}$$

are linearly independent. If so, find a linear dependence relation among them.

Solution. As in the previous example, we start by row reducing the augmented matrix of the related linear system:

$$\begin{bmatrix} -1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 2 & 0 & 2 & 0 \\ 3 & -2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 4 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This shows that the system has a free variable, and therefore, the given vectors are linearly dependent. In order to find a linear dependence relation, we must find a nontrivial solution of the homogeneous equation. Thus, we complete the row reduction:

$$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that the general solution of the homogeneous system is $x_1 = -x_3$, $x_2 = -2x_3$, and x_3 free. In particular, when $x_3 = 1$, we obtain the nontrivial solution $(-1, -2, 1)$. It is not difficult to check that these weights do yield a linear dependence relation:

$$-\begin{bmatrix} -1 \\ 0 \\ 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 2 \\ -1 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

□

6.2. **Sets of one or two vectors.** We know how to determine whether a set of vectors is linearly dependent. However, it is also important to understand what does that mean. Let's start with the simplest case: consider a set of one vector—say, $\mathbf{v}$. In this case, the linear independence of $\{\mathbf{v}\}$ hinges on whether or not $\mathbf{v}$ is the zero vector. If $\mathbf{v} \neq \mathbf{0}$, the vector equation $x_1\mathbf{v} = \mathbf{0}$ has only the trivial solution; hence, $\{\mathbf{v}\}$ is linearly independent. On the other hand, the set $\{\mathbf{0}\}$ is linearly dependent:

$$(1)\mathbf{0} = \mathbf{0}$$

is a linear dependence relation. The following two examples illustrate the possible scenarios for sets of two vectors.

Example 6.2.1. *Are the vectors*

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \textit{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1.5 \end{bmatrix}$$

linearly independent?

Solution. Notice that $\mathbf{v}_2 = -0.5\mathbf{v}_1$. Rewriting this relation in the form

$$0.5\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0},$$

we obtain a linear dependence relation between $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus, the two vectors are linearly dependent. □

Example 6.2.2. *Observe that the vectors*

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \quad \textit{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

are NOT proportional. Using this observation, deduce that these vectors are linearly independent?

Solution. Suppose that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}. \tag{6.1}$$

If $c_1 \neq 0$, this would imply that $\mathbf{v}_1 = (-c_2/c_1)\mathbf{v}_2$, which is impossible, because $\mathbf{v}_1$ and $\mathbf{v}_2$ are not proportional. Likewise, if $c_2 \neq 0$, (6.1) would imply that $\mathbf{v}_2 = (-c_1/c_2)\mathbf{v}_1$, which is also impossible. Thus, (6.1) can hold only when $c_1 = c_2 = 0$, that is, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. □

These two examples demonstrate that we can decide whether two given vectors are linearly dependent essentially "by inspection": we need only check whether they are proportional. That is:

Fact 6.2.3. *A set of two vectors* $\{\mathbf{v}_1, \mathbf{v}_2\}$ *is linearly dependent if and only if one of the vectors is a multiple of the other.*

6.3. **Sets of two or more vectors.** We now state and explain some facts about linearly dependent sets of at least two vectors. The first of those is so obvious that we present the proof as well.

Theorem 5. *If a set* $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ *in* $\mathbb{R}^n$ *contains the zero vector, then* S *is linearly dependent.*

Proof. By rearranging the vectors, if necessary, we may assume that $\mathbf{v}_1 = \mathbf{0}$. Then

$$1\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p = \mathbf{0}$$

is a linear dependence relation. Thus, S is linearly dependent. □

Theorem 6. *If* $p > n$, *any set* $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ *of* p n*-dimensional vectors is linearly dependent.*

We explain the reasoning behind this theorem by an example.

Example 6.3.1. *Explain why the vectors*

$$\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ -7 \\ -5 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$$

are linearly dependent.

Solution. We must decide whether the homogeneous linear system with coefficient matrix

$$A = \begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & 5 \end{bmatrix}$$

has free variables. Typically, this would require finding an echelon form of the augmented matrix

$$\begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ -2 & -7 & 5 & 1 & 0 \\ -4 & -5 & 7 & 5 & 0 \end{bmatrix},$$

but here we can save ourselves the work. A free variable corresponds to a non-pivot column among the first four columns of this matrix. But this matrix has at most three pivot positions, since it has only three rows and no row can contain more than one pivot position. Therefore, at least one among the first four columns is not pivot. Because the homogeneous system has a nontrivial solution, the vectors are linearly dependent. □

Theorem 7 (Characterization of linearly dependent sets)**.** *A set* $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$*,* $p \geq 2$*, is linearly dependent if and only if at least one of the vectors is a linear combination of the others.*

Remark 6.3.2. Notice that we are not saying that every vector is a linear combination of the others. Just one. For example, the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

form a linearly dependent set, because

$$3\mathbf{v}_1 - \mathbf{v}_2 + 0\mathbf{v}_3 = \mathbf{0}. \tag{6.2}$$

However, $\mathbf{v}_3$ cannot be expressed a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, because any such linear combination must be a multiple of $\mathbf{v}_1$ and $\mathbf{v}_3$ is not a multiple of $\mathbf{v}_1$. Of course, this doesn't mean that the theorem is false. Each of $\mathbf{v}_1$ and $\mathbf{v}_2$ is a linear combination of the remaining vectors:

$$\mathbf{v}_1 = (1/3)\mathbf{v}_2 + 0\mathbf{v}_3, \qquad \mathbf{v}_2 = 3\mathbf{v}_1 + 0\mathbf{v}_3. \tag{6.3}$$

This example also demonstrates why Theorem 7 holds. Given any linear dependence relation, we can "solve" it for one of the vectors with nonzero coefficients to express that vector as a linear combination of the rest—just as we "solved" (6.2) for $\mathbf{v}_1$ and $\mathbf{v}_2$ to obtain (6.3). Conversely, if we take the second identity in (6.3), we can move $\mathbf{v}_2$ to the right side of the "=" sign, thus obtaining the linear dependence relation (6.2). And a linear dependence relation it is, because the coefficient in front of $\mathbf{v}_2$ is (-1).

7. Linear Transformations

In this lecture, we introduce *linear transformations*, which some believe are the main object studied in linear algebra.

7.1. Brief review of functions. In calculus (and even before that), you encountered the notion of a function. A *function* $f : A \to B$ is a rule that assigns to each element of A a corresponding element of B. The set A is called the *domain* of f, and the set B is called the *codomain* of f. The set of outputs which actually occur is called the *range* of the function. Here are some functions:

$$f(x) = x^2, \qquad g(x) = 7, \qquad h(x) = \frac{1}{x^2+1}.$$

If we consider these three functions as functions from $\mathbb{R}$ to $\mathbb{R}$, then they all have the same domains and codomains. However, they have different ranges: the range of f is $[0, \infty)$; the range of g is the single number 7; and the range of h is the interval $(0, 1]$.

You might have seen also *vector functions* (which map sets of numbers to $\mathbb{R}^n$, $n \geq 2$) and *multivariable functions* (which map sets in $\mathbb{R}^n$, $n \geq 2$, to $\mathbb{R}$). For example, the functions

$$\mathbf{r}(t) = (\cos t, \sin t, t) \qquad \text{and} \qquad F(x, y, z) = x^2z + yz^2$$

are a vector function from $\mathbb{R}$ to $\mathbb{R}^3$ and a multivariable function from $\mathbb{R}^3$ to $\mathbb{R}$, respectively.

7.2. Matrix transformations. Let A be an $m \times n$ matrix. We can define a function $T : \mathbb{R}^n \to \mathbb{R}^m$ by

$$T(\mathbf{x}) = A\mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

We call such a function a *matrix transformation*. A simple example is the following transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$:

$$T(\mathbf{x}) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ x_1 + x_2 \\ x_1 + x_2 \\ 2x_1 \end{bmatrix}. \tag{7.1}$$

7.3. Linear transformations. Matrix transformations have a very special property: they are linear. That is, they belong to the following class of functions.

Definition 7.3.1. A *linear transformation* is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

- $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;
- $T(c\mathbf{x}) = cT(\mathbf{x})$ for any vector $\mathbf{x} \in \mathbb{R}^n$ and any scalar c.

Example 7.3.2. *Verify that the matrix transformation* (7.1) *is linear.*

Solution. We must check that T has the two properties in Definition 7.3.1. We start with the second property, since it is slightly easier to check. Let $\mathbf{x}$ be a vector in $\mathbb{R}^2$ and let c be a number. By (7.1) and the definition of scalar multiplication, we have

$$c\mathbf{x} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}, \qquad T(c\mathbf{x}) = \begin{bmatrix} 2cx_1 \\ cx_1 + cx_2 \\ cx_1 + cx_2 \\ 2cx_1 \end{bmatrix} = \begin{bmatrix} c(2x_1) \\ c(x_1 + x_2) \\ c(x_1 + x_2) \\ c(2x_1) \end{bmatrix} = c \begin{bmatrix} 2x_1 \\ x_1 + x_2 \\ x_1 + x_2 \\ 2x_1 \end{bmatrix} = cT(\mathbf{x}).$$

Thus, T has the second property in Definition 7.3.1.

Next, consider two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. We have

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix},$$

so

$$T(\mathbf{x} + \mathbf{y}) = \begin{bmatrix} 2(x_1 + y_1) \\ (x_1 + y_1) + (x_2 + y_2) \\ (x_1 + y_1) + (x_2 + y_2) \\ 2(x_1 + y_1) \end{bmatrix}. \tag{7.2}$$

On the other hand,

$$T(\mathbf{x}) + T(\mathbf{y}) = \begin{bmatrix} 2x_1 \\ x_1 + x_2 \\ x_1 + x_2 \\ 2x_1 \end{bmatrix} + \begin{bmatrix} 2y_1 \\ y_1 + y_2 \\ y_1 + y_2 \\ 2y_1 \end{bmatrix} = \begin{bmatrix} 2x_1 + 2y_1 \\ (x_1 + x_2) + (y_1 + y_2) \\ (x_1 + x_2) + (y_1 + y_2) \\ 2x_1 + 2y_1 \end{bmatrix}. \tag{7.3}$$

Since the vectors on the right sides of (7.2) and (7.3) are equal, we conclude that the matrix transformation T has the first property in Definition 7.3.1. Hence, T is linear. □

A glorified version of the above solution leads to the following fact.

Fact 7.3.3. *Every matrix transformation is linear.*

Remark 7.3.4. It is natural at this point to ask: are there any linear transformations that are not matrix transformations? Somewhat surprisingly, the answer to this question is: "That depends on what you mean." (Huh?) In fact, when you do your homework you will encounter the following true/false questions:

> **1.8.21.d.** *Every linear transformation is a matrix transformation.*
>
> **1.9.24.a.** *Not every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.*

According to Lay, the answer to both these questions is "False". What this means is "Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation, but not every linear transformation is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$." However, we won't be able to discuss the second part of this statement until later in the course.

Next, we consider two examples of linear transformation defined by different means.

Example 7.3.5. *Define $D : \mathbb{R}^3 \to \mathbb{R}^2$ as follows:*

- *given a vector $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$, use its entries to define a quadratic polynomial $f(x) = ax^2 + bx + c$;*
- *define $D(\mathbf{v})$ by the rule*

$$D(\mathbf{v}) = \begin{bmatrix} f(1) \\ f'(1) \end{bmatrix},$$

where $f'(x)$ is the first derivative of $f(x)$.

Show that D is linear.

Solution. Let us translate the above definition into an explicit formula in terms of a, b, c. We have

$$f(1) = a + b + c, \qquad f'(1) = 2a + b.$$

Hence,

$$D(\mathbf{v}) = \begin{bmatrix} a + b + c \\ 2a + b \end{bmatrix}.$$

We can now use this formula to argue similarly to Example 7.3.2, but that is unnecessary. Instead, we observe that

$$\begin{bmatrix} a + b + c \\ 2a + b \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Therefore, D is a matrix transformation and, by Fact 7.3.3, must be linear. □

Example 7.3.6. *Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation which rotates each input vector counterclockwise by a fixed angle θ. Show that R is linear.*

Solution. This time, we will argue geometrically. Recall that any rotation is a congruence in the plane. Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^2$. We will think of those as points in the plane. By the parallelogram rule, the sum $\mathbf{x} + \mathbf{y}$ is the fourth vertex of the parallelogram with vertices $\mathbf{x}, \mathbf{y}, \mathbf{0}$. R maps this parallelogram onto a congruent parallelogram with vertices $R(\mathbf{x}), R(\mathbf{y}), \mathbf{0}, R(\mathbf{x} + \mathbf{y})$. On the other hand, also by the parallelogram rule, the fourth vertex of the parallelogram with vertices $R(\mathbf{x}), R(\mathbf{y}), \mathbf{0}$ is $R(\mathbf{x}) + R(\mathbf{y})$. It follows that

$$R(\mathbf{x} + \mathbf{y}) = R(\mathbf{x}) + R(\mathbf{y}),$$

which is the first of the two conditions we needed to verify. The second condition,

$$R(c\mathbf{x}) = cR(\mathbf{x}),$$

can be checked similarly using the geometric interpretation of the scalar multiplication. □

7.4. **The standard matrix of a linear transformation.** We now proceed to show why every linear transformation is a matrix transformation. To this end, we define the *$n \times n$ identity matrix I_n*:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

It turns out that in order to understand a linear transformation defined on $\mathbb{R}^n$, it suffices to understand its effect on the columns of I_n, which we denote $\mathbf{e}_1, \ldots, \mathbf{e}_n$, that is,

$$I_n = [\mathbf{e}_1 \ \mathbf{e}_2 \ \cdots \ \mathbf{e}_n].$$

Example 7.4.1. *Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that*

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}.$$

Compute

$$T\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right), \qquad T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right), \qquad \textit{and} \qquad T(\mathbf{x}).$$

Solution. We have

$$T\left(\begin{bmatrix}-1\\3\end{bmatrix}\right) = T(-\mathbf{e}_1 + 3\mathbf{e}_2) = -T(\mathbf{e}_1) + 3T(\mathbf{e}_2) = -\begin{bmatrix}1\\0\\2\end{bmatrix} + 3\begin{bmatrix}-1\\1\\-3\end{bmatrix} = \begin{bmatrix}-4\\3\\-11\end{bmatrix},$$

$$T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = T(2\mathbf{e}_1 + \mathbf{e}_2) = 2T(\mathbf{e}_1) + T(\mathbf{e}_2) = 2\begin{bmatrix}1\\0\\2\end{bmatrix} + \begin{bmatrix}-1\\1\\-3\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix},$$

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) = x_1\begin{bmatrix}1\\0\\2\end{bmatrix} + x_2\begin{bmatrix}-1\\1\\-3\end{bmatrix} = \begin{bmatrix}x_1 - x_2\\x_2\\2x_1 - 3x_2\end{bmatrix}.$$

In particular, the latter shows that

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}1 & -1\\0 & 1\\2 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}.$$

□

The same principle works in general:

Theorem 8. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a unique matrix A such that*

$$T(\mathbf{x}) = A\mathbf{x} \qquad \textit{for all } \mathbf{x} \in \mathbb{R}^n.$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, that is,

$$A = \begin{bmatrix}T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n)\end{bmatrix}.$$

The matrix A in the theorem is called the *standard matrix* of the linear transformation T.

Example 7.4.2. *Find the standard matrix of the rotation R from Example 7.3.6.*

Solution. The standard matrix of R is $[R(\mathbf{e}_1)\ R(\mathbf{e}_2)]$, where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the columns of I_2. Geometrically $\mathbf{e}_1$ is represented by the point $(1,0)$ on the unit circle. Rotation by angle θ maps this point to the point on the unit circle with coordinates $(\cos\theta, \sin\theta)$. Similarly, $\mathbf{e}_2$ is represented by the point $(0,1)$, which R maps to the point $(\cos(\theta + 90°), \sin(\theta + 90°)) = (-\sin\theta, \cos\theta)$. Thus,

$$R(\mathbf{e}_1) = \begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}, \quad R(\mathbf{e}_2) = \begin{bmatrix}-\sin\theta\\\cos\theta\end{bmatrix},$$

and the standard matrix of R is

$$\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}.$$

□

Example 7.4.3. *Find the standard matrix of the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which first reflects points through the horizontal x_1-axis and then rotate them $-\pi/4$ radians (i.e., $45°$ in the clockwise direction). You may assume that T is linear.*

Solution. Let us trace the images of the points representing $\mathbf{e}_1$ and $\mathbf{e}_2$:

$$\mathbf{e}_1 : \qquad (1,0) \mapsto (1,\ \ 0) \mapsto (\cos(-\pi/4), \sin(-\pi/4)) = \left(1/\sqrt{2}, -1/\sqrt{2}\right),$$
$$\mathbf{e}_2 : \qquad (0,1) \mapsto (0,-1) \mapsto (\cos(-3\pi/4), \sin(-3\pi/4)) = \left(-1/\sqrt{2}, -1/\sqrt{2}\right).$$

Thus, the standard matrix of T is

$$[T(\mathbf{e}_1)\ T(\mathbf{e}_2)] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

□

7.5. **Two special classes of linear transformations.** We now consider two special classes of linear transformations. Recall that a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called *onto* if each vector $\mathbf{b} \in \mathbb{R}^m$ is the image of at least one $x \in \mathbb{R}^n$ (i.e., its range equals its codomain). A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called *one-to-one* if each vector $\mathbf{b} \in \mathbb{R}^m$ is the image of at most one $x \in \mathbb{R}^n$ (i.e., different inputs produce different outputs). Here we will be interested in one-to-one and onto linear transformations.

Theorem 9. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Then:*

1. *T is onto if and only if the columns of A span $\mathbb{R}^m$;*
2. *T is one-to-one if and only if the columns of A are linearly independent.*

Example 7.5.1. *Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be defined by the formula*

$$T(x_1, x_2) = (2x_2 - 3x_1, x_1 - 4x_2, 0, x_2).$$

(a) *Find the standard matrix of T.*
(b) *Is T one-to-one?*
(c) *Is T onto?*

Solution. (a) The standard matrix of T is

$$A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)] = \begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) T will be one-to-one if whenever the equation $A\mathbf{x} = \mathbf{b}$ is consistent, it has exactly one solution. This happens if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. By row reduction,

$$\begin{bmatrix} -3 & 2 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and T is one-to-one.

(c) T will be onto, if the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$. We know that this happens exactly when each row of A has a pivot position, which is not the case. Thus, T is not onto. □

8. Matrix Algebra

We now move to define the basic algebraic operations with matrices.

8.1. **Addition and scalar multiplication of matrices.** Just like vectors, matrices (of the same size) are added entry-by-entry:

$$\begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 9 & -5 & 0 \\ 5 & -9 & -1 \end{bmatrix}.$$

Likewise, any matrix can be multiplied by a scalar simply by multiplying each of the entries of the matrix by the scalar:

$$5\begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 0 & -5 \\ 20 & -25 & 10 \end{bmatrix}.$$

The properties of matrix addition and scalar multiplication are not unexpected, and follow from the corresponding properties of real numbers:

Proposition 8.1.1. *Let A, B, C be matrices of the same size, and let r, s be scalars. Then*

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$, *where* 0 *denotes the* ***zero matrix***, *whose entries are all zero*
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

8.2. **Matrix multiplication.** Under the right circumstances (meaning that the matrices have the right dimensions), it is also possible to multiply two matrices. This is how we do it:

Definition 8.2.1. Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix with columns $\mathbf{b}_1, \dots, \mathbf{b}_p$. Then the *product* AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \dots, A\mathbf{b}_p$, that is,

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}.$$

Example 8.2.2. *Compute AB, where*

$$A = \begin{bmatrix} -1 & 2 \\ -5 & 4 \\ 2 & -3 \end{bmatrix} \qquad \textit{and} \qquad B = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}.$$

Solution. First, we observe that the product AB does exist, because the number of columns A (2) is equal to the number of rows of B. If $B = [\mathbf{b}_1\ \mathbf{b}_2]$, then

$$A\mathbf{b}_1 = 3\begin{bmatrix} -1 \\ -5 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ 0 \end{bmatrix}, \qquad A\mathbf{b}_2 = (-2)\begin{bmatrix} -1 \\ -5 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \\ -7 \end{bmatrix},$$

whence

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -7 & 14 \\ 0 & -7 \end{bmatrix}.$$

□

You may ask why do we use this rather complicated definition? To answer this question, we need to look back to the last lecture. Recall that if $S : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then it has an associated $m \times n$ matrix

$$A = \begin{bmatrix} S(\mathbf{e}_1) & S(\mathbf{e}_2) & \cdots & S(\mathbf{e}_n) \end{bmatrix}.$$

Likewise, if $T : \mathbb{R}^p \to \mathbb{R}^n$ is a linear transformation, then it has an associated $n \times p$ matrix

$$B = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_p) \end{bmatrix}.$$

It turns out that the composition function $S \circ T : \mathbb{R}^p \to \mathbb{R}^m$ is also linear (it is a good exercise in linear transformations to prove this), and so has an associated $m \times p$ matrix C. The above definition is just the right one in order to have $C = AB$. Indeed,

$$\begin{aligned} C &= \begin{bmatrix} (S \circ T)(\mathbf{e}_1) & (S \circ T)(\mathbf{e}_2) & \cdots & (S \circ T)(\mathbf{e}_p) \end{bmatrix} \\ &= \begin{bmatrix} S(T(\mathbf{e}_1)) & S(T(\mathbf{e}_2)) & \cdots & S(T(\mathbf{e}_p)) \end{bmatrix} \\ &= \begin{bmatrix} S(\mathbf{b}_1) & S(\mathbf{b}_2) & \cdots & S(\mathbf{b}_p) \end{bmatrix} \\ &= \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}. \end{aligned}$$

Proposition 8.2.3. *Let A, B, C be matrices of sizes for which the expressions below are defined and let $r \in \mathbb{R}$. Then*

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. $I_m A = A = A I_n$, *where I_n denotes the $n \times n$ identity matrix*

An important property of the usual multiplication of numbers that is missing from the above list is *commutativity*, that is, $AB = BA$. The reason for this omission is that this property simply does not hold for matrices. Indeed, in many cases one of the products is not even defined while the other makes perfect sense. For example, if A is a 2×4 matrix and B is a 4×3 matrix, then AB is 2×3, but BA is undefined. In fact, two matrices A and B stand a chance to commute only if they are square matrices of the same dimension. Otherwise, either some of the products is undefined, or AB and BA have different dimensions. However, even when A and B are both $n \times n$, it is more likely than not to have $AB \neq BA$. For example,

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 0 & 0 \end{bmatrix},$$

but

$$BA = \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix}.$$

This doesn't mean that no two matrices commute, just that that is more of a coincidence than the rule. On the other hand, the $n \times n$ identity matrix I_n and the $n \times n$ zero matrix 0 commute with every $n \times n$ matrix. Also, the following two matrices commute:

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -1 & -2 \\ 4 & -1 \end{bmatrix}.$$

Indeed,

$$AB = \begin{bmatrix} -5 & -1 \\ 2 & -5 \end{bmatrix}, \qquad BA = \begin{bmatrix} -5 & -1 \\ 2 & -5 \end{bmatrix}.$$

The above definition of matrix multiplication is convenient for proofs and applications, but not for calculations. Suppose that A is an $m \times n$ matrix whose (i, j)th entry is denoted a_{ij} (recall (1.3)). Suppose also that B is an $n \times p$ matrix whose (i, j)th entry is denoted b_{ij}. Then the (i, j)th entry of the product AB is the dot product of the ith row of A with the jth column of B:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}. \tag{8.1}$$

Example 8.2.4. *Compute AB, where A and B are as in Example 8.2.2.*

Solution. Using (8.1), we get

$$\begin{bmatrix} -1 & 2 \\ -5 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} (-1)(3) + (2)(2) & (-1)(-2) + (2)(1) \\ (-5)(3) + (4)(2) & (-5)(-2) + (4)(1) \\ (2)(3) + (-3)(2) & (2)(-2) + (-3)(1) \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -7 & 14 \\ 0 & -7 \end{bmatrix}.$$

□

8.3. **Transposition.** Yet another operation on matrices is the transposition of matrices. If A is $m \times n$, then its *transpose* A^t is the $n \times m$ matrix whose rows are the columns of A. For example,

$$\begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & -3 \end{bmatrix}^t = \begin{bmatrix} -1 & 6 \\ 2 & -9 \\ -1 & -3 \end{bmatrix}.$$

Proposition 8.3.1. *The transposition of matrices has the following properties:*

1. $(A^t)^t = A$
2. $(A + B)^t = A^t + B^t$
3. $(rA)^t = rA^t$
4. $(AB)^t = B^tA^t$

8.4. **Partitioned matrices*.** The notion of partitioned matrices is key in many real-life applications, where the sizes of the involved matrices can run into the thousands and, sometimes, even into the millions. The idea of partitioned matrices is quite simple, we consider a large matrix as a matrix of smaller dimensions but whose entries are again matrices. For example, the 3×7 matrix

$$A = \begin{bmatrix} 3 & 0 & -1 & 5 & -9 & -2 & 1 \\ -5 & 2 & -3 & 1 & -3 & 1 & 0 \\ -8 & -6 & 0 & 0 & -1 & 2 & -4 \end{bmatrix}$$

can be viewed as a 2×3 *partitioned* (or *block*) *matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix},$$

with *blocks*

$$A_{11} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & -3 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} 5 & -9 \\ 1 & -3 \end{bmatrix}, \qquad A_{13} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} -8 & -6 & 0 \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} 0 & -1 \end{bmatrix}, \qquad A_{23} = \begin{bmatrix} 2 & -4 \end{bmatrix}.$$

Let us explain how to perform algebraic operations on partitioned matrices. Addition and scalar multiplication are easy: to add two partitioned matrices whose blocks have equal dimensions, we just add the respective blocks; to multiply a partitioned matrix by a scalar, we multiply each block by the scalar. Partitioned matrices can be multiplied by the usual row-column rule as if the blocks were scalars, except that when we multiply block entries we use matrix multiplication instead of the usual multiplication of numbers.

Example 8.4.1. *Compute AB, where*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix},$$

and the blocks of A and B are

$$A_{11} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 5 & -2 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} 0 & -4 \\ 3 & -1 \end{bmatrix}, \qquad B_{11} = \begin{bmatrix} -2 & 2 \\ 1 & 1 \\ -7 & 0 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 0 & -4 & -2 \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} 7 & -4 \end{bmatrix}, \qquad B_{21} = \begin{bmatrix} 1 & -2 \\ 5 & -3 \end{bmatrix}.$$

Solution. If all the matrix products and sums exist, AB is a block matrix with blocks

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix}.$$

We have

$$A_{11}B_{11} = \begin{bmatrix} -14 & 1 \\ 17 & 7 \end{bmatrix}, \qquad A_{12}B_{21} = \begin{bmatrix} -20 & 12 \\ -2 & -3 \end{bmatrix}, \qquad A_{11}B_{11} + A_{12}B_{21} = \begin{bmatrix} -34 & 13 \\ 15 & 4 \end{bmatrix},$$

$$A_{21}B_{11} = \begin{bmatrix} 10 & -4 \end{bmatrix}, \qquad A_{22}B_{21} = \begin{bmatrix} -13 & -2 \end{bmatrix}, \qquad A_{21}B_{11} + A_{22}B_{21} = \begin{bmatrix} -3 & -6 \end{bmatrix},$$

so

$$AB = \begin{bmatrix} -34 & 13 \\ 15 & 4 \\ -3 & -6 \end{bmatrix}.$$

Notice that we would have gotten the same result by regular matrix multiplication of the 3×5 matrix A and the 5×2 matrix B. □

8.5. **Elementary matrices*.** We now discuss an element of the algebra of matrices that the text uses mostly in proofs. Since proofs are not the focus of this course, we will seldom use elementary matrices. However, they are among those notions which are a must in any linear algebra course, and this is as good a place as any to mention them.

Definition 8.5.1. An *elementary matrix* is an $n \times n$ matrix obtained by performing a single elementary row operation on the I_n, the $n \times n$ identity matrix.

For example, the matrices

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are all elementary matrices. E_1 was obtained from I_3 by adding 2 times the first row to the second row; E_2 was obtained from I_3 by interchanging rows 1 and 2; and E_3 was obtained from I_3 by scaling the second row by a factor of 3. The following example demonstrates that we can use elementary matrices to represent elementary row operations by matrix multiplications.

Example 8.5.2. *Compute E_1A, E_2A, and E_3A, where E_1, E_2, and E_3 are the above matrices and*

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ -2 & 3 & 2 & 0 \\ 0 & 5 & 0 & -2 \end{bmatrix}.$$

Solution. We have

$$E_1A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 7 & 4 & -2 \\ 0 & 5 & 0 & -2 \end{bmatrix}, \qquad E_2A = \begin{bmatrix} -2 & 3 & 2 & 0 \\ 1 & 2 & 1 & -1 \\ 0 & 5 & 0 & -2 \end{bmatrix}, \qquad E_3A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ -6 & 9 & 6 & 0 \\ 0 & 5 & 0 & -2 \end{bmatrix}.$$

□

Remark 8.5.3. The outcomes in this example are not a coincidence. In general, when we multiply a matrix A on the left by an elementary matrix E (of the right dimension), the product EA is the matrix that results from performing on the rows of A the same elementary row operation that was used to produce E from the identity matrix I. This observation can be used to write any matrix A as a product

$$A = E_1E_2 \cdots E_kB,$$

where $E_1, E_2, \ldots, E_k$ are elementary matrices and B is the reduced echelon form of A. (This is just the row reduction algorithm in disguise.) This representation provides a very convenient way to think of matrices in a general, abstract setting. It is also extremely inefficient from a practical standpoint, which explains why elementary matrices are mainly a theoretical tool.

9. The Inverse of a Matrix

Every nonzero real number has a multiplicative inverse. That is, for every $x \neq 0$, there is a real number y (also nonzero) such that $xy = yx = 1$; y is the number we denote by x^{-1} or $1/x$. The number zero doesn't have a multiplicative inverse. The purpose of this lecture is to generalize this to square matrices. Not surprisingly, the matrix case is a little more complicated.

9.1. **Definition.** Let A be an $n \times n$ matrix. The *multiplicative inverse of* A, denoted A^{-1}, is an $n \times n$ matrix B such that $AB = BA = I_n$, where I_n is the $n \times n$ identity matrix.

Remark 9.1.1. If we talk about *the* inverse matrix A^{-1}, we should explain why it is unique. Suppose that B and C are two matrices such that

$$AB = BA = I, \qquad AC = CA = I.$$

Notice that we don't claim that they are distinct (in fact, out goal is to prove that they are not). Multiplying the identity $AB = I$ on the left by C, we get

$$C(AB) = CI = C.$$

Here we used that $CI = C$ for any matrix C. On the other hand, since $CA = I$, we have

$$C(AB) = (CA)B = IB = B.$$

It follows that in fact $B = C$, as desired.

If a square matrix A has an inverse, we say that it is *invertible* or *non-singular*; otherwise, we call A *singular* or *non-invertible*.

9.2. **Finding A^{-1}.** How do we determine whether a matrix is invertible, and if is, how do we compute the inverse? Let's begin with the 2×2 case.

Theorem 10. *Let*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If $ad - bc \neq 0$, A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, A is singular.

The number $\det A = ad - bc$ is called the *determinant of* A. We will learn more about determinants in the next lecture.

Example 9.2.1. *Compute A^{-1}, where $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$.*

Solution. The determinant of A is $\det A = (2)(2) - (1)(-1) = 5$. Since $5 \neq 0$, A is invertible and

$$A^{-1} = (1/5) \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ 0.2 & 0.4 \end{bmatrix}.$$

□

We demonstrate the general approach towards finding the inverse of a matrix by an example.

Example 9.2.2. *Compute the inverse of the matrix*

$$A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}.$$

Solution. Let $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ be the columns of A^{-1} (assuming it exists). By the definition of matrix multiplication,

$$AA^{-1} = A\big[\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3\big] = \big[A\mathbf{b}_1\ A\mathbf{b}_2\ A\mathbf{b}_3\big]. \tag{9.1}$$

On the other hand, by definition of the inverse, $AA^{-1} = I_3$, so the columns of the matrix on the right side of (9.1) must be $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. In other words, if A^{-1} is to exist, its columns $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ must be solutions, respectively, of the equations

$$A\mathbf{x} = \mathbf{e}_1, \qquad A\mathbf{x} = \mathbf{e}_2, \qquad A\mathbf{x} = \mathbf{e}_3. \tag{9.2}$$

Thus, we must solve three matrix equations with the same coefficient matrix, but with different right sides. Because the row reductions of their augmented matrices follow essentially the same steps, we will perform the calculations simultaneously. To this end, we form the matrix

$$\begin{bmatrix} -2 & -7 & -9 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{bmatrix},$$

whose first three columns are the columns of A (the common coefficient matrix of the equations (9.2)) and whose last three columns are the last columns of the augmented matrices of (9.2). Row reducing this matrix, we get

$$\begin{aligned}
\begin{bmatrix} -2 & -7 & -9 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{bmatrix} &\sim \begin{bmatrix} 1 & 3 & 4 & 0 & 0 & 1 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ -2 & -7 & -9 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 & 1 & -2 \\ 0 & -1 & -1 & 1 & 0 & 2 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 3 & 4 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 & 2 \\ 0 & -1 & -1 & 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & -1 & 4 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 3 & 0 & -4 & 4 & -15 \\ 0 & 1 & 0 & -2 & 1 & -6 \\ 0 & 0 & 1 & 1 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 & 1 & 3 \\ 0 & 1 & 0 & -2 & 1 & -6 \\ 0 & 0 & 1 & 1 & -1 & 4 \end{bmatrix}.
\end{aligned}$$

We conclude that the solutions of the equations (9.2) are, respectively,

$$\mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}.$$

Thus, we must have

$$A^{-1} = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & -6 \\ 1 & -1 & 4 \end{bmatrix}.$$

It is always to check our work. In this problem that is easy—we simply compute AA^{-1}:

$$AA^{-1} = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & -6 \\ 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3. \quad \checkmark$$

□

We want to make several observations regarding the above solution.

Remark 9.2.3. First, let us explain why the matrix equations (9.2) cannot have more than one solution, regardless of the matrix A. Suppose that $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ is a solution. Then, as we saw, $[\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3]$ is A^{-1}, which we know is unique. This is not to say that none of the equations (9.2) can have more than one solution. It can happen that, say, the equation $A\mathbf{x} = \mathbf{e}_2$ has infinitely many solutions, but in that case one of the other two equations will be inconsistent.

Remark 9.2.4. Next, we observe that had we determined that the equations (9.2) are not all consistent, it would have followed that A is singular. That is, because otherwise the columns of A^{-1} would have provided a solution of (9.2).

Remark 9.2.5. The more critical among you may ask why we checked that $AA^{-1} = I_3$ but did not check that $A^{-1}A = I_3$. After all, the inverse must satisfy both conditions. It turns out that if the $n \times n$ matrices A and B satisfy $AB = I_n$, then they automatically satisfy also $BA = I_n$. That is why it wasn't necessary to check the other half of the definition.

Example 9.2.2 and Remarks 9.2.3 and 9.2.4 are the motivation behind the following theorem.

Theorem 11. *An $n \times n$ matrix A is invertible if and only if it is row equivalent to I_n. Furthermore, any sequence of elementary row operations which reduce A to I_n also transforms I_n to A^{-1}.*

This leads to a rather straightforward method for computing A^{-1} or proving that it does not exist:

1. Row reduce $[A\ I]$.
2. If the reduced echelon form of $[A\ I]$ is of the form $[I\ B]$, then $A^{-1} = B$; otherwise, A is singular.

Example 9.2.6. *Let*

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & -2 & 3 \\ 2 & 1 & 1 & 3 \\ 2 & -1 & 3 & 0 \end{bmatrix}.$$

Compute A^{-1} or show that it does not exist.

Solution. By row reduction,

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 3 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 3 & 0 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & 5 & -2 & -2 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 0 & 1 & 0 \\ 0 & 2 & -2 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 5 & -2 & -2 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & -8 & 1 & 4 & 1 & -2 & 0 \\ 0 & 0 & 8 & -1 & -4 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & -8 & 1 & 4 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}.$$

At this point, we can tell that the reduced echelon form of $[A\ I_4]$ won't be of the form $[I_4\ B]$. Thus, A is singular. □

Another tool that is often useful for computing inverse matrices in a more theoretical setting are the general properties of inverse matrices summarized in the following proposition.

Proposition 9.2.7. *Let A and B be invertible $n \times n$ matrices. Then:*

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^t)^{-1} = (A^{-1})^t$

9.3. **The Invertible Matrix Theorem.** The following theorem gives a bunch of ways to check whether an $n \times n$ matrix is invertible. It puts together most of the material covered so far in the course.

Theorem 12 (Invertible Matrix Theorem)**.** *Let A be an $n \times n$ matrix. Then the following conditions are equivalent:*

1. *A is invertible.*
2. *A is row equivalent to I_n.*
3. *A has n pivot positions.*
4. *The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *The columns of A are linearly independent.*
6. *The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.*
7. *The equation $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^n$.*
8. *The columns of A span $\mathbb{R}^n$.*
9. *The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is onto.*
10. *There is a matrix C such that $CA = I_n$.*
11. *There is a matrix D such that $AD = I_n$.*

12. *A^t is invertible.*

The Invertible Matrix Theorem is extremely useful in theoretical considerations, but it also has practical uses. Most notably, we may be able to use it to “check” our answer if the method from the previous section yields a negative answer.

Example 9.3.1. *Use the Invertible Matrix Theorem to decide whether or not the following matrices are invertible:*

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 3 & 7 & 0 \\ 8 & 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 4 \\ 3 & 3 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 0 & 3 \\ 1 & 0 & 2 \end{bmatrix}.$$

Try to use as few calculations as possible.

Solution. Matrices B and C are singular, because their columns are not linearly independent: the columns of C include the zero vector $\mathbf{0}$, and the first two columns of B are equal (and hence, proportional). A is invertible, because it has three pivots:

$$A \sim \begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

Example 9.3.2. *If the equation $G\mathbf{x} = \mathbf{y}$ has more than one solution for some $\mathbf{y} \in \mathbb{R}^n$, can the columns of G span $\mathbb{R}^n$?*

Solution. No. If $G\mathbf{x} = \mathbf{y}$ has more than one solution for some $\mathbf{y} \in \mathbb{R}^n$, then this equation has more than one solution in the homogeneous case $\mathbf{y} = \mathbf{0}$. In other words, condition 4) of the Inverse Matrix Theorem fails. Then condition 8) also fails, that is, the columns of G do not span $\mathbb{R}^n$. □

9.4. **Invertible linear transformations.**

Definition 9.4.1. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called *invertible* if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = T(S(\mathbf{x})) = \mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n. \tag{9.3}$$

The following theorem demonstrates that if S exists, then there isn't much choice as to what it can be: S is the linear transformation whose standard matrix is the inverse of the standard matrix of T.

Theorem 13. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix A. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying* (9.3).

The proof is not difficult, but we will still skip it. You can find it on p. 131 of the text. Together with the Inverse Matrix Theorem, Theorem 13 leads to the following

Corollary 9.4.2. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then the following conditions are equivalent:*

1. *T is invertible.*
2. *T is onto.*
3. *T is one-to-one.*

10. Determinants

In the last lecture, we encountered the determinant of a 2×2. In this lecture, we introduce determinants of square matrices of any dimension and study their properties.

10.1. **Definition.** Let A be an $n \times n$ matrix, whose (i, j)th entry is denoted by a_{ij}. We define the *determinant of* A, denoted $\det A$, using the following recursive procedure:

1. If $n = 1$ and $A = [a_{11}]$, then $\det A = a_{11}$.
2. If $n \geq 2$, for each i and j, we introduce the matrix A_{ij}, which is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and jth column. Then

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}. \tag{10.1}$$

Formula (10.1) is known as the *expansion of* $\det A$ *along the first row*.

Example 10.1.1. *Compute the determinant of the matrix* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Solution. We want to apply (10.1). We have

$$a_{11} = a, \quad a_{12} = b, \quad a_{21} = c, \quad a_{22} = d, \quad A_{11} = \begin{bmatrix} d \end{bmatrix}, \quad A_{12} = \begin{bmatrix} c \end{bmatrix},$$

so (10.1) with $n = 2$ gives

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} = ad - bc.$$

Notice that this is exactly the expression we encountered in Theorem 10. □

Example 10.1.2. *Compute the determinant of the matrix*

$$A = \begin{bmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{bmatrix}.$$

Solution. It is common to write the determinant of a matrix as the matrix enclosed in vertical lines instead of brackets, that is,

$$\det A = \begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix}.$$

By (10.1),

$$\begin{aligned} \det A &= 6 \begin{vmatrix} 7 & 2 & -5 \\ 0 & 0 & 0 \\ 3 & 1 & 8 \end{vmatrix} - 0 \begin{vmatrix} 1 & 2 & -5 \\ 2 & 0 & 0 \\ 8 & 1 & 8 \end{vmatrix} + 0 \begin{vmatrix} 1 & 7 & -5 \\ 2 & 0 & 0 \\ 8 & 3 & 8 \end{vmatrix} - 5 \begin{vmatrix} 1 & 7 & 2 \\ 2 & 0 & 0 \\ 8 & 3 & 1 \end{vmatrix} \\ &= 6 \begin{vmatrix} 7 & 2 & -5 \\ 0 & 0 & 0 \\ 3 & 1 & 8 \end{vmatrix} - 5 \begin{vmatrix} 1 & 7 & 2 \\ 2 & 0 & 0 \\ 8 & 3 & 1 \end{vmatrix}. \end{aligned}$$

Next, we have to compute two 3×3 determinants:

$$\begin{vmatrix} 7 & 2 & -5 \\ 0 & 0 & 0 \\ 3 & 1 & 8 \end{vmatrix} = 7\begin{vmatrix} 0 & 0 \\ 1 & 8 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ 3 & 8 \end{vmatrix} + (-5)\begin{vmatrix} 0 & 0 \\ 3 & 1 \end{vmatrix}$$

$$= 7[(0)(8) - (0)(1)] - 2[(0)(8) - (0)(3)] + (-5)[(0)(1) - (0)(3)] = 0,$$

$$\begin{vmatrix} 1 & 7 & 2 \\ 2 & 0 & 0 \\ 8 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 0 & 0 \\ 3 & 1 \end{vmatrix} - 7\begin{vmatrix} 2 & 0 \\ 8 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 \\ 8 & 3 \end{vmatrix}$$

$$= 1[(0)(1) - (0)(3)] - 7[(2)(1) - (8)(0)] + 2[(2)(3) - (8)(0)] = -2.$$

Substituting these values back into the expression for $\det A$, we get

$$\det A = 6(0) - 5(-2) = 10.$$

□

10.2. **Cofactor expansions of a determinant.** In Example 10.1.2, we wrote more than our fair share of terms, which eventually turned out to be 0s. One may ask whether there is a way not to have to write those. The answer to that question is in the affirmative. Given an $n \times n$ matrix A with entries a_{ij}, the (i, j)*th cofactor of* A is the number

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

That is, up to a sign, the cofactor is the determinant $\det A_{ij}$. The sign $(-1)^{i+j}$ depends on the position of the entry a_{ij} in the following way:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In particular, the right side of (10.1) is just the sum of the products of the numbers in the first row of A and their respective cofactors:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

It turns out that we can replace the first row by any row or column of A:

Theorem 14. *The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column of A. More precisely, for any i, $1 \le i \le n$, we have*

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}, \tag{10.2}$$

and for any j, $1 \le j \le n$, we have

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}. \tag{10.3}$$

Formulas (10.2) and (10.3) are called the *cofactor expansion across the ith row* and the *cofactor expansion down the jth column*, respectively. We now show how to use these formulas to take advantage of possible zero entries.

Example 10.2.1. *Compute the determinant of the matrix from Example 10.1.2.*

Solution. First, we expand across the third row. If we ignore the terms in the expansion that correspond to zero entries—and there are three such entries in the third row of A, this gives

$$\det A = 2(-1)^{3+1}\begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{vmatrix}.$$

We now expand this 3×3 determinant across the first row:

$$\det A = 2(5)(-1)^{1+3}\begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix} = 10[(7)(1) - (2)(3)] = 10.$$

□

Corollary 10.2.2. *If A is a triangular matrix, then* $\det A$ *is the product of the entries on the main diagonal.*

As usual, instead of proving this, we demonstrate how it works by an example.

Example 10.2.3. *Compute the determinant of the matrix*

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 5 & -2 & 0 & 0 \\ -8 & 3 & -1 & 0 \\ 4 & -7 & -5 & 1 \end{bmatrix}.$$

Solution. We expand each determinant across the first row:

$$\begin{vmatrix} 3 & 0 & 0 & 0 \\ 5 & -2 & 0 & 0 \\ -8 & 3 & -1 & 0 \\ 4 & -7 & -5 & 1 \end{vmatrix} = 3\begin{vmatrix} -2 & 0 & 0 \\ 3 & -1 & 0 \\ -7 & -5 & 1 \end{vmatrix} = (3)(-2)\begin{vmatrix} -1 & 0 \\ -5 & 1 \end{vmatrix} = (3)(-2)(-1)\begin{vmatrix} 1 \end{vmatrix} = (3)(-2)(-1)(1) = 6.$$

□

10.3. **Properties of determinants.** Having various cofactor expansions to play with is only useful if the matrix contains many zeros. If there are no zero entries, no matter which cofactor expansion we use, we will end up performing tons of arithmetic operations. We can avoid this by using the properties of determinants to replace the given determinant by an equal one that does contain many zeros. Here is a list of properties, which are useful in this context.

Proposition 10.3.1. *Let A and B be square matrices. Then:*

1. *If B is obtained from A by a row replacement, then* $\det B = \det A$.
2. *If B is obtained from A by the interchange of two rows, then* $\det B = -\det A$.
3. *If B is obtained from A by multiplyting one of its rows by a number k, then* $\det B = k \det A$.
4. $\det A^t = \det A$.
5. $\det(AB) = (\det A)(\det B)$.

Together, the first three properties in this proposition give us a simple path from a given determinant to an equal one with lots of zeros—namely, (partial) **row reduction**.

Example 10.3.2. *Compute the determinant of the matrix*

$$A = \begin{bmatrix} 1 & -1 & 2 & -2 \\ -1 & 2 & 1 & 6 \\ 2 & 1 & 14 & 10 \\ -2 & 6 & 10 & 33 \end{bmatrix}.$$

Solution. Using row replacements and then an expansion down the first column, we find

$$\begin{vmatrix} 1 & -1 & 2 & -2 \\ -1 & 2 & 1 & 6 \\ 2 & 1 & 14 & 10 \\ -2 & 6 & 10 & 33 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 3 & 10 & 14 \\ 0 & 4 & 14 & 29 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} 1 & 3 & 4 \\ 3 & 10 & 14 \\ 4 & 14 & 29 \end{vmatrix}.$$

Another series of row replacements and an expansion down the first column give

$$\det A = \begin{vmatrix} 1 & 3 & 4 \\ 3 & 10 & 14 \\ 4 & 14 & 29 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 2 & 13 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} 1 & 2 \\ 2 & 13 \end{vmatrix} = (1)(13) - (2)(2) = 9.$$

□

Example 10.3.3. *Compute the determinant of the matrix*

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & 1 \\ 2 & -4 & 14 \end{bmatrix}.$$

Solution. First, we apply property 4) to replace A by A^t:

$$\det A = \begin{vmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & 1 & 14 \end{vmatrix}.$$

Next, we add twice row 1 to row 2:

$$\det A = \begin{vmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 2 & 1 & 14 \end{vmatrix}.$$

Clearly, an expansion across the second row now gives $\det A = 0$. □

10.4. **Determinants and inverse matrices*.** We know from Theorem 10 that a 2×2 matrix is invertible if and only if its determinant is nonzero. It turns out that the same applies to any matrix:

Theorem 15. *An $n \times n$ matrix A is invertible if and only if* $\det A \neq 0$*.*

Furthermore, we can use determinants to write a "formula" for the inverse of an invertible matrix.

Theorem 16 (Formula for A^{-1})**.** *Let A be an invertible $n \times n$ matrix and let C_{ij} denote its (i, j)th cofactor. Then*

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

Notice that when $n = 2$, this is exactly the formula we gave in Theorem 10. Unfortunately, the general formula is completely useless for any practical purposes, because it involves a gazillion arithmetic operations. Another formula using determinants—just as useless for practical purposes, but quite useful in theoretical mathematics—gives an expression for the general solution of a linear system in terms of the coefficients.

Theorem 17 (Cramer's rule)**.** *Let A be an $n \times n$ invertible matrix and let $\mathbf{b} \in \mathbb{R}^n$. Then the unique solution of the equation $A\mathbf{x} = \mathbf{b}$ is*

$$x_j = \frac{\det A_j(\mathbf{b})}{\det A} \qquad (j = 1, 2, \ldots, n),$$

where $A_j(\mathbf{b})$ is the matrix obtained from A by replacing its jth column by $\mathbf{b}$.

11. Vector spaces and subspaces

This and the following three lectures generalize much of content of Lectures 3, 6, and 7 to a more abstract setting: that of abstract vector spaces. We will then show how to reduce the more general problems to familiar ones about matrices and vectors in $\mathbb{R}^n$.

11.1. **Abstract vector spaces.**

A vector space is just a set[1] having some special properties.

Definition 11.1.1. A *vector space* is a set V (whose elements are called *vectors*) that has the following properties:

I. V is nonempty.
II. We can *add* any two vectors in V, that is, given $\mathbf{x}, \mathbf{y} \in V$, there is some rule that defines their *sum* $\mathbf{x} + \mathbf{y}$.
III. We can *multiply vectors in V by scalars*, that is, given $\mathbf{x} \in V$ and $c \in \mathbb{R}$, there is some rule that defines the *scalar multiple* $c\mathbf{x}$.
IV. The operations addition and multiplication by scalars have the 10 properties listed below. Here $\mathbf{x}, \mathbf{y}, \mathbf{z}$ denote vectors in V and c, d real numbers
   1. The sum $\mathbf{x} + \mathbf{y}$ is also a vector in V.
   2. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
   3. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
   4. There is a *zero vector* $\mathbf{0}$ in V such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$.
   5. For each vector $\mathbf{x}$ there is a vector $-\mathbf{x}$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.
   6. The product $c\mathbf{x}$ is also a vector in V.
   7. $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$
   8. $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$
   9. $c(d\mathbf{x}) = (cd)\mathbf{x}$
   10. $1\mathbf{x} = \mathbf{x}$

This definition may seem a little overwhelming at first, but it isn't that bad after all. The essential idea is this: a vector space is a set which isn't empty (I. holds) and on which we have two operations (II. and III. hold) that have properties similar to those of the operations in $\mathbb{R}^n$ (compare IV. and Proposition 3.1.1).

Example 11.1.2. *The set $\mathbb{R}^n$ of all n-dimensional vectors, with the operations defined in Lecture 3, is a vector space. It is the canonical example of a vector space.*

Example 11.1.3. *The set of all $m \times n$ matrices, with the operations defined in Lecture 8, is a vector space.*

Example 11.1.4. *Let V be a set containing a single element $\mathbf{0}$. We can turn V into a vector space by defining*

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \qquad \textit{and} \qquad c\mathbf{0} = \mathbf{0} \textit{ for all } c \in \mathbb{R}.$$

Example 11.1.5. *The set of all functions $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ becomes a vector space, if we define the functions $\mathbf{f} + \mathbf{g}$ and $c\mathbf{f}$ by*

$$(\mathbf{f} + \mathbf{g})(t) = \mathbf{f}(t) + \mathbf{g}(t) \qquad \textit{and} \qquad (c\mathbf{f})(t) = c\mathbf{f}(t).$$

Example 11.1.6. *The set $\mathbb{P}_n$ of all polynomials of degree at most n is a vector space.*

[1] If the term "set" doesn't sound familiar, you should check the short handout on sets posted on the class website.

Example 11.1.7. *Let $n \geq 1$. The set of all polynomials of degree exactly n is **not** a vector space, because it fails axiom IV.1.*

Explanation. For example, the polynomials $\mathbf{p}(t) = t^n + 1$ and $\mathbf{q}(t) = -t^n - 3$ have degree n, but their sum

$$\mathbf{p}(t) + \mathbf{q}(t) = (t^n + 1) + (-t^n - 3) = -2$$

is a polynomial of degree 0. □

Example 11.1.8. *The set $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ of all integers is not a vector space, because it fails axiom IV.6.*

11.2. **Subspaces.** When a vector space H sits inside another vector space V, we say that H is a *subspace* of V, that is, a subspace of V is a part of V that is a vector space of its own right. The formal defininition reads as follows.

Definition 11.2.1. Let V be a vector space. A *subspace* of V is any set H in V that has the following three properties:

1. The zero vector $\mathbf{0}$ is in H.
2. For each $\mathbf{x}, \mathbf{y} \in H$, the sum $\mathbf{x} + \mathbf{y}$ is in H.
3. For each $\mathbf{x} \in H$ and $c \in \mathbb{R}$, the scalar multiple $c\mathbf{x}$ is in H.

Remark 11.2.2. Properties 2) and 3) are the important ones. They mean that H is *closed* under addition and multiplication by scalars, that is, if we apply either operation to vectors from H, the result is back in H. Property 1) is much less important, and in fact, it is almost a corollary of property 3). The main reason to have property 1) is to rule out the possibility that H is empty; in fact, for any nonempty set H, 1) follows from 3).

Example 11.2.3. *For any vector space V, the sets $\{\mathbf{0}\}$ and V are both subspaces of V.*

Example 11.2.4. *The space $\mathbb{P}_n$ from Example 11.1.6 is a subspace of the space of all real functions (Example 11.1.5).*

Example 11.2.5. *The space $\mathbb{P}_n$ is a subspace of $\mathbb{P}_{n+1}$.*

Example 11.2.6. *The space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ (simply because $\mathbb{R}^2$ does not sit inside $\mathbb{R}^3$). On the other hand, the set*

$$H = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^3$.

Example 11.2.7. *The set*

$$H = \left\{ \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

is not a subspace of $\mathbb{R}^3$.

Example 11.2.8. *If $\mathbf{v}_1, \mathbf{v}_2$ are vectors in $\mathbb{R}^n$, the set $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a subspace of $\mathbb{R}^n$.*

Solution. Recall that H is the set of all vectors of the form

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2,$$

where x_1 and x_2 are scalars. H contains the zero vector, because

$$\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2.$$

To show that H is closed under addition, we consider two generic elements of H:

$$\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 \qquad \text{and} \qquad \mathbf{y} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2.$$

We then have

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1\mathbf{v}_1 + x_2\mathbf{v}_2) + (y_1\mathbf{v}_1 + y_2\mathbf{v}_2) \\ &= (x_1\mathbf{v}_1 + y_1\mathbf{v}_1) + (x_2\mathbf{v}_2 + y_2\mathbf{v}_2) = (x_1 + y_1)\mathbf{v}_1 + (x_2 + y_2)\mathbf{v}_2, \end{aligned}$$

whence $(\mathbf{x} + \mathbf{y}) \in H$. Finally, H is closed under scalar multiplication because for any $c \in \mathbb{R}$, we have

$$c\mathbf{x} = c(x_1\mathbf{v}_1 + x_2\mathbf{v}_2) = (cx_1)\mathbf{v}_1 + (cx_2)\mathbf{v}_2$$

is an element of H. □

Clearly, we can generalize the notions of linear combination and Span to abstract vector spaces. Let V be a vector space. An expression of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p,$$

where $c_1, \dots, c_p \in \mathbb{R}$ and $\mathbf{v}_1, \dots, \mathbf{v}_p \in V$, is called a *linear combination of* $\mathbf{v}_1, \dots, \mathbf{v}_p$ *with weights* $c_1, \dots, c_p$. We also define $\operatorname{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ as the set of all possible linear combinations of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$; it is called the *subspace spanned* (or *generated*) by $\mathbf{v}_1, \dots, \mathbf{v}_p$, because of the following theorem.

Theorem 18. *If V is a vector space and $\mathbf{v}_1, \dots, \mathbf{v}_p$ are vectors in V, then $\operatorname{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of V.*

11.3. **Column and null spaces of a matrix.**

Definition 11.3.1. The *column space* of a matrix A, denoted $\operatorname{Col} A$, is the set of all linear combinations of the columns of A.

In other words, if A is an $m \times n$ matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, then $\operatorname{Col} A = \operatorname{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Thus, $\operatorname{Col} A$ is a subspace of $\mathbb{R}^m$ (the columns of A are m-dimensional vectors).

Example 11.3.2. *Find three vectors in the column space of A, where*

$$A = \begin{bmatrix} 2 & -3 & 4 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & 2 & -1 & -3 \\ 3 & 0 & 1 & 4 \end{bmatrix}.$$

Solution. For example, the following three are all linear combinations of the columns of A:

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \qquad \mathbf{a}_4 = \begin{bmatrix} -1 \\ -1 \\ -3 \\ 4 \end{bmatrix}, \qquad \mathbf{a}_1 + \mathbf{a}_4 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 7 \end{bmatrix}.$$

□

Example 11.3.3. *Determine whether* **b** *is in the column space of A, where*

$$A = \begin{bmatrix} 2 & -3 & 4 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & 2 & -1 & -3 \\ 3 & 0 & 1 & 4 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 1 \\ 7 \end{bmatrix}.$$

Solution. We must decide whether **b** is a linear combination of the columns of A, or equivalently, whether the matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent. Row reducing the augmented matrix, we obtain

$$\begin{bmatrix} 2 & -3 & 4 & -1 & 5 \\ 1 & -1 & 0 & -1 & 1 \\ 0 & 2 & -1 & -3 & 1 \\ 3 & 0 & 1 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 2 & -3 & 4 & -1 & 5 \\ 0 & 2 & -1 & -3 & 1 \\ 3 & 0 & 1 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 0 & -1 & 4 & 1 & 3 \\ 0 & 2 & -1 & -3 & 1 \\ 0 & 3 & 1 & 7 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & -4 & -1 & -3 \\ 0 & 2 & -1 & -3 & 1 \\ 0 & 3 & 1 & 7 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & -4 & -1 & -3 \\ 0 & 0 & 7 & -1 & 7 \\ 0 & 0 & 13 & 10 & 13 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & -4 & -1 & -3 \\ 0 & 0 & 1 & -1/7 & 1 \\ 0 & 0 & 13 & 10 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & -4 & -1 & -3 \\ 0 & 0 & 1 & -1/7 & 1 \\ 0 & 0 & 0 & 83/7 & 0 \end{bmatrix}.$$

Since the last column of the augmented matrix is not a pivot column, it follows that the equation $A\mathbf{x} = \mathbf{b}$ is consistent, and therefore, **b** is in $\operatorname{Col} A$. (If you complete the row reduction, you will find out that $(2, 1, 1, 0)$ is a solution of $A\mathbf{x} = \mathbf{b}$ and you will be able to check that, indeed, $\mathbf{b} = 2\,\mathrm{col}_1(A) + \mathrm{col}_2(A) + \mathrm{col}_3(A)$.) □

Definition 11.3.4. The *null space* of a matrix A, denoted $\operatorname{Nul} A$, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

It is easy to check whether a given vector is in the null space of a given matrix.

Example 11.3.5. *Determine whether* **b** *is in* $\operatorname{Nul} A$*, where*

$$A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Solution. All we have to do is check whether **b** is a solution to $A\mathbf{x} = \mathbf{0}$, that is, check whether $A\mathbf{b} = \mathbf{0}$:

$$A\mathbf{b} = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0-3+0 \\ 0+0-5+0 \\ 3+0+1+0 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 4 \end{bmatrix}.$$

Thus, $\mathbf{b} \notin \operatorname{Nul} A$. □

So far, we saw that it is easy to describe the column space of a matrix, but not so easy to decide whether a particular vector belongs to it—the latter was equivalent to solving a linear system. We also saw that it is not difficult to decide whether a particular vector belongs to the null set, but we know that it is difficult to describe it—that would mean to describe the general solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Yet, if we gather what we know about solution sets of linear systems (recall Fact 5.1.3, in particular), we reach the following conclusion.

Theorem 19. *The null space of an* $m \times n$ *matrix* A *is a subspace of* $\mathbb{R}^n$.

12. Bases and Coordinates

Before we can proceed to discuss the properties of vector spaces and subspaces, we want to find an efficient way of describing them. This leads us to the notion of a "basis".

12.1. **Linear independence.** Before we get to the notion of basis, we need to extend the notion of linear (in)dependence from Lecture 6 to arbitrary vector spaces. Let V be a vector space. A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V is called *linearly independent* if the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{12.1}$$

holds only when $c_1 = \cdots = c_p = 0$. The set vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is called *linearly dependent* if there exist coefficients $c_1, \dots, c_p$, **not all zero**, such that (12.1) holds; in such a case, (12.1) is called a *linear dependence relation*.

Example 12.1.1. *Let $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t^2 - 1$, and $\mathbf{p}_3(t) = 3t^2 + 2$. Then the set $\{\mathbf{p}_1, \mathbf{p}_2\}$ is linearly independent in $\mathbb{P}_2$; the set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly dependent in $\mathbb{P}_2$.*

Solution. If c_1 and c_2 are any two real numbers, we have

$$(c_1\mathbf{p}_1 + c_2\mathbf{p}_2)(t) = c_1\mathbf{p}_1(t) + c_2\mathbf{p}_2(t) = c_1(1) + c_2(t^2 - 1) = c_2t^2 + (c_1 - c_2),$$

so $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 = \mathbf{0}$ means that the polynomial $c_2t^2 + (c_1 - c_2)$ is the zero polynomial, that is, all its coefficients vanish:

$$c_2 = 0, \quad c_1 - c_2 = 0 \qquad \Rightarrow \qquad c_1 = c_2 = 0.$$

This shows that the equation $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 = \mathbf{0}$ holds only when $c_1 = c_2 = 0$, which means that $\{\mathbf{p}_1, \mathbf{p}_2\}$ is linearly independent.

We now turn to second part of the problem. We this is easier, because it suffices to find a linear dependence relation among the three given "vectors". Here is one such relation:

$$(-5)\mathbf{p}_1 + (-3)\mathbf{p}_2 + \mathbf{p}_3 = \mathbf{0}.$$

Indeed, for every t, we have

$$((-5)\mathbf{p}_1 + (-3)\mathbf{p}_2 + \mathbf{p}_3)(t) = (-5)1 + (-3)(t^2 - 1) + (3t^2 + 2) = 0.$$

□

12.2. **Bases.**

Definition 12.2.1. Let V be a vector space and H be a subspace of V. A set $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a *basis* for H if:

1. $H = \operatorname{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$;
2. $\mathcal{B}$ is linearly independent.

Example 12.2.2. *The vectors*

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^n$.

This basis is called the *standard basis for $\mathbb{R}^n$*.

Solution. We have to check two things: that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly independent and that every vector in $\mathbf{x} \in \mathbb{R}^n$ is a linear combination of $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Both these are pretty straightforward. First, suppose that

$$x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n = \mathbf{0}.$$

The expression on the left equals

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

which is the zero vector only when $x_1 = x_2 = \cdots = x_n = 0$. That is, $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly independent. It is also clear that any $\mathbf{x} \in \mathbb{R}^n$ can be expressed as a linear combination of $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Indeed,

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

□

Example 12.2.3. *The polynomials* $1, t, t^2, \ldots, t^n$ *form a basis for* $\mathbb{P}_n$.

This basis is called the *standard basis for* $\mathbb{P}_n$.

Solution. Write $\mathbf{p}_j(t) = t^j$, $j = 0, 1, \ldots, n$. An arbitrary polynomial in $\mathbb{P}_n$ is of the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

for some real numbers $a_0, a_1, \ldots, a_n$. But then

$$\mathbf{p}(t) = a_0\mathbf{p}_0(t) + a_1\mathbf{p}_1(t) + \cdots + a_n\mathbf{p}_n(t),$$

so we find that $\mathbf{p}_0, \ldots, \mathbf{p}_n$ span $\mathbb{P}_n$. It's also easy to show that $\mathbf{p}_0, \ldots, \mathbf{p}_n$ are linearly independent. (Try this as an exercise!) □

Remark 12.2.4. We now want to consider the notion of basis from two different standpoints. We will make use the following theorem, which is a generalization of Theorem 7.

Theorem 20. *A set* $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ *in a vector space* V *is linearly dependent if and only if some vector is a linear combination of the remaining vectors.*

First, we argue that a basis is a "maximal" linear independent set. Indeed, if $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for H, a subspace of a space V, then any vector in H is a linear combination of the vectors in $\mathcal{B}$ (by condition 1) in the definition of basis). Thus, if we were to "increase" $\mathcal{B}$ by adding another vector, the resulting set of $p + 1$ vectors would be linearly dependent by Theorem 20. But if the new set is linearly dependent, it's not a basis.

On the other hand, we may think of bases as "minimal" spanning sets. Indeed, suppose that $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for H and consider the set we obtain by removing one vector from $\mathcal{B}$—say, $\mathcal{B}' = \{\mathbf{b}_2, \ldots, \mathbf{b}_p\}$. If $\mathcal{B}'$ were a spanning set for H, we would have that $\mathbf{b}_1$ (which lies in H) is a linear combination of $\mathbf{b}_2, \ldots, \mathbf{b}_p$ and Theorem 20 would imply that $\mathcal{B}$ is a linearly dependent set. Since we know that $\mathcal{B}$ is linearly independent (it's a basis), it follows that $\mathcal{B}'$ is not a spanning set for H.

Example 12.2.5. *Find a basis for* Nul A*, where*

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & 2.5 & 1.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution. Since we are given the reduced echelon form of A, we can tell that the general solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{cases} x_1 = -6x_3 - 5x_4 \\ x_2 = -2.5x_3 - 1.5x_4 \\ x_3, x_4 \text{ free} \end{cases},$$

or in vector form

$$\mathbf{x} = x_3 \begin{bmatrix} -6 \\ -2.5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -1.5 \\ 0 \\ 1 \end{bmatrix}.$$

Thus,

$$\operatorname{Nul} A = \operatorname{Span} \left\{ \begin{bmatrix} -6 \\ -2.5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -1.5 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

That is, the vectors

$$\begin{bmatrix} -6 \\ -2.5 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -5 \\ -1.5 \\ 0 \\ 1 \end{bmatrix}$$

span Nul A. Since these vectors are also linearly independent (there are two of them and they are not proportional), it follows that they form a basis for Nul A. □

This example is typical of how we find a basis for the null space of matrix. We state the general situation in the first part of the next theorem. The second part of the theorem describes how to find a basis for the column space.

Theorem 21. *Let A be a matrix. Then:*

1. *The spanning set for* Nul A *which we obtain by writing the general solution to* $A\mathbf{x} = \mathbf{0}$ *in vector form is in fact a basis for* Nul A*.*
2. *The pivot columns of A form a basis for* Col A*.*

Example 12.2.6. *Find a basis for the column space of the matrix from Example 12.2.5.*

Solution. Since first two columns for A are pivot columns, by the second part of Theorem 21, a basis for Col A is

$$\left\{ \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}.$$

Take notice that we use the **pivot columns of the matrix** A **itself**, not the pivot columns of the reduced matrix! □

Remark 12.2.7. Theorem 21 can be used to construct a basis for any subspace of $\mathbb{R}^n$ that is described as the subspace spanned by certain vectors. Consider the space $H = \mathrm{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_r\}$. We can think of it as the column space of the matrix $A = [\mathbf{a}_1 \cdots \mathbf{a}_r]$, which reduces the problem to constructing a basis for $\mathrm{Col}\, A$. Theorem 21 tells us how to construct such a basis: we can take the pivot columns of A.

12.3. **Coordinate systems.** Let V be a vector space. The main advantage of a basis $\mathcal{B}$ over a mere spanning set for a subspace H of V is the following fact.

Theorem 22 (Unique representation theorem)**.** *Let V be a vector space and let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ be a basis for a subspace H of V. Then every vector $\mathbf{v} \in H$ has a unique representation as a linear combination of the vectors in $\mathcal{B}$, that is, there are unique scalars $c_1, \dots, c_p$ such that*

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p. \tag{12.2}$$

Proof. Suppose that a vector $\mathbf{v} \in H$ has two (potentially different) representations:

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p \qquad \text{and} \qquad \mathbf{v} = d_1\mathbf{b}_1 + d_2\mathbf{b}_2 + \cdots + d_p\mathbf{b}_p. \tag{12.3}$$

Subtracting these two representations gives

$$\mathbf{0} = \mathbf{v} - \mathbf{v} = (c_1 - d_1)\mathbf{b}_1 + (c_2 - d_2)\mathbf{b}_2 + \cdots + (c_p - d_p)\mathbf{b}_p.$$

Since $\mathcal{B}$ is a linearly independent set, the weights on the right side of the last equation must all be zeros, that is,

$$c_1 - d_1 = 0, \qquad c_2 - d_2 = 0, \qquad \dots, \qquad c_p - d_p = 0.$$

So, the representations (12.3) must in fact coincide. □

Definition 12.3.1. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ be a basis for a vector space V. For each vector $\mathbf{v} \in V$, the unique set of coefficients $c_1, \dots, c_p$ in (12.2) are called the *coordinates of* $\mathbf{v}$ *with respect to the basis* $\mathcal{B}$. The vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is called the *coordinate vector of* $\mathbf{v}$ *with respect to* $\mathcal{B}$, or the $\mathcal{B}$*-coordinate vector of* $\mathbf{v}$.

Example 12.3.2. *Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis in $\mathbb{R}^3$. Compute $[\mathbf{v}_i]_{\mathcal{E}}$ for the vectors*

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Solution. We have

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = (-1)\mathbf{e}_1 + 0\mathbf{e}_2 + 3\mathbf{e}_3,$$

so

$$[\mathbf{v}_1]_{\mathcal{E}} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \mathbf{v}_1.$$

Similarly,

$$[\mathbf{v}_2]_{\mathcal{E}} = \mathbf{v}_2, \qquad [\mathbf{v}_3]_{\mathcal{E}} = \mathbf{v}_3.$$

□

This example is the reason we call the basis $\mathcal{E}$ "standard": the standard basis is the only basis for $\mathbb{R}^n$, with respect to which the coordinates of any vector are that same vector.

Example 12.3.3. *The set*

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} \right\}.$$

is basis for $\mathbb{R}^3$.

(a) *Find the vector* $\mathbf{v} \in \mathbb{R}^3$*, whose* $\mathcal{B}$*-coordinates are* $\begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$.

(b) *Find the* $\mathcal{B}$*-coordinates of* $\mathbf{v} = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}$.

Solution. (a) By the definition of $\mathcal{B}$-coordinates,

$$\mathbf{v} = 5 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -1 \end{bmatrix}.$$

(b) Let c_1, c_2, c_3 be the $\mathcal{B}$-coordinates of $\mathbf{v}$. Then (c_1, c_2, c_3) is the solution (we know that there is only one, because $\mathcal{B}$ is a basis) of the vector equation

$$x_1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}. \tag{12.4}$$

By row reduction,

$$\begin{bmatrix} 0 & 1 & 2 & 0 \\ 2 & 1 & -1 & 5 \\ 1 & 0 & -3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 4 \\ 2 & 1 & -1 & 5 \\ 0 & 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 5 & -3 \\ 0 & 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Thus, the solution of (12.4) is $(1, 2, -1)$ and $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. □

12.4. **The coordinate mapping.** Let V be a vector space with basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$. Then for every $\mathbf{x} \in V$ we can compute its $\mathcal{B}$-coordinate vector $[\mathbf{x}]_{\mathcal{B}}$, which is a good old-fashioned vector in $\mathbb{R}^p$. That is, we have a function $T : V \to \mathbb{R}^p$, defined by

$$T(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}.$$

This function has the following properties:

(i) if $\mathbf{x} \neq \mathbf{y}$, then $T(\mathbf{x}) \neq T(\mathbf{y})$ (i.e., T is one-to-one);
(ii) each $\mathbf{y} \in \mathbb{R}^n$ is a value of T (i.e., T is onto);

(iii) if $\mathbf{x}, \mathbf{y} \in V$ and $c \in \mathbb{R}$, then $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(c\mathbf{x}) = cT(\mathbf{x})$ (i.e., T is linear).

The mathematical term for such a function (one-to-one, onto, and linear) is *isomorphism*. It is possible to define an isomorphism between any two vector spaces, but we won't go there in this course. Instead, we will say that a space V as above is *isomorphic to* $\mathbb{R}^p$.

The practical side of the notion of isomorphism is that respective vectors in isomorphic spaces have identical properties. For instance, if we want to know whether vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in some isoteric vector space V are linearly independent, we simply have to decide whether or not $[\mathbf{x}]_{\mathcal{B}}, [\mathbf{y}]_{\mathcal{B}}, [\mathbf{z}]_{\mathcal{B}}$ (which are vectors in $\mathbb{R}^p$) are linearly independent. This is much easier, since we now have row reduction at our disposal.

Example 12.4.1. *Are the polynomials* $1 + t^3$, $3 + t - 2t^2$, *and* $-t + 3t^2 - t^3$ *linearly independent?*

Solution. Let $\mathcal{B} = \{1, t, t^2, t^3\}$ be the standard basis for $\mathbb{P}_3$. Then

$$\left[1 + t^3\right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \left[3 + t - 2t^2\right]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \qquad \left[-t + 3t^2 - t^3\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \\ 3 \\ -1 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \\ 0 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

the coordinate vectors are linearly independent. Thus, the original polynomials (vectors in $\mathbb{P}_3$) are also linearly independent. □

Example 12.4.2. *Is* $7 + t - t^2$ *in* $\operatorname{Span}\left\{1 + t^3, 3 + t - 2t^2, -t + 3t^2 - t^3\right\}$*?*

Solution. Again, let $\mathcal{B} = \{1, t, t^2, t^3\}$ be the standard basis for $\mathbb{P}_3$. Then

$$\left[7 + t - t^2\right]_{\mathcal{B}} = \begin{bmatrix} 7 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 1 & 3 & 0 & 7 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & 3 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 7 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & 3 & -1 \\ 0 & -3 & -1 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 7 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we find that

$$\left[7 + t - t^2\right]_{\mathcal{B}} = (1)\left[1 + t^3\right]_{\mathcal{B}} + (2)\left[3 + t - 2t^2\right]_{\mathcal{B}} + (1)\left[-t + 3t^2 - t^3\right]_{\mathcal{B}}.$$

Thus, by isomorphism, we also have (which, of course, is also easy to check directly, now that we know what to check)

$$7 + t - t^2 = 1(1 + t^3) + 2(3 + t - 2t^2) + 1(-t + 3t^2 - t^3).$$

□

13. Rank and Dimension

13.1. **Dimension.** Next, we approach the important notion of dimension of a vector space.

Definition 13.1.1. Let V be a vector space. Then the *dimension of* V, denoted $\dim V$, is the number of vectors in any basis for V. The dimension of the zero space $\{\mathbf{0}\}$ is defined to be zero.

In other words, the dimension of a vector space V is a number that we attach to V, which tells us "how complicated" V is. However, before we proceed further with the study of dimensions of vector spaces, we must make sure that this notion is well-defined. What that means is that we must rule out the possibility for a vector space to have two different bases containing different number of vectors. To this end, we state the following theorem.

Theorem 23. *If a vector space V has a basis of p vectors, then every basis for V must consist of p vectors.*

We will not give a formal proof of this theorem, but we will sketch the main idea.

Example 13.1.2. *Let H be the subspace of $\mathbb{R}^3$ spanned by the vectors*

$$\mathbf{b}_1 = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.$$

Explain why any set in H consisting of three vectors must be linearly dependent (and hence, not a basis).

Solution. Consider three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in H. Since $H = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2\}$, each $\mathbf{v}_i$ must be a linear combination of $\mathbf{b}_1, \mathbf{b}_2$, that is,

$$\begin{cases} \mathbf{v}_1 = c_{11}\mathbf{b}_1 + c_{12}\mathbf{b}_2 \\ \mathbf{v}_2 = c_{21}\mathbf{b}_1 + c_{22}\mathbf{b}_2 \\ \mathbf{v}_3 = c_{31}\mathbf{b}_1 + c_{32}\mathbf{b}_2 \end{cases} \tag{13.1}$$

for some numbers $c_{11}, \dots, c_{32}$. We will show that these formulas imply the linear dependence of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Consider the homogeneous equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}. \tag{13.2}$$

Using (13.1), we can rewrite (13.2) as

$$x_1\big(c_{11}\mathbf{b}_1 + c_{12}\mathbf{b}_2\big) + x_2\big(c_{21}\mathbf{b}_1 + c_{22}\mathbf{b}_2\big) + x_3\big(c_{31}\mathbf{b}_1 + c_{32}\mathbf{b}_2\big) = \mathbf{0}$$
$$\Updownarrow$$
$$\big(c_{11}x_1 + c_{21}x_2 + c_{31}x_3\big)\mathbf{b}_1 + \big(c_{12}x_1 + c_{22}x_2 + c_{32}x_3\big)\mathbf{b}_2 = \mathbf{0}.$$

Hence, every solution of the linear system

$$\begin{cases} c_{11}x_1 + c_{21}x_2 + c_{31}x_3 = 0 \\ c_{12}x_1 + c_{22}x_2 + c_{32}x_3 = 0 \end{cases} \tag{13.3}$$

is also a solution of (13.2). However, (13.3) is a homogeneous linear system of two equations in three unknowns, and thus it must have nontrivial solutions (it is consistent and must have at least one free variable). It follows that (13.2) must also have nontrivial solutions. But that is exactly the definition of linear dependence of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$—we reached the desired contradiction. □

Hopefully, you are convinced that a similar argument will yield the following observation: if V is a vector space that has a spanning set of p vectors, then any set in V consisting of more than p vectors is linearly dependent. We now see why a vector space cannot have two bases consisting of different numbers of vectors. Suppose that $\mathcal{B}$ and $\mathcal{C}$ are bases of p and q vectors, respectively. If say, $p < q$, using that $\mathcal{B}$ spans V, we deduce that every set of q vectors is linearly dependent, and hence, not a basis; in particular, $\mathcal{C}$ can't be a basis. On the other hand, if $q < p$, using that $\mathcal{C}$ spans V, we deduce that every set of p vectors is linearly dependent, and hence, not a basis; in particular, $\mathcal{B}$ can't be a basis. That is, the only case where both $\mathcal{B}$ and $\mathcal{C}$ can be bases is the case $p = q$.

Example 13.1.3. *The dimension of* $\mathbb{R}^n$ *is* n*, because the standard basis consists of* n *vectors.*

Example 13.1.4. *The dimension of* $\mathbb{P}_n$ *is* $n+1$*, because the standard basis* $\{1, t, t^2, \ldots, t^n\}$ *consists of* $n+1$ *vectors.*

Example 13.1.5. *Find the dimension of the following subspace of* $\mathbb{R}^4$*:*

$$H = \left\{ \begin{bmatrix} a - 4b - 2c \\ 2a + 5b - 4c \\ -a + 2c \\ -3a + 7b + 6c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Solution. First, we need to find a basis for H. We have

$$\begin{bmatrix} a - 4b - 2c \\ 2a + 5b - 4c \\ -a + 2c \\ -3a + 7b + 6c \end{bmatrix} = \begin{bmatrix} a \\ 2a \\ -a \\ -3a \end{bmatrix} + \begin{bmatrix} -4b \\ 5b \\ 0 \\ 7b \end{bmatrix} + \begin{bmatrix} -2c \\ -4c \\ 2c \\ 6c \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix} + b \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} + c \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix},$$

so

$$H = \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix} \right\}.$$

As we said in the last lecture, we can interpret this as $H = \operatorname{Col} A$, where

$$A = \begin{bmatrix} 1 & -4 & -2 \\ 2 & 5 & -4 \\ -1 & 0 & 2 \\ -3 & 7 & 6 \end{bmatrix}.$$

Row reduction yields

$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so a basis for $\operatorname{Col} A$ is given by the first and second columns of A:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} \right\}.$$

Hence, $\dim H = 2$. □

Example 13.1.6 (Subspaces of $\mathbb{R}^2$)**.** The subspaces of $\mathbb{R}^2$ can be classified according to dimension as follows:

- 0-dimensional subspaces. Only the zero subspace $\{\mathbf{0}\}$.
- 1-dimensional subspaces. These have bases consisting of a single vector, so they are lines through the origin.
- 2-dimensional subspaces. Only the entire space $\mathbb{R}^2$.

Example 13.1.7 (Subspaces of $\mathbb{R}^3$)**.** The subspaces of $\mathbb{R}^3$ can be classified according to dimension as follows:

- 0-dimensional subspaces. Only the zero subspace $\{\mathbf{0}\}$.
- 1-dimensional subspaces. These have bases consisting of a single vector, so they are lines through the origin.
- 2-dimensional subspaces. These have bases consisting of two vectors. Geometrically, they are planes through the origin.
- 3-dimensional subspaces. Only the entire space $\mathbb{R}^3$.

We finish this section with the following theorem.

Theorem 24 (Basis theorem)**.** *Let V be an n-dimensional vector space. Any linearly independent set of exactly n vectors in V is automatically a basis for V. Any spanning set of n vectors in V is also automatically a basis for V.*

In other words, if we know the dimension of the space, then to check that a set of that many vectors is a basis, we need to check only one of the two properties required of a basis. The second property will then be automatic.

13.2. **The row space of a matrix.**

Definition 13.2.1. The *row space* of a matrix A, denoted $\operatorname{Row} A$, is the set of all linear combinations of the rows of A.

Example 13.2.2. *Observe that the matrices*

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 6 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 2 & 6 \\ 2 & 3 \end{bmatrix}$$

have the same column space.

For the same reasons, we have the following theorem.

Theorem 25. *If $A \sim B$, then $\operatorname{Row} A = \operatorname{Row} B$. In particular, the nonzero rows of any echelon form of A form a basis for $\operatorname{Row} A$.*

Example 13.2.3. *Find a basis for the row space of the matrix A in Example 12.2.5.*

Solution. We use the pivot rows of the echelon form of A:

$$\operatorname{Row} A = \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2.5 \\ 1.5 \end{bmatrix} \right\}.$$

□

13.3. **The Rank Theorem.**

Definition 13.3.1. The *rank* of a matrix A is the dimension of $\operatorname{Col} A$.

Example 13.3.2. *Let*

$$A = \begin{bmatrix} 1 & 6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & -5 \\ 0 & 0 & 0 & -5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the dimensions of $\operatorname{Col} A$ *and* $\operatorname{Nul} A$*.*

Solution. A basis for $\operatorname{Col} A$ consists of the pivot columns of A. Since A is already in echelon form (although, not in reduced echelon form), we see that a basis for $\operatorname{Col} A$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ -5 \\ 0 \end{bmatrix} \right\}.$$

Hence, $\operatorname{rank} A = \dim(\operatorname{Col} A) = 3$.

To find $\dim(\operatorname{Nul} A)$, we need to count the number of vectors in a basis for $\operatorname{Nul} A$. One way to do this is to actually compute a basis for $\operatorname{Nul} A$. However, this is unnecessary, as we know that that basis will contain as many vectors as there are free variables (see Theorem 21). Since A has two non-pivot columns, there will be two free variables, whence $\dim(\operatorname{Nul} A) = 2$. □

Notice that in this example we have $\dim(\operatorname{Col} A) + \dim(\operatorname{Nul} A) = 5$ (the number of columns of A). This, of course, is not an accident: $\dim(\operatorname{Col} A)$ is the number of pivot columns and $\dim(\operatorname{Nul} A)$ is the number of non-pivot columns. This simple observation applies to any matrix, leading to the following theorem.

Theorem 26 (Rank theorem)**.** *Let A be an $m \times n$ matrix. Then*

$$\operatorname{rank}(A) + \dim(\operatorname{Nul} A) = n.$$

Example 13.3.3. *If A is a 6×8 matrix, what is the smallest possible dimension of* $\operatorname{Nul} A$*?*

Solution. We know that $\dim(\operatorname{Nul} A) = 8 - \operatorname{rank} A$. Since the number of pivot columns is at most 6, the rank of A is at most 6. Thus, $\dim(\operatorname{Nul} A) \geq 2$. □

Example 13.3.4. *Suppose a nonhomogeneous system of nine linear equations in ten unknowns has a solution for all possible constants on the right sides of the equations. Is it possible to find two nonzero solutions of the associated homogeneous system that are not multiples of each other?*

Solution. Let A be the coefficient matrix. Since $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^9$, A must have a pivot position in every row. Thus, the number of pivot columns is equal to the number of rows—nine. It follows that $\operatorname{rank} A = 9$, whence $\dim(\operatorname{Nul} A) = 10 - \operatorname{rank} A = 1$. On the other hand, two non-proportional solutions of $A\mathbf{x} = \mathbf{0}$ give a two-vector linearly independent set in $\operatorname{Nul} A$, which can only happen if $\dim(\operatorname{Nul} A) \geq 2$. Since we know that $\dim(\operatorname{Nul} A) = 1$, it follows that $A\mathbf{x} = \mathbf{0}$ can't have two non-proportional nonzero solutions. □

13.4. **The Invertible Matrix Theorem revisited.** Finally, the notions of rank and dimension allow us to add even more criteria for invertibility to IMT.

Theorem 12 (Invertible Matrix Theorem). *Let A be an $n \times n$ matrix. Then the following conditions are equivalent:*

1. *A is invertible.*
2. *A is row equivalent to I_n.*
3. *A has n pivot positions.*
4. *The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *The columns of A are linearly independent.*
6. *The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.*
7. *The equation $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^n$.*
8. *The columns of A span $\mathbb{R}^n$.*
9. *The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is onto.*
10. *There is a matrix C such that $CA = I_n$.*
11. *There is a matrix D such that $AD = I_n$.*
12. *A^t is invertible.*
13. *The columns of A form a basis for $\mathbb{R}^n$.*
14. $\operatorname{Col} A = \mathbb{R}^n$.
16. $\operatorname{rank} A = n$.
17. $\operatorname{Nul} A = \{\mathbf{0}\}$.
18. $\dim(\operatorname{Nul} A) = 0$.
19. $\det A \neq 0$.

14.1. **The change-of-coordinates matrix.** Our next goal is to understand the relation between the coordinates relative to different bases. We will be interested in the following situation. Suppose that V is an n-dimensional vector space and $\mathcal{A}$ and $\mathcal{B}$ are two bases for V. Then for any given vector $\mathbf{v} \in V$ we can compute its coordinate vectors relative to $\mathcal{A}$ and $\mathcal{B}$: $[\mathbf{v}]_{\mathcal{A}}$ and $[\mathbf{v}]_{\mathcal{B}}$. The question at hand is: how are $[\mathbf{v}]_{\mathcal{A}}$ and $[\mathbf{v}]_{\mathcal{B}}$ related? Let's look at an example.

Example 14.1.1. *Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be two bases for $\mathbb{R}^2$, such that*

$$\mathbf{b}_1 = 2\mathbf{a}_1 - \mathbf{a}_2 \qquad \textit{and} \qquad \mathbf{b}_2 = 3\mathbf{a}_1 - 2\mathbf{a}_2. \tag{14.1}$$

If $\mathbf{v} = 2\mathbf{b}_1 + 3\mathbf{b}_2$, find the coordinates of $\mathbf{v}$ with respect to $\mathcal{A}$ and $\mathcal{B}$. That is, find $[\mathbf{v}]_{\mathcal{A}}$ and $[\mathbf{v}]_{\mathcal{B}}$.

Solution. It's easy to find the $\mathcal{B}$-coordinates: by the very definition of "coordinates" we have $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. In order to find the $\mathcal{A}$-coordinates, we first express $\mathbf{v}$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. We have

$$\begin{aligned} \mathbf{v} &= 2\mathbf{b}_1 + 3\mathbf{b}_2 \\ &= 2(2\mathbf{a}_1 - \mathbf{a}_2) + 3(3\mathbf{a}_1 - 2\mathbf{a}_2) && \text{by (14.1)} \\ &= 13\mathbf{a}_1 - 8\mathbf{a}_2 && \text{by Properties 3.1.1.} \end{aligned}$$

We can now find the $\mathcal{A}$-coordinates of $\mathbf{v}$: $[\mathbf{v}]_{\mathcal{A}} = \begin{bmatrix} 13 \\ -8 \end{bmatrix}$. □

The above solution is pretty generic-looking, so let's try to uncover the general principle behind it. Clearly, the key to this solution were equations (14.1). Note that these can also be stated in the form

$$[\mathbf{b}_1]_{\mathcal{A}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \text{and} \qquad [\mathbf{b}_2]_{\mathcal{A}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}. \tag{14.2}$$

Next, we recall from §12.4 that the $\mathcal{A}$-coordinate mapping is "linear". In particular, we have

$$[2\mathbf{b}_1 + 3\mathbf{b}_2]_{\mathcal{A}} = 2[\mathbf{b}_1]_{\mathcal{A}} + 3[\mathbf{b}_2]_{\mathcal{A}} = 2\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 13 \\ -8 \end{bmatrix}.$$

Finally, we note that

$$2[\mathbf{b}_1]_{\mathcal{A}} + 3[\mathbf{b}_2]_{\mathcal{A}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{A}} & [\mathbf{b}_2]_{\mathcal{A}} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{A}} & [\mathbf{b}_2]_{\mathcal{A}} \end{bmatrix} [\mathbf{v}]_{\mathcal{B}},$$

that is,

$$[\mathbf{v}]_{\mathcal{A}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{A}} & [\mathbf{b}_2]_{\mathcal{A}} \end{bmatrix} [\mathbf{v}]_{\mathcal{B}}.$$

This identity is a special case of the following theorem.

Theorem 27. *Let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ and $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be two bases for a vector space V. Then there is a unique $n \times n$ matrix P such that for any vector $\mathbf{v} \in V$ we have*

$$[\mathbf{v}]_{\mathcal{A}} = P[\mathbf{v}]_{\mathcal{B}}. \tag{14.3}$$

Furthermore, the columns of P are the $\mathcal{A}$-coordinate vectors of the vectors in $\mathcal{B}$, that is,

$$P = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{A}} \, [\mathbf{b}_2]_{\mathcal{A}} \cdots [\mathbf{b}_n]_{\mathcal{A}} \end{bmatrix}.$$

The matrix P constructed in the theorem is called the *change-of-coordinates matrix* from $\mathcal{B}$ to $\mathcal{A}$. The textbook uses the notation $\underset{\mathcal{A}\leftarrow\mathcal{B}}{P}$, but I will refrain from using that or any other "fancy" notation.

Remark 14.1.2. Since the columns of the matrix P in Theorem 27 form a basis for $\mathbb{R}^n$, the IMT implies that P is invertible. Thus, we have

$$P^{-1}[\mathbf{v}]_{\mathcal{A}} = P^{-1}\big(P[\mathbf{v}]_{\mathcal{B}}\big) = \big(P^{-1}P\big)[\mathbf{v}]_{\mathcal{B}} = I_n[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{B}}.$$

That is, the matrix $Q = P^{-1}$ has the property that

$$[\mathbf{v}]_{\mathcal{B}} = Q[\mathbf{v}]_{\mathcal{A}} \qquad \text{for all } \mathbf{v} \in V,$$

meaning that Q is the change-of-coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$. This is worth stating explicitly.

Fact 14.1.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be bases for a vector space V. If P is the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{A}$, then P^{-1} is the change-of-coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$.*

Example 14.1.4. *Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be two bases for a three-dimensional vector space V, and suppose that*

$$\mathbf{b}_1 = 2\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3, \qquad \mathbf{b}_2 = 3\mathbf{a}_2 + \mathbf{a}_3, \qquad \mathbf{b}_3 = -3\mathbf{a}_1 + 2\mathbf{a}_3. \tag{14.4}$$

(a) *Compute the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{A}$ and $[\mathbf{v}]_{\mathcal{A}}$, where $\mathbf{v} = \mathbf{b}_1 - 2\mathbf{b}_2 + 2\mathbf{b}_3$.*
(b) *Compute the change-of-coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$ and $[\mathbf{v}]_{\mathcal{B}}$, where $\mathbf{v} = \mathbf{a}_1 + \mathbf{a}_2 - 3\mathbf{a}_3$.*

Solution. (a) We have

$$[\mathbf{b}_1]_{\mathcal{A}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \qquad [\mathbf{b}_2]_{\mathcal{A}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \qquad [\mathbf{b}_2]_{\mathcal{A}} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix},$$

so the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{A}$ is

$$P = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

The $\mathcal{A}$-coordinates of $\mathbf{v}$ are

$$[\mathbf{v}]_{\mathcal{A}} = P[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}.$$

(b) By the above fact,

$$[\mathbf{v}]_{\mathcal{B}} = P^{-1}[\mathbf{v}]_{\mathcal{A}}.$$

Since

$$P^{-1} = \begin{bmatrix} 1/4 & -1/8 & 3/8 \\ 1/12 & 7/24 & 1/8 \\ -1/6 & -1/12 & 1/4 \end{bmatrix},$$

we get

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1/4 & -1/8 & 3/8 \\ 1/12 & 7/24 & 1/8 \\ -1/6 & -1/12 & 1/4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}.$$

□

14.2. **Further examples and applications.**

Example 14.2.1. *The set* $\mathcal{B} = \{-2, t+1, t^2+t\}$ *is a basis for* $\mathbb{P}_2$. *Find the change-of-coordinates matrix from* $\mathcal{B}$ *to the standard basis* $\mathcal{C} = \{1, t, t^2\}$. *Then compute the* $\mathcal{B}$*-coordinates of* $\mathbf{f}(t) = t^2$ *and* $\mathbf{g}(t) = 2t^2 - t + 4$.

Solution. We write

$$\mathbf{b}_1(t) = -2, \qquad \mathbf{b}_2(t) = t+1, \qquad \mathbf{b}_3(t) = t^2 + t.$$

Then

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \qquad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad [\mathbf{b}_3]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

so the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ is

$$P = \begin{bmatrix} -2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$ is

$$P^{-1} = \begin{bmatrix} -0.5 & 0.5 & -0.5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using the latter matrix, we find that

$$[\mathbf{f}]_{\mathcal{B}} = P^{-1}[\mathbf{f}]_{\mathcal{C}} = \begin{bmatrix} -0.5 & 0.5 & -0.5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -1 \\ 1 \end{bmatrix}$$

and

$$[\mathbf{g}]_{\mathcal{B}} = P^{-1}[\mathbf{g}]_{\mathcal{C}} = \begin{bmatrix} -0.5 & 0.5 & -0.5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3.5 \\ -3 \\ 2 \end{bmatrix}.$$

□

In Example 14.1.4 the $\mathcal{A}$-coordinates of the vectors in $\mathcal{B}$ were essentially given to us via (14.4), and in Example 14.2.1 we could compute $\mathcal{C}$-coordinates of the vectors in $\mathcal{B}$, because $\mathcal{C}$ was the standard basis. How to proceed when we have neither data like (14.4), nor a convenient basis to work with? When we are dealing with vectors in $\mathbb{R}^n$, we can argue as in the following example.

Example 14.2.2. *The sets*

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} \qquad \textit{and} \qquad \mathcal{C} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right\}$$

are bases for $\mathbb{R}^2$. *Find the change-of-coordinates matrix from* $\mathcal{B}$ *to* $\mathcal{C}$.

Solution. To find the change-of-coordinates matrix P, we need to compute the $\mathcal{C}$-coordinates of the vectors in $\mathcal{B}$, that is, we need to solve the systems

$$C\mathbf{x} = \mathbf{b}_1, \qquad C\mathbf{x} = \mathbf{b}_2,$$

where $\mathbf{b}_1, \mathbf{b}_2$ are the vectors in $\mathcal{B}$ and

$$C = \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}.$$

As in Example 9.2.2, we can save some work by row-reducing the following 4×2 matrix:

$$\begin{bmatrix} 3 & -2 & 1 & 2 \\ 1 & -1 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 & 3 \\ 3 & -2 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 & 3 \\ 0 & 1 & -8 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & -4 \\ 0 & 1 & -8 & -7 \end{bmatrix}.$$

Thus,

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} -5 \\ -8 \end{bmatrix}, \qquad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}, \qquad P = \begin{bmatrix} -5 & -4 \\ -8 & -7 \end{bmatrix}.$$

□

Remark 14.2.3. In general, to find the change-of-coordinates matrix from a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ for $\mathbb{R}^n$ to a basis $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$, we row reduce the $n \times 2n$ matrix $[\mathbf{c}_1 \ \cdots \ \mathbf{c}_n \ \mathbf{b}_1 \ \cdots \ \mathbf{b}_n]$ to a matrix of the form $[I_n \ P]$. The matrix P is the change-of-coordinates matrix.

Example 14.2.4. *The matrices*

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the space $\mathbb{M}_{2\times 2}$ of 2×2 real matrices. (This is the ***standard basis for*** *$\mathbb{M}_{2\times 2}$.)*

(a) *Show that the following matrices also form a basis for $\mathbb{M}_{2\times 2}$:*

$$M_1 = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} -1 & 0 \\ -3 & 4 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}.$$

(b) *Find the change-of-coordinates matrix from $\mathcal{B} = \{M_1, M_2, M_3, M_4\}$ to the standard basis $\mathcal{E}$.*

Solution. We will answer both questions simultaneously. We note that

$$M_1 = E_{11} - E_{12} + 2E_{21} + E_{22} \qquad \Rightarrow \qquad [M_1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}.$$

Similarly,

$$[M_2]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \qquad [M_3]_{\mathcal{E}} = \begin{bmatrix} -1 \\ 0 \\ -3 \\ 4 \end{bmatrix}, \qquad [M_4]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 3 \\ -1 \\ 2 \end{bmatrix}.$$

Therefore, if $\mathcal{B}$ is really a basis, the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis will be

$$P = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 2 & 0 & 3 \\ 2 & 3 & -3 & -1 \\ 1 & 1 & 4 & 2 \end{bmatrix}.$$

In order to check that $\mathcal{B}$ is a basis, we recall that the coordinate mapping is an isomorphism between $\mathbb{M}_{2\times 2}$ and $\mathbb{R}^4$. Thus, $\mathcal{B}$ is a basis for $\mathbb{M}_{2\times 2}$ if and only if the coordinate vectors form a basis for $\mathbb{R}^4$.

By the IMT, the latter is equivalent to the invertibility of the matrix P. We will check that P is invertible by showing that $\det P \neq 0$:

$$\begin{vmatrix} 1 & 0 & -1 & 2 \\ -1 & 2 & 0 & 3 \\ 2 & 3 & -3 & -1 \\ 1 & 1 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 2 \\ -1 & 2 & 0 & 3 \\ -1 & 3 & 0 & -7 \\ 5 & 1 & 0 & 10 \end{vmatrix} = (-1)(-1)^{1+3} \begin{vmatrix} -1 & 2 & 3 \\ -1 & 3 & -7 \\ 5 & 1 & 10 \end{vmatrix}$$

$$= - \begin{vmatrix} -1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & 11 & 25 \end{vmatrix} = -(-1)(-1)^{1+1} \begin{vmatrix} 1 & -10 \\ 11 & 25 \end{vmatrix} = 135 \neq 0.$$

□

15. Eigenvectors and Eigenvalues

15.1. **Definitions.** Let A be an $n \times n$ matrix. It defines a linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ from $\mathbb{R}^n$ into $\mathbb{R}^n$. Our goal in this lecture and the next will be to understand the vectors in $\mathbb{R}^n$, for which the action of this transformation is as simple as possible.

Definition 15.1.1. Let A be an $n \times n$ matrix. A scalar λ is called an *eigenvalue of* A if the equation $A\mathbf{x} = \lambda\mathbf{x}$ has a nontrivial solution; any such $\mathbf{x}$ is called an *eigenvector of* A *corresponding to* λ. The set of all eigenvectors corresponding to a given eigenvalue λ is called the *eigenspace of* A *corresponding to* λ.

In other words, if $\mathbf{x}$ is an eigenvector of A, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ just "stretches" $\mathbf{x}$ by the amount and in the direction determined by the respective eigenvalue λ. For example, if

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix},$$

we have $A\mathbf{x} = 2\mathbf{x}$, so 2 is an eigenvalue of A and $\mathbf{x}$ is an eigenvector corresponding to that eigenvalue.

Example 15.1.2. *Let*

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Determine whether $\mathbf{x}$ *and* $\mathbf{y}$ *are eigenvectors of A. If so, find their respective eigenvalues.*

Solution. We have

$$A\mathbf{x} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = -2\mathbf{x}, \qquad A\mathbf{y} = \begin{bmatrix} 10 \\ -12 \\ 5 \end{bmatrix} \neq \lambda\mathbf{x}.$$

Thus, $\mathbf{x}$ is an eigenvector for the eigenvalue $\lambda = -2$; $\mathbf{y}$ is not an eigenvector. □

Example 15.1.3. *Show that* 1 *is an eigenvalue of the matrix A in Example 15.1.2 and find the corresponding eigenspace.*

Solution. To show that 1 is an eigenvalue, we have to find a nontrivial solution of the equation

$$A\mathbf{x} = \mathbf{x} \qquad \Leftrightarrow \qquad (A - I)\mathbf{x} = 0.$$

We have

$$\begin{bmatrix} A - I & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 & 0 \\ -4 & -7 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 9 & 9 & 0 \\ 0 & -9 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that $(A - I)\mathbf{x} = \mathbf{0}$ does have nontrivial solutions, and thus, 1 is indeed an eigenvalue. The eigenspace is the solution set, described by

$$\begin{cases} x_1 = x_3 \\ x_2 = -x_3 \\ x_3 \text{ free} \end{cases} \qquad \mathbf{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

so the eigenspace is

$$\text{Span}\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}.$$

□

Example 15.1.4. *It is known that* 2 *is an eigenvalue of the matrix*

$$A = \begin{bmatrix}4 & -1 & 6\\2 & 1 & 6\\2 & -1 & 8\end{bmatrix}.$$

Find the corresponding eigenspace.

Solution. The eigenspace consists of the solutions of f the equation

$$A\mathbf{x} = 2\mathbf{x} \qquad \Leftrightarrow \qquad (A-2I)\mathbf{x} = 0.$$

We have

$$\begin{bmatrix}A-2I & \mathbf{0}\end{bmatrix} = \begin{bmatrix}2 & -1 & 6 & 0\\2 & -1 & 6 & 0\\2 & -1 & 6 & 0\end{bmatrix} \sim \begin{bmatrix}2 & -1 & 6 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -0.5 & 3 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix},$$

so the solution set is described by

$$\begin{cases}x_1 = 0.5x_2 - 3x_3\\x_2, x_3 \text{ free}\end{cases} \qquad \mathbf{x} = x_2\begin{bmatrix}0.5\\1\\0\end{bmatrix} + x_3\begin{bmatrix}-3\\0\\1\end{bmatrix}.$$

The eigenspace is

$$\text{Span}\left\{\begin{bmatrix}0.5\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\end{bmatrix}\right\}.$$

□

15.2. **The characteristic equation.** It should be clear that the above approach will lead to a description of any eigenspace corresponding to a **known** eigenvalue. It is more difficult to compute the actual eigenvalues. Observe that λ is an eigenvalue if and only if

$$(A-\lambda I)\mathbf{x} = \mathbf{0} \tag{15.1}$$

has a nontrivial solution. By the Invertible Matrix Theorem, this is equivalent to

$$\det(A-\lambda I) = 0. \tag{15.2}$$

For a fixed matrix A, (15.2) is an equation for λ called the *characteristic equation of* A.

Example 15.2.1. *Find the eigenvalues of*

$$A = \begin{bmatrix}2 & 3\\3 & -6\end{bmatrix}.$$

Solution. We have

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = (2-\lambda)(-6-\lambda) - 9 = \lambda^2 + 4\lambda - 21,$$

so the roots of the characteristic equation are −7 and 3. These are the eigenvalues. □

In general, the characteristic equation of an $n \times n$ matrix is a polynomial equation of the form

$$(-1)^n \lambda^n + \cdots = 0.$$

(the polynomial $\det(A - \lambda I)$ is called the *characteristic polynomial of* A). You probably know from algebra that a polynomial equation of degree n has exactly n (complex) roots, counting multiplicities. Unfortunately, when $n > 4$ there is no general formula that can be used to compute those roots.[2]

[2]In case you need a quick review of algebraic polynomials and such, you can check the brief handout posted on the website.

16. Diagonalization

16.1. Diagonal and diagonalizable matrices.

Definition 16.1.1. An $n \times n$ matrix D is called *diagonal* if all its off-diagonal entries are zeros, that is, if D is of the form

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}.$$

A matrix A is called *diagonalizable* if it is *similar* to a diagonal matrix, that is, if there are an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Our goal in this lecture is to learn how to determine whether a given matrix A is diagonalizable, and when that is possible how to compute the matrices P and D. Of course, you might ask what is the benefit of that? Here is an example:

Example 16.1.2. *Compute D^{10} and A^{10}, where*

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 4 & 3 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}^{-1}.$$

Solution. We have

$$D^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \qquad D^3 = \begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix}, \qquad D^4 = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}, \qquad \ldots, \qquad D^{10} = \begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix}.$$

It should be clear by the third or fourth matrix in this series that in general we have

$$D^k = \begin{bmatrix} 1 & 0 \\ 0 & (-2)^k \end{bmatrix}.$$

On the other hand,

$$A^2 = \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix}, \qquad A^3 = \begin{bmatrix} 10 & 9 \\ -18 & -17 \end{bmatrix}, \qquad A^4 = \begin{bmatrix} -14 & -15 \\ 30 & 31 \end{bmatrix}, \qquad \ldots.$$

There may be a pattern (in fact, there is), but it is definitely much harder to see it. On the other hand,

$$A^{10} = \left(PDP^{-1}\right)^{10} = \underbrace{PDP^{-1}PDP^{-1}\cdots PDP^{-1}}_{10\text{ times}} = PDI_2DI_2\cdots I_2DP^{-1} = PD^{10}P^{-1}$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1022 & -1023 \\ 2046 & 2047 \end{bmatrix}.$$

□

16.2. The Diagonalization Theorem.

The answer to the question whether a given matrix is diagonalizable is given by the following theorem.

Theorem 28 (Diagonalization theorem)**.** *An $n \times n$ matrix is diagonalizable if and only if A has n linearly independent eigenvectors. In fact, $A = PDP^{-1}$ with D diagonal if and only if the diagonal entries of D are eigenvalues of A and the columns of P are eigenvectors of A.*

Idea of proof. We illustrate the idea of the proof in the 3×3 case. Suppose that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} \mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}, \qquad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

If $A = PDP^{-1}$, we have $AP = PD$. On the other hand,

$$PD = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 v_{11} & \lambda_2 v_{12} & \lambda_3 v_{13} \\ \lambda_1 v_{21} & \lambda_2 v_{22} & \lambda_3 v_{23} \\ \lambda_1 v_{31} & \lambda_2 v_{32} & \lambda_3 v_{33} \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{v}_1 \ \lambda_2 \mathbf{v}_2 \ \lambda_3 \mathbf{v}_3 \end{bmatrix},$$

and by the definition of matrix multiplication,

$$AP = \begin{bmatrix} A\mathbf{v}_1 \ A\mathbf{v}_2 \ A\mathbf{v}_3 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} A\mathbf{v}_1 \ A\mathbf{v}_2 \ A\mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{v}_1 \ \lambda_2 \mathbf{v}_2 \ \lambda_3 \mathbf{v}_3 \end{bmatrix} \qquad \Rightarrow \qquad A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \quad A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2, \quad A\mathbf{v}_3 = \lambda_3 \mathbf{v}_3.$$

In other words: the first column of P is an eigenvector of A whose respective eigenvalue is the first diagonal entry of D; the second column of P is an eigenvector of A whose respective eigenvalue is the second diagonal entry of D; and the third column of P is an eigenvector of A whose respective eigenvalue is the third diagonal entry of D. □

16.3. **Diagonalization of matrices.** Based on the Diagonalization theorem, we can form the following strategy for diagonalization of a matrix A:

1. Find the eigenvalues of A.
2. For each eigenvalue, find a basis for the corresponding eigenspace.
3. If Step 2 results in fewer than n eigenvectors, the matrix is not diagonalizable. If Step 2 results in exactly n eigenvectors, say $\mathbf{v}_1, \dots, \mathbf{v}_n$, then $A = PDP^{-1}$, where

$$P = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}, \qquad D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

and $\mathbf{v}_1, \dots, \mathbf{v}_n$ and $\lambda_1, \dots, \lambda_n$ are ordered so that $\mathbf{v}_j$ is an eigenvector of A with eigenvalue λ_j.

Example 16.3.1. *If possible, diagonalize the matrix*

$$A = \begin{bmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. Step 1. The characteristic polynomial of A is

$$\begin{vmatrix} 3-\lambda & -4 & 0 \\ 2 & -3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 3-\lambda & -4 \\ 2 & -3-\lambda \end{vmatrix}$$
$$= (1-\lambda)\big[(3-\lambda)(-3-\lambda) + 8\big]$$
$$= (1-\lambda)(\lambda^2 - 1) = -(\lambda-1)^2(\lambda+1).$$

Thus, the characteristic equation is

$$-(\lambda - 1)^2(\lambda + 1) = 0$$

and the eigenvalues (listed according to their multiplicities) are $1, 1, -1$.

Step 2. First, we find a basis for $\text{Nul}(A - I)$. We have

$$\begin{bmatrix} A - I & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 2 & -4 & 0 & 0 \\ 2 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so the general solution of $(A - I)\mathbf{x} = \mathbf{0}$ is

$$\begin{cases} x_1 = 2x_2 \\ x_2, x_3 \text{ free} \end{cases}$$

The vector form of the solution is

$$\mathbf{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

so a basis for the eigenspace for $\lambda = 1$ is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Next, we find a basis for the eigenspace for $\lambda = -1$. We have

$$\begin{bmatrix} A + I & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 4 & -4 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so the general solution of $(A + I)\mathbf{x} = \mathbf{0}$ is

$$\begin{cases} x_1 = x_2 \\ x_3 = 0 \\ x_2 \text{ free} \end{cases}$$

The vector form of the solution is

$$\mathbf{x} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

so a basis for the eigenspace for $\lambda = -1$ is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Step 3. Since Step 2 resulted in three vectors, A is diagonalizable. The two matrices P and D in the diagonalization are

$$P = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Check. To check our work, we compute AP and PD. We must obtain the same result. We have

$$AP = \begin{bmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$PD = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

so $AP = PD$ indeed. □

In the above solution, we made silent use of the following theorem:

Theorem 29. *Eigenvectors of A corresponding to different eigenvalues are linearly independent.*

This is why we did not check that the set

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is linearly independent (technically, we should have): it was obtained by combining bases for two different eigenspaces.

Another useful fact to remember is the following:

Theorem 30. *The dimension of the eigenspace corresponding to an eigenvalue λ does not exceed the multiplicity of λ as a root of the characteristic equation.*

This has two important corollaries:

Theorem 31. *If A has n distinct eigenvalues, then A is diagonalizable.*

The second corollary of Theorem 30 is more of a piece of practical advice: in diagonalization always start with the eigenspaces for multiple eigenvalues. If one of those turns out to have dimension strictly less than the multiplicity of the respective eigenvalue, we automatically know that the matrix is NOT diagonalizable and we can stop. This is illustrated in the following example.

Example 16.3.2. *If possible, diagonalize the matrix*

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Solution. Step 1. The characteristic polynomial is

$$\begin{vmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -6-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} - 4\begin{vmatrix} -4 & -3 \\ 3 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} -4 & -6-\lambda \\ 3 & 3 \end{vmatrix}$$

$$= (2-\lambda)\big[(-6-\lambda)(1-\lambda)+9\big] - 4\big(4\lambda-4+9\big) + 3\big(-12+3\lambda+18\big)$$

$$= -\lambda^3 - 3\lambda^2 + 4 = -(\lambda-1)(\lambda+2)^2.$$

Thus, the eigenvalues are $1, -2, -2$.

Step 2. Because -2 is a double eigenvalue, we first consider its eigenspace. We have

$$\big[A+2I\ \mathbf{0}\big] = \begin{bmatrix} 4 & 4 & 3 & 0 \\ -4 & -4 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 4 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 4 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, $\mathrm{Nul}(A+2I)$ is one-dimensional and it follows that A is not diagonalizable. □

17. Complex Eigenvalues

We know from the last lecture that if the characteristic polynomial of a matrix A has complex[3] eigenvalues, the matrix is not diagonalizable. This is because the dimension of an eigenspace does not exceed the multiplicity of the corresponding eigenvalue. If there are complex eigenvalues, the number of real eigenvalues (counted with multiplicities) will be strictly less than the size of the matrix. The purpose of this lecture is to show how to "almost diagonalize" a matrix with comlex eigenvalues.

17.1. Linear algebra over the complex numbers.

So far in these lectures, we have dealt strictly with real numbers, or as is common to say, *we have done linear algebra over the real numbers*. However, we could have easily worked over the complex numbers $\mathbb{C}$ and almost nothing would have changed (the only thing that *would* have changed is the geometric interpretations of various concepts). Working over the complex numbers has at least one great benefit: a polynomial of degree n has exactly n complex roots in $\mathbb{C}$, so the issue discussed in this lecture simply does not arise: if every complex eigenspace has the maximum possible dimension, the matrix is diagonalizable over $\mathbb{C}$.

Example 17.1.1. *Compute the complex eigenvalues and eigenvectors of the matrix* $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Solution. The characteristic polynomial of A is

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1,$$

so the complex eigenvalues are $i, -i$. Since

$$\begin{bmatrix} -i & -1 & 0 \\ 1 & -i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -i & 0 \\ -i & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the eigenspace for $\lambda = i$ contains the solutions $\mathbf{x}$ of

$$x_1 - ix_2 = 0 \qquad \Leftrightarrow \qquad \mathbf{x} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad \Leftrightarrow \qquad \mathbf{x} \in \text{Span}\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}.$$

Similarly, the eigenspace for $\lambda = -i$ is $\text{Span}\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$.

Notice that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i \begin{bmatrix} i \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -i \end{bmatrix} = (-i) \begin{bmatrix} -i \\ 1 \end{bmatrix},$$

so the vectors $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ are indeed eigenvectors for $\lambda = i$ and $\lambda = -i$. □

Example 17.1.2. *Diagonalize (over $\mathbb{C}$) the matrix A from Example 17.1.1.*

[3] See Appendix B of the textbook, if you feel that you need to "brush up" your knowledge of complex numbers.

Solution. We have $A = PDP^{-1}$, where

$$P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

□

17.2. **Complex eigenvalues and eigenvectors of real matrices.** Recall that if $z = x + iy$ is a complex number, the number $\bar{z} = x - iy$ is called *its complex conjugate* and the real numbers $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$ are called the *real part of* x and the *imaginary part of* x. We can extend these definitions to vectors in $\mathbb{C}^n$ (i.e., n-dimensional vectors with complex entries):

$$\text{for } \mathbf{z} = \begin{bmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ \vdots \\ x_n + iy_n \end{bmatrix}: \qquad \bar{\mathbf{z}} = \begin{bmatrix} x_1 - iy_1 \\ x_2 - iy_2 \\ \vdots \\ x_n - iy_n \end{bmatrix}, \quad \operatorname{Re} \mathbf{z} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \operatorname{Im} \mathbf{z} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

Our technique for "almost diagonalizing" real matrices having complex eigenvalues uses the following fact.

Fact 17.2.1. *Let A be an $n \times n$ matrix with real entries. If λ is a (complex) eigenvalue of A, then so is $\bar{\lambda}$. Moreover, if λ and $\bar{\lambda}$ are a pair of complex conjugate eigenvalues of A and if $\mathbf{z}$ is an eigenvector for λ, then $\bar{z}$ is an eigenvector for $\bar{\lambda}$.*

Notice that this is consistent with our findings in Example 17.1.1: we found the complex conjugate eigenvalues $\pm i$, whose respective eigenvectors were also complex conjugate:

$$\overline{\begin{bmatrix} i \\ 1 \end{bmatrix}} = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

In order to explain clearly our approach towards diagonalization of real matrices with complex eigenvalues, we require an example of diagonalization of such a matrix over $\mathbb{C}$, where the size of the matrix is at least 4×4 or 5×5.

Example 17.2.2. *If possible, diagonalize (over $\mathbb{C}$) the matrix*

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 2 & -1 \\ 1 & 3 & 0 & 1 & 2 \end{bmatrix}.$$

Solution. Step 1. The characteristic polynomial of A is

$$\begin{vmatrix} 2-\lambda & 0 & 0 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 & 0 \\ 0 & 1 & 1-\lambda & 0 & 0 \\ 2 & 0 & 1 & 2-\lambda & -1 \\ 1 & 3 & 0 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 0 & 0 & 0 \\ 1 & 1-\lambda & 0 & 0 \\ 0 & 1 & 2-\lambda & -1 \\ 3 & 0 & 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & -1 \\ 0 & 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2(1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2(1-\lambda)\left[(2-\lambda)^2+1\right],$$

so the eigenvalues of A are $1, 2, 2$, and the two roots of $(2-\lambda)^2+1=0$:

$$(2-\lambda)^2 = -1 \qquad \Leftrightarrow \qquad 2-\lambda = \pm i \qquad \Leftrightarrow \qquad \lambda = 2 \pm i.$$

Step 2. $\lambda = 2$: We have

$$\begin{bmatrix} A-2I & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & -1 & 0 \\ 1 & 3 & 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -6 & 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -5 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.6 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0.6 & 0.2 & 0 \\ 0 & 0 & 1 & 0.6 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -0.8 & -0.6 & 0 \\ 0 & 1 & 0 & 0.6 & 0.2 & 0 \\ 0 & 0 & 1 & 0.6 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so the parametric vector form of the solution of $(A-2I)\mathbf{x} = \mathbf{0}$ and a basis for the eigenspace for $\lambda = 2$ are:

$$\mathbf{x} = \begin{bmatrix} 0.8x_4 + 0.6x_5 \\ -0.6x_4 - 0.2x_5 \\ -0.6x_4 - 0.2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_4\begin{bmatrix} 0.8 \\ -0.6 \\ -0.6 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} 0.6 \\ -0.2 \\ -0.2 \\ 0 \\ 1 \end{bmatrix}, \qquad \left\{\begin{bmatrix} 4 \\ -3 \\ -3 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \\ 0 \\ 5 \end{bmatrix}\right\}.$$

$\lambda = 1$: We have

$$[A - I \;\; \mathbf{0}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & -1 & 0 \\ 1 & 3 & 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 3 & 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so the parametric vector form of the solution of $(A - I)\mathbf{x} = \mathbf{0}$ and a basis for the eigenspace for $\lambda = 1$ are:

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 2x_5 \\ -x_5 \\ x_5 \end{bmatrix} = x_5 \begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \qquad \left\{ \begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

$\lambda = 2 - i$: We have

$$[A - (2-i)I \;\; \mathbf{0}] = \begin{bmatrix} i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 1 & -1+i & 0 & 0 & 0 \\ 2 & 0 & 1 & i & -1 & 0 \\ 1 & 3 & 0 & 1 & i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 1 & -1+i & 0 & 0 & 0 \\ 2 & 0 & 1 & i & -1 & 0 \\ 1 & 3 & 0 & 1 & i & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 1 & -1+i & 0 & 0 & 0 \\ 0 & 0 & 1 & i & -1 & 0 \\ 0 & 3 & 0 & 1 & i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1+i & 0 & 0 & 0 \\ 0 & 0 & 1 & i & -1 & 0 \\ 0 & 3 & 0 & 1 & i & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1+i & 0 & 0 & 0 \\ 0 & 0 & 1 & i & -1 & 0 \\ 0 & 0 & 0 & 1 & i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & i & -1 & 0 \\ 0 & 0 & 0 & 1 & i & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & -1 & 0 \\ 0 & 0 & 0 & 1 & i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & i & 0 \\ 0 & 0 & 0 & i & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so the parametric vector form of the solution of $(A - (2 - i)I)\mathbf{x} = \mathbf{0}$ and a basis for the eigenspace for $\lambda = 2 - i$ are:

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -ix_5 \\ x_5 \end{bmatrix} = x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -i \\ 1 \end{bmatrix}, \qquad \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ -i \\ 1 \end{bmatrix} \right\}.$$

$\lambda = 2 + i$: By Fact 17.2.1, a basis for this eigenspace is

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ i \\ 1 \end{bmatrix} \right\}.$$

Step 3. A is diagonalizable and $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 4 & 3 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 & 0 \\ -3 & -1 & 2 & 0 & 0 \\ 5 & 0 & -1 & -i & i \\ 0 & 5 & 1 & 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2-i & 0 \\ 0 & 0 & 0 & 0 & 2+i \end{bmatrix}.$$

□

Now, suppose that A is a matrix that is diagonalizable over $\mathbb{C}$, but not over $\mathbb{R}$. Is it possible to represent A as $A = PCP^{-1}$, where C is a real matrix that may not be diagonal but is still pretty simple to work with? Not only is the answer to this question "yes", but the matrices P and C in this representation can be easily derived from the matrices P and D in the diagonalization of A over the complex numbers. Since A is diagonalizable over $\mathbb{C}$, we can find (working as in the above example) real eigenvalues $\lambda_1, \ldots, \lambda_r$ and pairs of complex conjugate eigenvalues $\mu_1, \bar{\mu}_1, \ldots, \mu_s, \bar{\mu}_s$, altogether n of them. We can also find r linearly independent **real** eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_r$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_r$ and s linearly independent pairs of complex eigenvectors $\mathbf{z}_1, \bar{\mathbf{z}}_1, \ldots, \mathbf{z}_s, \bar{\mathbf{z}}_s$ corresponding to the pairs of complex eigenvalues $\mu_1, \bar{\mu}_1, \ldots, \mu_s, \bar{\mu}_s$. For instance, in Example 17.2.2, we have $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = 1$, $\mu_1 = 2 - i$, $\bar{\mu}_1 = 2 + i$, and

$$\mathbf{x}_1 = \begin{bmatrix} 4 \\ -3 \\ -3 \\ 5 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 0 \\ 5 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -i \\ 1 \end{bmatrix}, \quad \bar{\mathbf{z}}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ i \\ 1 \end{bmatrix}.$$

Then the matrix P in the desired representation has columns $\mathbf{x}_1, \ldots, \mathbf{x}_r, \operatorname{Re}\mathbf{z}_1, \operatorname{Im}\mathbf{z}_1, \ldots, \operatorname{Re}\mathbf{z}_s, \operatorname{Im}\mathbf{z}_s$; for the above example, this means that

$$P = \begin{bmatrix} 4 & 3 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 & 0 \\ -3 & -1 & 2 & 0 & 0 \\ 5 & 0 & -1 & 0 & -1 \\ 0 & 5 & 1 & 1 & 0 \end{bmatrix}.$$

The matrix C is a block matrix of the form (a *block-diagonal matrix*)

$$\begin{bmatrix} \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & \lambda_r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & C_1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & C_s \end{bmatrix},$$

where $C_1, \ldots, C_s$ are 2×2 blocks of the form

$$C_j = \begin{bmatrix} \operatorname{Re}\mu_j & \operatorname{Im}\mu_j \\ -\operatorname{Im}\mu_j & \operatorname{Re}\mu_j \end{bmatrix}.$$

In particular, for the matrix A in Example 17.2.2, we have

$$C = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

Example 17.2.3. *Let A be a* 7×7 *real matrix with eigenvalues* $1, 1, -2, 1+i, 1+i, 1-i, 1-i$, *and respective eigenvectors*

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 0 \\ -2 \\ -1 \\ 0 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1+i \\ 0 \\ 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 2i \\ 3-i \\ 0 \\ 0 \\ -5 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1-i \\ 0 \\ 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ -2i \\ 3+i \\ 0 \\ 0 \\ -5 \end{bmatrix}.$$

Represent A in the form $A = PCP^{-1}$, *where P and C are real matrices and C is block-diagonal.*

Answer.

$$P = \begin{bmatrix} 1 & -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & -2 & 1 & 1 & 0 & 3 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 4 & -1 & 0 & 0 & 0 \\ 3 & -3 & 2 & 2 & 0 & -5 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

□

18. Eigenvalues, Eigenvectors, and Linear Transformations

18.1. **Linear transformations between vector spaces.**

Definition 18.1.1. A *linear transformation* from a vector space V to a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W so that

1. $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$;
2. $T(c\mathbf{x}) = cT(\mathbf{x})$ for all $\mathbf{x} \in V$ and all scalars c.

Example 18.1.2. *Let A be an $m \times n$ matrix and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the matrix transformation given by $T(\mathbf{x}) = A\mathbf{x}$. By Fact 7.3.3, T is a linear transformation from the vector space $\mathbb{R}^n$ to the vector space $\mathbb{R}^m$.*

Example 18.1.3. *Let D be the transformation from $\mathbb{P}_n$ to $\mathbb{P}_n$ given by the differentiation operation:*

$$[D(\mathbf{f})](t) = \mathbf{f}'(t).$$

D is a linear transformation from $\mathbb{P}_n$ to $\mathbb{P}_n$.

Solution. We need to check that

$$D(\mathbf{f} + \mathbf{g}) = D(\mathbf{f}) + D(\mathbf{g}) \qquad \text{and} \qquad D(c\mathbf{f}) = cD(\mathbf{f}),$$

whenever $\mathbf{f}, \mathbf{g}$ are in $\mathbb{P}_n$ and $c \in \mathbb{R}$. But the first of these two properties says simply that

$$(\mathbf{f} + \mathbf{g})' = \mathbf{f}' + \mathbf{g}',$$

which is a known property of the derivative (not only for polynomials, but for any two functions). Similarly, the second required property holds because for any function $\mathbf{f}(t)$ and any constant c, $(c\mathbf{f})' = c\mathbf{f}'$. □

Example 18.1.4. *Let I be the transformation from $\mathbb{P}_n$ to $\mathbb{P}_{n+1}$ given by*

$$[I(\mathbf{f})](t) = \int_0^t \mathbf{f}(x)\,dx.$$

Show that I is a linear transformation from $\mathbb{P}_n$ to $\mathbb{P}_{n+1}$.

Solution. Exercise. □

Example 18.1.5. *Let I be the transformation from $\mathbb{P}_n$ to $\mathbb{R}$ given by*

$$I(\mathbf{f}) = \int_0^1 \mathbf{f}(x)\,dx.$$

Show that I is a linear transformation from $\mathbb{P}_n$ to $\mathbb{R}$.

Solution. Exercise. □

18.2. **The matrix of a linear transformation.** Recall that in §7.4 we defined the standard matrix of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Now we want to do the same for linear transformations between any two vector spaces. Our main tool in this will be the notion of basis.

Suppose that T is a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Then we can pick a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ for V and a basis $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_m\}$ for W. Once we have fixed $\mathcal{B}$ and $\mathcal{C}$, we can compute the coordinate vectors $[\mathbf{x}]_{\mathcal{B}}$ and $[T(\mathbf{x})]_{\mathcal{C}}$ of any vector $\mathbf{x}$ in V relative to $\mathcal{B}$ and of its image $T(\mathbf{x})$ (which is in W) relative to $\mathcal{C}$. With this notation in hand, we can state the following fact.

Fact 18.2.1. *Assume the above notation. Then there is a unique $m \times n$ matrix M such that*

$$[T(\mathbf{x})]_{\mathcal{C}} = M[\mathbf{x}]_{\mathcal{B}} \qquad \text{for all } \mathbf{x} \in V.$$

Moreover, the matrix M is

$$M = \left[\, [T(\mathbf{b}_1)]_{\mathcal{C}}\, [T(\mathbf{b}_2)]_{\mathcal{C}} \cdots [T(\mathbf{b}_n)]_{\mathcal{C}} \,\right].$$

Definition 18.2.2. The matrix M above is called the *matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$.*

Example 18.2.3. *Let T be a linear transformation from $\mathbb{R}_n$ to $\mathbb{R}_m$, let $\mathcal{B}$ be the standard basis for $\mathbb{R}_n$, and let $\mathcal{C}$ be the standard basis for $\mathbb{R}_m$. Then the matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$ is simply the standard matrix defined in §7.4.*

Example 18.2.4. *Let $\mathcal{B}$ and $\mathcal{C}$ be two bases for the vector space V and let I be the identity transformation: $I(\mathbf{x}) = \mathbf{x}$. Then the matrix for I relative to the bases $\mathcal{B}$ and $\mathcal{C}$ is simply the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.*

Example 18.2.5. *Let D be a linear transformation from $\mathbb{P}_3$ to $\mathbb{P}_3$ defined in Example 18.1.3 and let both $\mathcal{B}$ and $\mathcal{C}$ be the standard basis $\{1, t, t^2, t^3\}$ for $\mathbb{P}_3$. Find the matrix M for D relative to the bases $\mathcal{B}$ and $\mathcal{C}$.*

Solution. First, we must find the images under D of the polynomials in $\mathcal{B}$. We have

$$D(1) = (1)' = 0, \qquad D(t) = (t)' = 1, \qquad D(t^2) = (t^2)' = 2t, \qquad D(t^3) = (t^3)' = 3t^2.$$

Next, we have to find the $\mathcal{C}$-coordinates of the outputs we just computed:

$$\left[0\right]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \left[1\right]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \left[2t\right]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \qquad \left[3t^2\right]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}.$$

These are the columns of the matrix M of D, that is,

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that if

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

is an arbitrary polynomial in $\mathbb{P}_3$, we have

$$\left[\mathbf{p}\right]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

and

$$M\left[\mathbf{p}\right]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{bmatrix} = \left[a_1 + 2a_2 t + 3a_3 t^2\right]_{\mathcal{C}} = \left[\mathbf{p}'\right]_{\mathcal{C}}.$$

□

Notice that in the last example we had $V = W = \mathbb{P}_3$ and $\mathcal{B} = \mathcal{C}$. In general, when the T is a linear transformation from V to itself (i.e., $W = V$) and we choose $\mathcal{B} = \mathcal{C}$, the matrix M relative to $\mathcal{B}$ and $\mathcal{B}$ is called the *$\mathcal{B}$-matrix of T*.

18.3. **Eigenvalues and eigenvectors of a linear transformation.** For the remainder of this lecture, we will concentrate on linear transformations from an n-dimensional vector space V into itself. Let $\mathcal{B}$ be a basis for V and let M be the $\mathcal{B}$-matrix of a linear transformation $T : V \to V$. The *eigenvalues of T* and the *eigenvectors of T* are then merely the eigenvalues and the eigenvectors of M. Similarly, we say that a transformation T is *diagonalizable* if its matrix M is diagonalizable.

Example 18.3.1. *Let D be a linear transformation from $\mathbb{P}_3$ to $\mathbb{P}_3$ considered in Example 18.2.5. Find its eigenvalues and a basis for each eigenspace of D. Is D diagonalizable?*

Solution. We already know the $\mathcal{B}$-matrix of D:

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since

$$\det(M - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 2 & 0 \\ 0 & 0 & -\lambda & 3 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = (-\lambda)^4 = \lambda^4,$$

T has a quadruple eigenvalue $\lambda = 0$. To find a basis for the respective eigenspace, we solve the homogeneous equation $M\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} M & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \\ x_1 \text{ free} \end{cases} \quad \Leftrightarrow \quad \mathbf{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

That is, the eigenspace for $\lambda = 0$ is one-dimensional and T is not diagonalizable. □

Remark 18.3.2. A remark is in order regarding our definition of eigenvalues and eigenvectors of a linear transformation T. We defined those in terms of the $\mathcal{B}$-matrix of T, which depends on the choice of the basis $\mathcal{B}$. Does that mean that the eigenvalues and the eigenvectors of T will change if we choose a different basis? The answer is "No." If we choose a different basis, say $\mathcal{C}$, we will obtain a different matrix in place of M, but that matrix will have the same eigenvalues and eigenvectors as M, because it turns out to be of the form PMP^{-1} for some invertible matrix P (in fact, P is the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$). We won't engage in further discussion how to justify these claims.

19.1. **Definitions.** If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the single entry of the product $\mathbf{x}^t\mathbf{y}$ is called the *inner product* (or the *dot product*) of $\mathbf{x}$ and $\mathbf{y}$ and is denoted $\mathbf{x} \cdot \mathbf{y}$, that is,

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Having defined the inner product of two vectors, we can define the *norm* (or *length*) of a vector $\mathbf{x}$:

$$||\mathbf{x}|| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

The *distance* between two vectors is simply the length of their difference:

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

For example, if

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{z} = \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

we have

$$\mathbf{x} \cdot \mathbf{y} = (1)(2) + (1)(1) = 3, \qquad \mathbf{y} \cdot \mathbf{z} = 0, \qquad ||\mathbf{x}|| = \sqrt{2}, \qquad \operatorname{dist}(\mathbf{x}, \mathbf{y}) = 1.$$

Proposition 19.1.1. *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ *and let* $c \in \mathbb{R}$. *Then:*

1. $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
2. $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}$
3. $(c\mathbf{x}) \cdot \mathbf{y} = c(\mathbf{x} \cdot \mathbf{y}) = \mathbf{x} \cdot (c\mathbf{y})$
4. $||\mathbf{x}|| \geq 0$*;* $||\mathbf{x}|| = 0 \implies \mathbf{x} = \mathbf{0}$
5. $||c\mathbf{x}|| = |c|||\mathbf{x}||$
6. $|\mathbf{x} \cdot \mathbf{y}| \leq ||\mathbf{x}||\,||\mathbf{y}||$ *(Cauchy–Schwarz inequality)*
7. $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$ *(triangle inequality)*

Remark 19.1.2. In the two- and three-dimensional cases, we can give an alternative definition of the dot product (one that you may be familiar with):

$$\mathbf{x} \cdot \mathbf{y} = ||\mathbf{x}||\,||\mathbf{y}||\, \cos\theta,$$

where θ is the angle between the arrows representing $\mathbf{x}$ and $\mathbf{y}$. This geometric definition turns out to be equivalent to the algebraic definition given above. Sometimes, this connection is used to define "angles" between vectors in more than three dimensions: for nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we define the *angle* θ *between* $\mathbf{x}$ *and* $\mathbf{y}$ via its cosine:

$$\theta = \arccos\left(\frac{\mathbf{x} \cdot \mathbf{y}}{||\mathbf{x}||\,||\mathbf{y}||}\right). \tag{19.1}$$

Note that this is well-defined because of the Cauchy–Schwarz inequality (Property 6) above), which implies that

$$-1 < \frac{\mathbf{x} \cdot \mathbf{y}}{||\mathbf{x}||\,||\mathbf{y}||} < 1.$$

If this inequality failed for some $\mathbf{x}$ and $\mathbf{y}$, the inverse cosine function in (19.1) would be undefined.

Remark 19.1.3. We say that a vector $\mathbf{u}$ is *unit* if its length is 1, that is, if $||\mathbf{u}|| = 1$. If $\mathbf{x} \neq 0$, we can always find a unit vector $\mathbf{u}$ that points in the same direction as $\mathbf{x}$. Indeed, if $\mathbf{x} \neq 0$, its norm $||\mathbf{x}||$ is a positive number and $\mathbf{u} = \frac{1}{||\mathbf{x}||}\mathbf{x}$ is a vector that points in the same direction as $\mathbf{x}$. Moreover, if we denote $\frac{1}{||\mathbf{x}||}$ by c, we have

$$||\mathbf{u}||^2 = (c\mathbf{x}) \cdot (c\mathbf{x}) = c^2(\mathbf{x} \cdot \mathbf{x}) = c^2||\mathbf{x}||^2 = 1.$$

The vector $\mathbf{u}$ is called the *normalization* of $\mathbf{x}$.

19.2. **Orthogonality.**

Definition 19.2.1. We say that $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* (or *perpendicular*), and write $\mathbf{x} \perp \mathbf{y}$, if $\mathbf{x} \cdot \mathbf{y} = 0$. We say that a vector $\mathbf{x}$ is *orthogonal to a set S in* $\mathbb{R}^n$, and write $\mathbf{x} \perp S$, if $\mathbf{x}$ is orthogonal to every vector in S.

Remark 19.2.2. The zero vector is orthogonal to every vector in $\mathbb{R}^n$; $\mathbf{0}$ is the only vector with this property, that is, if $\mathbf{x} \perp \mathbf{y}$ for all $\mathbf{y} \in \mathbb{R}^n$, then $\mathbf{x} = \mathbf{0}$.

The following theorem is an important fact about orthogonal vectors in $\mathbb{R}^n$. It is also well-known to you in the case $n = 2$.

Theorem 32 (Pythagorean Theorem)**.** *Two vectors* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *are orthogonal if and only if* $||\mathbf{x}+\mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$.

Proof. Using the properties of the inner product, we find

$$||\mathbf{x} + \mathbf{y}||^2 = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + 2(\mathbf{x} \cdot \mathbf{y}) + \mathbf{y} \cdot \mathbf{y} = ||\mathbf{x}||^2 + 2(\mathbf{x} \cdot \mathbf{y}) + ||\mathbf{y}||^2. \tag{19.2}$$

Thus, the identity $||\mathbf{x} + \mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$ holds if and only if the middle term on the right side of (19.2) vanishes. However,

$$2(\mathbf{x} \cdot \mathbf{y}) = 0 \qquad \Leftrightarrow \qquad \mathbf{x} \cdot \mathbf{y} = 0 \qquad \Leftrightarrow \qquad \mathbf{x} \perp \mathbf{y}.$$

□

Definition 19.2.3. Let H be a subspace of $\mathbb{R}^n$. The set of all vectors $\mathbf{x} \perp H$ is called the *orthogonal complement of H*; it is denoted $H^\perp$ (read "H perp").

The orthogonal complement of a subspace has the following useful properties.

Proposition 19.2.4. *Let H be a subspace of* $\mathbb{R}^n$. *Then:*

1. *A vector* $\mathbf{x}$ *is in $H^\perp$ if and only if* $\mathbf{x}$ *is orthogonal to all vectors in a set S that spans H.*
2. *$H^\perp$ is a subspace of* $\mathbb{R}^n$.

The following theorem uses the inner product to relate null and column spaces to each other. We present its proof in full, since it is a nice exercise in matrix algebra.

Theorem 33. *Let A be an $m \times n$ matrix. Then* $(\mathrm{Col}\, A)^\perp = \mathrm{Nul}(A^t)$.

Proof. Let

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

A vector $\mathbf{x}$ is in $(\operatorname{Col} A)^\perp$ if $\mathbf{x}$ is perpendicular to every column of A, that is, if

$$\mathbf{a}_1 \cdot \mathbf{x} = \mathbf{a}_2 \cdot \mathbf{x} = \cdots = \mathbf{a}_n \cdot \mathbf{x} = 0.$$

In other words, $\mathbf{x} \in (\operatorname{Col} A)^\perp$ if

$$\begin{cases} a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_m = 0 \\ a_{12}x_1 + a_{22}x_2 + \cdots + a_{m2}x_m = 0 \\ \qquad\qquad\vdots \\ a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{mn}x_m = 0 \end{cases}$$

or equivalently, if $\mathbf{x}$ is in the null space of the matrix

$$A^t = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

□

Definition 19.2.5. A set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is called *orthogonal* if every two distinct vectors in S are orthogonal, that is, if $\mathbf{v}_i \perp \mathbf{v}_j$ whenever $i \neq j$.

Example 19.2.6. *Show that the set $\mathcal{B} = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is orthogonal, where*

$$\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \qquad \textit{and} \qquad \mathbf{z} = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}.$$

Solution. Indeed, we have:

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= (1)(0) + (-2)(1) + (1)(2) = 0, \\ \mathbf{x} \cdot \mathbf{z} &= (1)(-5) + (-2)(-2) + (1)(1) = 0, \\ \mathbf{y} \cdot \mathbf{z} &= (0)(-5) + (1)(-2) + (2)(1) = 0. \end{aligned}$$

□

We are interested in orthogonal sets of vectors, because they prove to be extremely useful in the construction of "nice" bases for subspaces of $\mathbb{R}^n$. First, an orthogonal set is automatically linearly independent, as long as none of its vectors is the zero vector.

Theorem 34. *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ be an orthogonal set of nonzero vectors in $\mathbb{R}^n$. Then S is linearly independent.*

Proof. Suppose that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$

Taking the dot product of both sides of the equation with $\mathbf{v}_j$, we get

$$(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) \cdot \mathbf{v}_j = \mathbf{0} \cdot \mathbf{v}_j,$$

which reduces to

$$c_1(\mathbf{v}_1 \cdot \mathbf{v}_j) + c_2(\mathbf{v}_2 \cdot \mathbf{v}_j) + \cdots + c_p(\mathbf{v}_p \cdot \mathbf{v}_j) = 0.$$

Since $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ when $i \neq j$, the equation further reduces to

$$c_j(\mathbf{v}_j \cdot \mathbf{v}_j) = 0.$$

and since $\mathbf{v}_j \neq \mathbf{0}$, we may divide both sides by $\mathbf{v}_j \cdot \mathbf{v}_j$ to conclude that $c_j = 0$. Thus, the only solution to the original equation is the trivial solution, and S is linearly independent. □

We just saw that if a collection of nonzero vectors is orthogonal, then it is linearly independent. Therefore, it is a basis for its span. A basis for a subspace consisting of orthogonal vectors is called an *orthogonal basis*.

Example 19.2.7. *The set $\mathcal{B}$ in Example 19.2.6 is an orthogonal basis for $\mathbb{R}^3$.*

Recall that, in general, computing the coordinates of a vector with respect to a given basis reduces to solving a linear system. When the basis is orthogonal, this computation turns out to be much easier.

Theorem 35. *Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ be an orthogonal basis for a subspace H of $\mathbb{R}^n$. Then for any $\mathbf{y} \in H$, the unique weights in the linear combination*

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p \tag{19.3}$$

are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j} \qquad (j = 1, \ldots, p). \tag{19.4}$$

Proof. Taking the dot product of both sides of (19.3) with $\mathbf{v}_j$, we obtain

$$\begin{aligned}
\mathbf{y} \cdot \mathbf{v}_j &= (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) \cdot \mathbf{v}_j && \text{by substitution}\\
&= c_1(\mathbf{v}_1 \cdot \mathbf{v}_j) + \cdots + c_j(\mathbf{v}_j \cdot \mathbf{v}_j) + \cdots + c_p(\mathbf{v}_p \cdot \mathbf{v}_j) && \text{by Properties 2 and 3}\\
&= c_1(0) + \cdots + c_j(\mathbf{v}_j \cdot \mathbf{v}_j) + \cdots + c_p(0) && \text{by orthogonality}\\
&= c_j(\mathbf{v}_j \cdot \mathbf{v}_j).
\end{aligned}$$

Notice that the last inner product is nonzero, since otherwise $\mathcal{B}$ would have been linearly dependent. Dividing both sides of the above equation by $\mathbf{v}_j \cdot \mathbf{v}_j$, we find that

$$c_j = \frac{\mathbf{y} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}.$$

□

Example 19.2.8. *Find the coordinates of*

$$\mathbf{a} = \begin{bmatrix} 6 \\ 8 \\ 1 \end{bmatrix}$$

with respect to the orthogonal basis $\mathcal{B}$ from Examples 19.2.6 and 19.2.7.

Solution. Back in the day (that is, on Monday), we would have done the following:

$$\begin{bmatrix} 1 & 0 & -5 & 6 \\ -2 & 1 & -2 & 8 \\ 1 & 2 & 1 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & -1.5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1.5 \end{bmatrix}.$$

Now we can simply apply Theorem 35:

$$\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{x}}{\mathbf{x}\cdot\mathbf{x}}\right)\mathbf{x} + \left(\frac{\mathbf{a}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}}\right)\mathbf{y} + \left(\frac{\mathbf{a}\cdot\mathbf{z}}{\mathbf{z}\cdot\mathbf{z}}\right)\mathbf{z}$$
$$= \frac{-9}{6}\mathbf{x} + \frac{10}{5}\mathbf{y} + \frac{-45}{30}\mathbf{z} = -1.5\mathbf{x} + 2\mathbf{y} - 1.5\mathbf{z}.$$

□

19.3. **Orthonormal sets.**

Definition 19.3.1. A set $\mathcal{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is *orthonormal* if it is an orthogonal set of unit vectors, that is, for any pair of indices i and j, we have

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{19.5}$$

Clearly, the vectors in an orthonormal set are always nonzero (**0** is not a unit vector). Thus, by Theorem 34, an orthonormal set $\mathcal{U}$ is always a basis for its span. In such situations, we talk about an *orthonormal basis*.

Example 19.3.2. *The standard basis* $\mathcal{E} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ *is an orthonormal basis for* $\mathbb{R}^n$.

Orthonormal bases are very convenient in theoretical considerations and in computer-based calculations, because most of the nice formulas involving orthogonal bases get even simpler when the basis is, in fact, orthonormal. For example, the representation (19.3), (19.4) of a vector relative to an orthogonal basis becomes

$$\mathbf{y} = (\mathbf{y}\cdot\mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y}\cdot\mathbf{u}_p)\mathbf{u}_p$$

in the case of an orthonormal basis. Unfortunately, these "nicer" formulas turn out quite ugly in paper-and-pencil calculations.

19.4. **Orthogonal matrices.**

Definition 19.4.1. An $n \times n$ invertible matrix U is called *orthogonal* if $U^{-1} = U^t$, that is, if $UU^t = U^tU = I$.

Recall that, in general, computing of the inverse of a matrix A involves roughly twice the amount of work it takes to row reduce A to its reduced echelon form. In the case of an orthogonal matrix, all we have to do is rearrange the entries. Here we give some alternative descriptions of orthogonal matrices.

Theorem 36. *Let U be an $n \times n$ matrix. Then the following two statements are equivalent:*

1. *U is orthogonal;*
2. *the columns of U are orthonormal.*

Sketch of proof. Let U be a 3×3 matrix:

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix}, \qquad U^t = \begin{bmatrix} u_{11} & u_{21} & u_{31} \\ u_{12} & u_{22} & u_{32} \\ u_{13} & u_{23} & u_{33} \end{bmatrix}.$$

Computing U^tU by the row-column rule, we obtain

$$U^tU = \begin{bmatrix} u_{11}u_{11} + u_{21}u_{21} + u_{31}u_{31} & u_{11}u_{12} + u_{21}u_{22} + u_{31}u_{32} & u_{11}u_{13} + u_{21}u_{23} + u_{31}u_{33} \\ u_{12}u_{11} + u_{22}u_{21} + u_{32}u_{31} & u_{12}u_{12} + u_{22}u_{22} + u_{32}u_{32} & u_{12}u_{13} + u_{22}u_{23} + u_{32}u_{33} \\ u_{13}u_{11} + u_{23}u_{21} + u_{33}u_{31} & u_{13}u_{12} + u_{23}u_{22} + u_{33}u_{32} & u_{13}u_{13} + u_{23}u_{23} + u_{33}u_{33} \end{bmatrix}.$$

If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ denote the columns of U, we can also write the last matrix in the form

$$U^tU = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{u}_1 & \mathbf{u}_1 \cdot \mathbf{u}_2 & \mathbf{u}_1 \cdot \mathbf{u}_3 \\ \mathbf{u}_2 \cdot \mathbf{u}_1 & \mathbf{u}_2 \cdot \mathbf{u}_2 & \mathbf{u}_2 \cdot \mathbf{u}_3 \\ \mathbf{u}_3 \cdot \mathbf{u}_1 & \mathbf{u}_3 \cdot \mathbf{u}_2 & \mathbf{u}_3 \cdot \mathbf{u}_3 \end{bmatrix}.$$

If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are orthonormal, they satisfy (19.5) and the last matrix turns into I_3. On the other hand, if U is orthogonal, we have $U^tU = I_3$; comparing the last matrix with I_3 entry-by-entry, we recover formulas (19.5), that is, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are orthonormal. □

Theorem 37. *Let U be an $n \times n$ matrix. Then the following statements are equivalent:*

1. *U is orthogonal;*
2. *U preserves dot products: $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;*
3. *U preserves lengths: $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.*

Partial proof. We will discuss only the equivalence of 1) and 2). First, suppose that $U^tU = I$. We want to show that 2) holds for all $\mathbf{x}$ and $\mathbf{y}$. We will use the various properties of the matrix algebra that we have learned so far:

$$\begin{aligned} (U\mathbf{x}) \cdot (U\mathbf{y}) &= (U\mathbf{x})^t(U\mathbf{y}) && \text{by Definition 19.1} \\ &= (\mathbf{x}^tU^t)(U\mathbf{y}) && \text{by Property 8.3.1.4} \\ &= \mathbf{x}^t(U^tU)\mathbf{y} && \text{by Property 8.2.3.1} \\ &= \mathbf{x}^tI\mathbf{y} && \text{by } U^tU = I \\ &= \mathbf{x}^t\mathbf{y} && \text{by Property 8.2.3.5} \\ &= \mathbf{x} \cdot \mathbf{y} && \text{by Definition 19.1.} \end{aligned}$$

Now, suppose that U preserves dot-products. We will show that the columns of U are orthonormal (which is equivalent to the orthogonality of U, by the previous theorem). We need to make two key observations:

(i) the standard basis is orthonormal (Example 19.3.2);
(ii) $U\mathbf{e}_j = \mathbf{u}_j$, where $\mathbf{u}_j$ is the jth column of U.

Then for any pair of indices i, j, we have

$$\mathbf{u}_i \cdot \mathbf{u}_j \underset{\text{(ii)}}{=} (U\mathbf{e}_i) \cdot (U\mathbf{e}_j) \underset{2)}{=} \mathbf{e}_i \cdot \mathbf{e}_j \underset{\text{(i)}}{=} \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

that is, the columns of U are orthonormal. □

20. Orthogonal Projections

20.1. **Definition.** To start this lecture, we want to address the following question: given a subspace H and a vector $\mathbf{y}$ in $\mathbb{R}^n$, is it possible to write $\mathbf{y}$ as $\mathbf{y} = \mathbf{v} + \mathbf{z}$, where $\mathbf{v} \in H$ and $\mathbf{z} \in H^\perp$? The answer to this question is given by the following theorem.

Theorem 38 (Orthogonal decomposition theorem)**.** *Let H be a subspace of $\mathbb{R}^n$. Then any $\mathbf{y} \in \mathbb{R}^n$ has a unique representation in the form*

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}, \qquad \hat{\mathbf{y}} \in H, \ \mathbf{z} \in H^\perp. \tag{20.1}$$

Moreover, if $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p$ is an orthogonal basis for H, we have

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 + \left(\frac{\mathbf{y}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\right)\mathbf{v}_2 + \cdots + \left(\frac{\mathbf{y}\cdot\mathbf{v}_p}{\mathbf{v}_p\cdot\mathbf{v}_p}\right)\mathbf{v}_p \tag{20.2}$$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

Definition 20.1.1. Let H be a subspace of $\mathbb{R}^n$. For any $\mathbf{y} \in \mathbb{R}^n$, the unique vector $\hat{\mathbf{y}}$ appearing in (20.1) is called the *projection of $\mathbf{y}$ onto H* and is denoted $\operatorname{proj}_H \mathbf{y}$. When H is the span of a single vector $\mathbf{x}$ (i.e., $H = \operatorname{Span}\{\mathbf{x}\}$), we may also write $\operatorname{proj}_{\mathbf{x}} \mathbf{y}$.

Example 20.1.2. *Let*

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find the orthogonal projection of $\mathbf{y}$ onto the subspace H of $\mathbb{R}^3$ spanned by $\mathbf{v}_1, \mathbf{v}_2$.

Solution. Since

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (2)(-1) + (5)(1) + (-1)(3) = 0,$$

$\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for H. Applying (20.2), we find that

$$\begin{aligned} \operatorname{proj}_H \mathbf{y} &= \left(\frac{(1)(2) + (1)(5) + (1)(-1)}{2^2 + 5^2 + (-1)^2}\right)\mathbf{v}_1 + \left(\frac{(1)(-1) + (1)(1) + (1)(3)}{(-1)^2 + 1^2 + 3^2}\right)\mathbf{v}_2 \\ &= (1/5)\mathbf{v}_1 + (3/11)\mathbf{v}_2 = \begin{bmatrix} 7/55 \\ 14/11 \\ 34/55 \end{bmatrix}. \end{aligned}$$

□

20.2. **Properties.** The following fact should be intuitively obvious.

Fact 20.2.1. *If $\mathbf{y} \in H$, then* $\operatorname{proj}_H \mathbf{y} = \mathbf{y}$.

You may remember the term "orthogonal projection" from geometry. There, one usually talks about the "orthogonal projection of a point on a line" (or a plane) and means the point on the line (or the plane) where that line (or plane) intersects the line perpendicular to it and passing through the given point. You may also remember that the intersection point—that is, the "orthogonal projection"—is the point on the given line (or plane) that is the closest to the given point. It turns out that the same is true for orthogonal projections in $\mathbb{R}^n$ as defined above.

Theorem 39 (Best approximation theorem)**.** *Let H be a subspace of $\mathbb{R}^n$ and let $\mathbf{y} \in \mathbb{R}^n$. Then the best approximation to $\mathbf{y}$ from H is $\operatorname{proj}_H \mathbf{y}$, that is, if $\mathbf{w}$ is a vector in H, we have*

$$\|\mathbf{y} - \mathbf{w}\| \geq \|\mathbf{y} - \operatorname{proj}_H \mathbf{y}\|. \tag{20.3}$$

Moreover, when $\mathbf{w} \neq \operatorname{proj}_H \mathbf{y}$, (20.3) *holds with "$>$" in place of "$\geq$".*

Proof. Let $\mathbf{z} = \mathbf{y} - \operatorname{proj}_H \mathbf{y}$. By Theorem 38, $\mathbf{z}$ is in $H^\perp$. Now, let $\mathbf{w}$ be any vector in H. The vector $\operatorname{proj}_W \mathbf{y} - \mathbf{w}$ is in H and hence it is orthogonal to $\mathbf{z}$. Then, by the Pythagorean theorem,

$$\|\mathbf{y} - \mathbf{w}\|^2 = \|(\mathbf{y} - \operatorname{proj}_W \mathbf{y}) + (\operatorname{proj}_W \mathbf{y} - \mathbf{w})\|^2 = \|\mathbf{z}\|^2 + \|\operatorname{proj}_W \mathbf{y} - \mathbf{w}\|^2 \geq \|\mathbf{z}\|^2,$$

which establishes (20.3). □

Example 20.2.2. *Find the best approximation to $\mathbf{y}$ from $H = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}.$$

Solution. According to the best approximation theorem, the best approximation to $\mathbf{y}$ from H is $\operatorname{proj}_H \mathbf{y}$. Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for H,

$$\operatorname{proj}_H \mathbf{y} = \left(\frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 + \left(\frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right) \mathbf{v}_2 = (3/2) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (5/2) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}.$$

□

If we define the *distance between a vector* $\mathbf{y}$ *and a subspace* H as the minimum distance $\|\mathbf{y} - \mathbf{w}\|$ between $\mathbf{y}$ and a vector $\mathbf{w} \in H$, we can use orthogonal projections to measure distances between vectors and subspaces.

Example 20.2.3. *Find the distance between $\mathbf{y}$ and the subspace $H = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where*

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Solution. As in the preceding example,

$$\operatorname{proj}_H \mathbf{y} = \left(\frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 + \left(\frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right) \mathbf{v}_2 = (-1/14) \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + (1/6) \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1/21 \\ -2/21 \\ -10/21 \end{bmatrix}.$$

Thus, the distance between $\mathbf{y}$ and H is

$$\begin{aligned} \operatorname{dist}(\mathbf{y}, H) = \|\mathbf{y} - \operatorname{proj}_H \mathbf{y}\| &= \sqrt{(22/21)^2 + (44/21)^2 + (11/21)^2} \\ &= \sqrt{(11/21)^2\left(2^2 + 4^2 + 1^2\right)} = \frac{11\sqrt{21}}{21}. \end{aligned}$$

□

Remark 20.2.4. Note that when $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for H, we have a simpler formula for the projection onto H:

$$\operatorname{proj}_H \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p.$$

20.3. **The Gram–Schmidt process.** We have seen that having an orthogonal basis for a subspace H of $\mathbb{R}^n$ is quite convenient. But what if we only have a basis $\mathcal{B}$ for H that is not orthogonal? Is there some way to transform it into an orthogonal basis? The answer is yes, and there is an algorithm for doing this—the *Gram–Schmidt process*.

Theorem 40 (Gram–Schmidt process)**.** *Let* $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_p\}$ *be a basis for a subspace* H *for* $\mathbb{R}^n$*, and define:*

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right) \mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \left(\frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 - \left(\frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right) \mathbf{v}_2 - \cdots - \left(\frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\right) \mathbf{v}_{p-1}.
\end{aligned}$$

That is, $\mathbf{v}_1 = \mathbf{x}_1$ *and for* $j \geq 2$*,*

$$\mathbf{v}_j = \mathbf{x}_j - \big(\textit{projection of } \mathbf{x}_j \textit{ onto the subspace spanned by } \mathbf{v}_1, \ldots, \mathbf{v}_{j-1}\big).$$

Then $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ *is an orthogonal basis for* H*. Moreover, for each* $k \leq p$*, we have*

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\}.$$

20.4. **Examples.**

Example 20.4.1. *Find an orthogonal basis for* $H = \text{Span}\left\{\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}\right\}$.

Solution. Denote the given vectors by $\mathbf{x}_1$ and $\mathbf{x}_2$. Then

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - (15/30) \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1.5 \\ 1.5 \end{bmatrix}.$$

□

Example 20.4.2. *Find an orthogonal basis for the column space of the matrix*

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}.$$

Solution. Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$. Orthogonalization of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ yields:

$$\mathbf{v}_1 = \mathbf{a}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{a}_2 - \left(\frac{\mathbf{a}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - (-40/20) \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{a}_3 - \left(\frac{\mathbf{a}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 - \left(\frac{\mathbf{a}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right) \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - (30/20) \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - (-10/20) \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}.$$

□

Remark 20.4.3. The more observant among you might have noticed that in the solution of the previous example we did not check whether the columns of A are linearly independent. Strictly speaking, we should have done that, since Theorem 40 (as stated) applies only to bases for H and not to spanning sets for H. However, such an initial check is unnecessary. In fact, if we apply the Gram–Schmidt process to a spanning set $\mathbf{x}_1, \dots, \mathbf{x}_p$, which is not linearly independent, at some point we will obtain a zero vector $\mathbf{v}_j$. If that happens, we can discard $\mathbf{x}_j$ and recompute $\mathbf{v}_j$ using $\mathbf{x}_{j+1}$ instead of $\mathbf{x}_j$. At the end, we will end up with an orthogonal basis consisting of fewer than p vectors.

Example 20.4.4. *Find an orthogonal basis for the column space of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 4 & 1 \\ -1 & 0 & -2 & 2 \\ 1 & 1 & 3 & 1 \end{bmatrix}.$$

Solution. Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$. We know for sure that the columns of A are linearly dependent, because of Theorem 6. Still, we apply the Gram–Schmidt process to $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$:

$$\mathbf{v}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{a}_2 - \left(\frac{\mathbf{a}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - (3/3) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{a}_3 - \left(\frac{\mathbf{a}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 - \left(\frac{\mathbf{a}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right) \mathbf{v}_2 = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} - (9/3) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - (2/2) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The last equation means that $\mathbf{a}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Since $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$, it follows that $\mathbf{a}_3$ is also a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, we could have thrown $\mathbf{a}_3$ away from the very beginning and worked only with $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$. We will act as if we had done that and we will repeat the last step using $\mathbf{a}_4$ instead of $\mathbf{a}_3$:

$$\mathbf{v}_3 = \mathbf{a}_4 - \left(\frac{\mathbf{a}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{a}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - (0/3)\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - (3/2)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \\ 1 \end{bmatrix}.$$

Finally, we will replace $\mathbf{v}_3$ by its multiple

$$\mathbf{v}_3' = 2\mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

We obtain the orthogonal basis

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

□

20.5. **Orthonormal bases.** Given any basis for a subspace H of $\mathbb{R}^n$, it is not difficult to find an orthonormal basis $\mathcal{U}$ for H: first we apply the Gram–Schmidt process to find an orthogonal basis, say $\mathcal{B}$; then, we normalize every vector in $\mathcal{B}$ to obtain an orthonormal set. Here is an example.

Example 20.5.1. *Find an orthonormal basis for the column space of the matrix*

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}.$$

Solution. We showed that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

form an orthogonal basis for $\operatorname{Col} A$. Since

$$\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \|\mathbf{v}_3\| = \sqrt{20},$$

the normalizations of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are

$$\mathbf{u}_1 = \begin{bmatrix} 3/\sqrt{20} \\ 1/\sqrt{20} \\ -1/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1/\sqrt{20} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -3/\sqrt{20} \\ 1/\sqrt{20} \\ 1/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}.$$

Thus, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\operatorname{Col} A$. □

21. Least-Squares Problems

21.1. **Introduction.** Consider an inconsistent linear system $A\mathbf{x} = \mathbf{b}$. In this lecture, we describe a method for computing "the closest thing to a solution" of such a system. Instead of trying to find a vector $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{b}$, our strategy will be to seek an $\mathbf{x}$ for which $A\mathbf{x}$ and $\mathbf{b}$ are as close as possible, that is, we will try to minimize the distance $\|A\mathbf{x} - \mathbf{b}\|$.

Definition 21.1.1. Let A be an $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$. A *least-squares solution* of $A\mathbf{x} = \mathbf{b}$ is a vector $\mathbf{x}_0 \in \mathbb{R}^n$ such that

$$\|A\mathbf{x}_0 - \mathbf{b}\| \le \|A\mathbf{x} - \mathbf{b}\| \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n. \tag{21.1}$$

Remark 21.1.2. Note that the above definition does not assume that $A\mathbf{x} = \mathbf{b}$ is inconsistent. However, in the case of a consistent system, a least-square solution is just a regular solution. It is the inconsistent case that is of interest.

You might wonder what is the purpose of computing a least-square solution of an inconsistent linear system. After all, we have (supposedly) already solved the system once and found that the solution set is empty. Why all the effort to fabricate a solution where there is none? Well, suppose that you are performing an experiment, where you take measurements of a physical quantity $F(t)$ at various times: say, $t = 1$, $t = 2$, ..., $t = 99$. Let's say that $F(t)$ is known to be approximately a cubic polynomial in t, that is, the mathematical model for $F(t)$ is

$$F(t) = at^3 + bt^2 + ct + d,$$

where a, b, c, d are fixed (but unknown to you) numbers. You want to recover the values of a, b, c, d from your data. Your 99 measurements ammount to the following system for a, b, c, d:

$$\begin{cases} 1^3a + 1^2b + 1c + d = F(1) \\ 2^3a + 2^2b + 2c + d = F(2) \\ \qquad\vdots \\ 99^3a + 99^2b + 99c + d = F(99) \end{cases}$$

In a perfect world, where both the mathematical model and your measuring equipment are perfect, the first four equations would yield a unique solution for a, b, c, d, which would also satisfy the remaining 95 equations. In reality, the first four equations will yield a unique solution for a, b, c, d, which will fail to satisfy any of the remaining 95 equations. It is here that the least-squares solution comes to the rescue: in the above example, the least-squares solution will be the "best fit" to your empirical data.

21.2. **Computing least-squares solutions.** Theoretically, finding the least-squares solution is straightforward. After all, the collection of vectors of the form $A\mathbf{x}$ is $\operatorname{Col} A$, so (21.1) is equivalent to

$$A\mathbf{x}_0 = \operatorname{proj}_{\operatorname{Col} A} \mathbf{b}.$$

Hence, to compute the least square solution $\mathbf{x}_0$, we must simply solve the equation

$$A\mathbf{x} = \operatorname{proj}_{\operatorname{Col} A} \mathbf{b},$$

which is always consistent. The best way to appreciate the shortcomings of this approach is to work out an example.

Example 21.2.1. *Find the least-squares solution to the equation $A\mathbf{x} = \mathbf{b}$, where*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 7 \end{bmatrix}.$$

Solution. First, we must find $\hat{\mathbf{b}} = \operatorname{proj}_{\operatorname{Col} A} \mathbf{b}$, which requires an orthogonal basis for $\operatorname{Col} A$. Using the Gram-Schmidt process, we get

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - (10/4) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -0.5 \\ 0.5 \\ 1.5 \end{bmatrix}.$$

Thus,

$$\begin{aligned} \hat{\mathbf{b}} &= \left(\frac{\mathbf{b} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 + \left(\frac{\mathbf{b} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 \\ &= (14/4) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + (10/5) \begin{bmatrix} -1.5 \\ -0.5 \\ 0.5 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 2.5 \\ 4.5 \\ 6.5 \end{bmatrix}. \end{aligned}$$

Finally, we solve the equation $A\mathbf{x} = \hat{\mathbf{b}}$ (which is automatically consistent) for $\mathbf{x}$:

$$\begin{bmatrix} 1 & 1 & 0.5 \\ 1 & 2 & 2.5 \\ 1 & 3 & 4.5 \\ 1 & 4 & 6.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0.5 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1.5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, the least-squares solution is $(-1.5, 2)$. □

A much easier (but less apparent) approach to finding least-squares solutions is provided by the following theorem.

Theorem 41. *Let A be an $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$. Then the set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ is the same as the set of solutions of the equation*

$$(A^t A)\mathbf{x} = A^t \mathbf{b}. \tag{21.2}$$

In particular, (21.2) *is always consistent.*

Equation (21.2) is called the *normal equation for* $\mathbf{x} = \mathbf{b}$.

Example 21.2.2. *Find the least-squares solution to the equation* $A\mathbf{x} = \mathbf{b}$, *where*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 7 \end{bmatrix}.$$

Solution. We have

$$A^tA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}, \qquad A^t\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 14 \\ 45 \end{bmatrix},$$

so the solution of the normal equation is $(-1.5, 2)$:

$$\begin{bmatrix} 4 & 10 & 14 \\ 10 & 30 & 45 \end{bmatrix} \sim \begin{bmatrix} 1 & 2.5 & 3.5 \\ 1 & 3 & 4.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2.5 & 3.5 \\ 0 & 0.5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2.5 & 3.5 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1.5 \\ 0 & 1 & 2 \end{bmatrix}.$$

□

21.3. **Least-squares error.** If $\mathbf{x}_0$ is a least-squares solution of $A\mathbf{x} = \mathbf{b}$, the number $\|\mathbf{b} - A\mathbf{x}_0\|$ is called the *least-squares error (of approximation)*. The least-squares error measures how close the least-squares solution is to being a genuine solution. For example, suppose that we ran the experiment from the Introduction twice. If each time we computed a least-squares solution and if the least-squares errors of approximation turned out to be 3.789 and 0.886, respectively, then we would feel more confident in the second set of measurements than in the first.

Example 21.3.1. *Compute the least-squares error in Examples 21.2.1 and 21.2.2.*

Solution. If $\mathbf{x}_0$ denotes the least-squares solution we found earlier, we have

$$A\mathbf{x}_0 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -1.5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 2.5 \\ 4.5 \\ 6.5 \end{bmatrix}, \qquad \mathbf{b} - A\mathbf{x}_0 = \begin{bmatrix} 1.5 \\ -2.5 \\ 0.5 \\ 0.5 \end{bmatrix},$$

so the least-squares error is

$$\|\mathbf{b} - A\mathbf{x}_0\| = \sqrt{2.25 + 6.25 + 0.25 + 0.25} = 3.$$

□

22. Inner Product Spaces

In this lecture, we generalize the concepts of inner product and orthogonality from $\mathbb{R}^n$ to abstract vector spaces.

22.1. **Definitions.** Let V be a vector space and suppose that for each pair of vectors $\mathbf{x}, \mathbf{y} \in V$, there is a rule for computing a real number $\langle \mathbf{x}, \mathbf{y} \rangle$ (thus, $\langle \mathbf{x}, \mathbf{y} \rangle$ is a real-valued function of two arguments from V). Suppose that the function $\langle \mathbf{x}, \mathbf{y} \rangle$ satisfies the following axioms:

1. $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
2. $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
3. $\langle c\mathbf{x}, \mathbf{y} \rangle = c \langle \mathbf{x}, \mathbf{y} \rangle$
4. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

Then $\langle \mathbf{x}, \mathbf{y} \rangle$ is called an *inner product* on V and V is called an *inner product space*. We also define the *norm* (or *length*) of a vector $\mathbf{x}$:

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle};$$

and the *distance* between two vectors:

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle}.$$

Further, we say that two vectors $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* (or *perpendicular*), and write $\mathbf{x} \perp \mathbf{y}$, if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. We say that a vector $\mathbf{x}$ is *orthogonal to a set S in V*, and write $\mathbf{x} \perp S$, if $\mathbf{x}$ is orthogonal to every vector in S. We can also use the inner product to measure angles in inner product spaces: for nonzero vectors $\mathbf{x}, \mathbf{y} \in V$, we define the *angle θ between* $\mathbf{x}$ *and* $\mathbf{y}$ via its cosine:

$$\theta = \arccos\left(\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \, \|\mathbf{y}\|}\right). \tag{22.1}$$

Example 22.1.1. *The dot product* $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^t\mathbf{y}$ *is an inner product on* $\mathbb{R}^n$.

Solution. In this case, axioms 1–4 above are simply properties 1–4 in Proposition 19.1.1. □

Example 22.1.2. *Let*

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & n \end{bmatrix}.$$

The function $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^t A \mathbf{y}$ *is an inner product on* $\mathbb{R}^n$.

Solution. If we write this inner product in terms of the components of $\mathbf{x}$ and $\mathbf{y}$, we have

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3 + \cdots + nx_ny_n.$$

Then axiom 1 claims that the expressions

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3 + \cdots + nx_ny_n$$

and

$$\langle \mathbf{y}, \mathbf{x} \rangle = y_1x_1 + 2y_2x_2 + 3y_3x_3 + \cdots + ny_nx_n$$

are equal for all choices of $\mathbf{x}$ and $\mathbf{y}$, which of course is true. To verify axiom 2, we must compare

$$\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = (x_1 + y_1)z_1 + 2(x_2 + y_2)z_2 + \cdots + n(x_n + y_n)z_n$$

and

$$\langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle = (x_1 z_1 + 2x_2 z_2 + \cdots + n x_n z_n) + (y_1 z_1 + 2y_2 z_2 + \cdots + n y_n z_n).$$

Using the basic arithmetic properties of real numbers, we see that the right sides of these two equations are equal, so axiom 2 holds too. The proof that axiom 3 holds is similar and we leave it as an exercise. Finally, we have

$$\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + 2x_2^2 + 3x_3^2 + \cdots + n x_n^2.$$

All the terms in the sum on the right side of this identity are non-negative, so $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$. Furthermore, that sum can only be zero if all the terms equal zero, that is, if $x_1 = x_2 = \cdots = x_n = 0$. This establishes axiom 4. □

Example 22.1.3. *For continuous functions* $\mathbf{f}(t)$ *and* $\mathbf{g}(t)$ *on* $[0, 1]$*, define*

$$\langle \mathbf{f}(t), \mathbf{g}(t) \rangle = \int_0^1 \mathbf{f}(t)\mathbf{g}(t)\, dt.$$

Then $\langle \mathbf{f}(t), \mathbf{g}(t) \rangle$ *is an inner product on the space of continuous functions on* $[0, 1]$.

Solution. We have

$$\langle \mathbf{f}(t), \mathbf{g}(t) \rangle = \int_0^1 \mathbf{f}(t)\mathbf{g}(t)\, dt = \int_0^1 \mathbf{g}(t)\mathbf{f}(t)\, dt = \langle \mathbf{g}(t), \mathbf{f}(t) \rangle,$$

so axiom 1 holds. We leave axiom 2 as an exercise, and move to axiom 3. We have

$$\langle c\mathbf{f}(t), \mathbf{g}(t) \rangle = \int_0^1 c\mathbf{f}(t)\mathbf{g}(t)\, dt = c \int_0^1 \mathbf{f}(t)\mathbf{g}(t)\, dt = c \langle \mathbf{f}(t), \mathbf{g}(t) \rangle,$$

so axiom 3 holds. Finally, we have

$$\langle \mathbf{f}(t), \mathbf{f}(t) \rangle = \int_0^1 \mathbf{f}(t)^2\, dt \geq 0,$$

because $\mathbf{f}(t)^2 \geq 0$. To complete the proof of axiom 4, we must show that the integral on the right side of the above equality is zero only when $\mathbf{f}(t) = 0$ for all t. This is true for continuous functions, a fact usually proved in rigorous calculus courses (such as Math 473 here, at TU). □

Example 22.1.4. *Find the distance between* $\sin t$ *and* $\cos t$ *and the length of* e^t *in the inner product space described in the previous example.*

Solution. We have

$$\begin{aligned}
\operatorname{dist}(\sin t, \cos t) &= \int_0^1 (\sin t - \cos t)^2\, dt = \int_0^1 (1 - 2\sin t \cos t)\, dt \\
&= \int_0^1 (1 - \sin 2t)\, dt = \left[t + \tfrac{1}{2}\cos 2t\right]_0^1 = \tfrac{1}{2}(1 + \cos 2), \\
\|e^t\| &= \left(\int_0^1 (e^t)^2\, dt\right)^{1/2} = \left(\int_0^1 e^{2t}\, dt\right)^{1/2} = \sqrt{\left[\tfrac{1}{2}e^{2t}\right]_0^1} = \sqrt{\frac{e^2 - 1}{2}}.
\end{aligned}$$

□

Example 22.1.5. *Show that the functions* $\sin \pi t$ *and* $\cos \pi t$ *and are orthogonal in the inner product space described in the previous two examples.*

Solution. We have

$$\langle \sin \pi t, \cos \pi t\rangle = \int_0^1 \sin \pi t \cos \pi t \, dt = \int_0^1 \tfrac{1}{2} \sin 2\pi t \, dt = \left[\tfrac{-1}{4\pi} \cos 2\pi t\right]_0^1 = 0,$$

so the two functions are orthogonal. □

Several important properties of the dot product in $\mathbb{R}^n$ extend to any inner product on any vector space. Two such properties are the triangle inequality and the Cauchy–Schwarz inequality.

Proposition 22.1.6. *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ *and let* $c \in \mathbb{R}$. *Then:*

1. $|\langle \mathbf{x}, \mathbf{y}\rangle| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$ *(Cauchy–Schwarz inequality)*
2. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ *(triangle inequality)*

22.2. **Orthogonal and orthonormal bases.**

Definition 22.2.1. A set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in an inner product space V is called *orthogonal* if every two distinct vectors in S are orthogonal, that is, if $\mathbf{v}_i \perp \mathbf{v}_j$ whenever $i \neq j$.

A set $\mathcal{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in an inner product space V is *orthonormal* if it is an orthogonal set of unit vectors, that is, for any pair of indices i and j, we have

$$\left\langle \mathbf{u}_i, \mathbf{u}_j \right\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{22.2}$$

Similarly to $\mathbb{R}^n$, orthogonal sets in other inner product spaces are linearly independent and provide very nice bases for the subspaces they span. We have the following theorem, which generalizes Theorems 34 and 35.

Theorem 42. *Let* V *be an inner product space and let* $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ *be an orthogonal set of nonzero vectors in* V. *Then* S *is linearly independent, and hence, a basis for* $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$. *Furthermore, for any* $\mathbf{y} \in H$, *the unique weights in the linear combination*

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p \tag{22.3}$$

are given by

$$c_j = \frac{\left\langle \mathbf{y}, \mathbf{v}_j \right\rangle}{\left\langle \mathbf{v}_j, \mathbf{v}_j \right\rangle} \qquad (j = 1, \ldots, p). \tag{22.4}$$

We can also generalize the Gram–Schmidt orthogonalization process to abstract vector spaces.

Theorem 43 (Gram–Schmidt process)**.** *Let* $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_p\}$ *be a basis for a subspace* H *for an inner product space* V, *and define:*

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\langle \mathbf{x}_2, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\right) \mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\langle \mathbf{x}_3, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\right) \mathbf{v}_1 - \left(\frac{\langle \mathbf{x}_3, \mathbf{v}_2\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\right) \mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \left(\frac{\left\langle \mathbf{x}_p, \mathbf{v}_1\right\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\right)\mathbf{v}_1 - \left(\frac{\left\langle \mathbf{x}_p, \mathbf{v}_2\right\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\right)\mathbf{v}_2 - \cdots - \left(\frac{\left\langle \mathbf{x}_p, \mathbf{v}_{p-1}\right\rangle}{\left\langle \mathbf{v}_{p-1}, \mathbf{v}_{p-1}\right\rangle}\right)\mathbf{v}_{p-1}.
\end{aligned}$$

That is, $\mathbf{v}_1 = \mathbf{x}_1$ *and for* $j \geq 2$,

$$\mathbf{v}_j = \mathbf{x}_j - \big(\textit{projection of } \mathbf{x}_j \textit{ onto the subspace spanned by } \mathbf{v}_1, \ldots, \mathbf{v}_{j-1}\big).$$

Then $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ *is an orthogonal basis for H. Moreover, for each* $k \leq p$*, we have*

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\}.$$

Example 22.2.2. *Find an orthogonal basis for* $\mathbb{P}_3$ *relative to the inner product*

$$\langle \mathbf{p}(t), \mathbf{q}(t) \rangle = \int_{-1}^{1} \mathbf{p}(t)\mathbf{q}(t)\, dt.$$

Solution. We start with the standard basis $\{1, t, t^2, t^3\}$. Then $\mathbf{p}_0(t) = 1$ and

$$\mathbf{p}_1(t) = t - \frac{\langle t, \mathbf{p}_0(t) \rangle}{\langle \mathbf{p}_0(t), \mathbf{p}_0(t) \rangle}\mathbf{p}_0(t).$$

Since

$$\langle t, \mathbf{p}_0(t) \rangle = \int_{-1}^{1} (t)(1)\, dt = \int_{-1}^{1} t\, dt = 0,$$

we get $\mathbf{p}_1(t) = t$. Then

$$\mathbf{p}_2(t) = t^2 - \frac{\langle t^2, \mathbf{p}_0(t) \rangle}{\langle \mathbf{p}_0(t), \mathbf{p}_0(t) \rangle}\mathbf{p}_0(t) - \frac{\langle t^2, \mathbf{p}_1(t) \rangle}{\langle \mathbf{p}_1(t), \mathbf{p}_1(t) \rangle}\mathbf{p}_1(t).$$

Since

$$\begin{aligned}
\langle t^2, \mathbf{p}_0(t) \rangle &= \int_{-1}^{1} (t^2)(1)\, dt = \int_{-1}^{1} t^2\, dt = 2\int_{0}^{1} t^2\, dt = \frac{2}{3}, \\
\langle t^2, \mathbf{p}_1(t) \rangle &= \int_{-1}^{1} (t^2)(t)\, dt = \int_{-1}^{1} t^3\, dt = 0, \\
\langle \mathbf{p}_0(t), \mathbf{p}_0(t) \rangle &= \int_{-1}^{1} (1)(1)\, dt = \int_{-1}^{1} 1\, dt = 2,
\end{aligned}$$

we get $\mathbf{p}_2(t) = t^2 - \frac{1}{3}$. Then

$$\mathbf{p}_3(t) = t^3 - \frac{\langle t^3, \mathbf{p}_0(t) \rangle}{\langle \mathbf{p}_0(t), \mathbf{p}_0(t) \rangle}\mathbf{p}_0(t) - \frac{\langle t^3, \mathbf{p}_1(t) \rangle}{\langle \mathbf{p}_1(t), \mathbf{p}_1(t) \rangle}\mathbf{p}_1(t) - \frac{\langle t^3, \mathbf{p}_2(t) \rangle}{\langle \mathbf{p}_2(t), \mathbf{p}_2(t) \rangle}\mathbf{p}_2(t).$$

Since

$$\begin{aligned}
\langle t^3, \mathbf{p}_0(t) \rangle &= \int_{-1}^{1} (t^3)(1)\, dt = \int_{-1}^{1} t^3\, dt = 0, \\
\langle t^3, \mathbf{p}_1(t) \rangle &= \int_{-1}^{1} (t^3)(t)\, dt = \int_{-1}^{1} t^4\, dt = 2\int_{0}^{1} t^4\, dt = \frac{2}{5}, \\
\langle t^3, \mathbf{p}_2(t) \rangle &= \int_{-1}^{1} (t^3)(t^2 - 1/3)\, dt = \int_{-1}^{1} \left(t^5 - (1/3)t^3\right) dt = 0, \\
\langle \mathbf{p}_1(t), \mathbf{p}_1(t) \rangle &= \int_{-1}^{1} (t)(t)\, dt = \int_{-1}^{1} t^2\, dt = \frac{2}{3},
\end{aligned}$$

we get $\mathbf{p}_3(t) = t^3 - \frac{2/5}{2/3}t = t^3 - \frac{3}{5}t$. □

22.3. **Fourier series*.** Let us consider the space of continuous functions on $[-\pi, \pi]$ with an inner product similar to that in Example 22.1.3:

$$\langle \mathbf{f}(t), \mathbf{g}(t) \rangle = \int_{-\pi}^{\pi} \mathbf{f}(t)\mathbf{g}(t)\, dt.$$

Relative to this inner product the functions $\sin x, \sin 2x, \sin 3x, \ldots$ and $1, \cos x, \cos 2x, \cos 3x, \ldots,$ are mutually orthogonal. For example, consider the product of a sine and a cosine ($1 = \cos 0x$, so we can lump the function 1 with the cosines):

$$\begin{aligned}\langle \sin mx, \cos nx \rangle &= \int_{-\pi}^{\pi} (\sin mx)(\cos nx)\, dx \\ &= \int_{-\pi}^{\pi} \tfrac{1}{2}\big[\sin(mx - nx) + \sin(mx + nx)\big]\, dx \\ &= \int_{-\pi}^{\pi} \tfrac{1}{2}\big[\sin(m-n)x + \sin(m+n)x\big]\, dx = 0,\end{aligned}$$

because the integrand in the last integral is an odd function. That is, every sine function is orthogonal to any cosine function. If we consider two cosines with different n's, we get

$$\begin{aligned}\langle \cos mx, \cos nx \rangle &= \int_{-\pi}^{\pi} (\cos mx)(\cos nx)\, dx \\ &= \int_{-\pi}^{\pi} \tfrac{1}{2}\big[\cos(mx - nx) + \cos(mx + nx)\big]\, dx \\ &= \int_{-\pi}^{\pi} \tfrac{1}{2}\big[\cos(m-n)x + \cos(m+n)x\big]\, dx \\ &= \left[\frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = 0.\end{aligned}$$

On the other hand, for $n \geq 1$,

$$\langle \cos nx, \cos nx \rangle = \int_{-\pi}^{\pi} \cos^2 nx\, dx = \int_{-\pi}^{\pi} \frac{1 + \cos 2nx}{2}\, dx = \left[\frac{x}{2} + \frac{\sin 2nx}{4n}\right]_{-\pi}^{\pi} = \pi;$$

and

$$\langle 1, 1 \rangle = \int_{-\pi}^{\pi} 1^2\, dx = 2\pi.$$

Thus,

$$\langle \cos mx, \cos nx \rangle = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n \geq 1, \\ 2\pi & \text{if } m = n = 0. \end{cases}$$

That is, every two different cosine functions are orthogonal. Finally, a similar calculation shows that

$$\langle \sin mx, \sin nx \rangle = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n; \end{cases}$$

that is, every two different sine functions are also orthogonal. Altogether, we have shown that every two distinct functions in the above infinite set are orthogonal. Therefore, the set

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$$

is an orthogonal (and therefore linearly independent) set of functions. This shows, in particular, that the space of continuous functions on $[-\pi, \pi]$ is infinite dimensional.

Given a continuous function $\mathbf{f}(t)$ on $[-\pi, \pi]$ and an integer $n \geq 1$, let us compute its projection onto the subspace spanned by

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx.$$

If we denote the projection by $\mathbf{p}(t)$, then

$$\begin{aligned}
\mathbf{p}(t) &= \frac{\langle \mathbf{f}(t), 1\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle \mathbf{f}(t), \cos t\rangle}{\langle \cos t, \cos t\rangle} \cos t + \frac{\langle \mathbf{f}(t), \sin t\rangle}{\langle \sin t, \sin t\rangle} \sin t \\
&\quad + \frac{\langle \mathbf{f}(t), \cos 2t\rangle}{\langle \cos 2t, \cos 2t\rangle} \cos 2t + \frac{\langle \mathbf{f}(t), \sin 2t\rangle}{\langle \sin 2t, \sin 2t\rangle} \sin 2t + \cdots \\
&\quad + \frac{\langle \mathbf{f}(t), \cos nt\rangle}{\langle \cos nt, \cos nt\rangle} \cos nt + \frac{\langle \mathbf{f}(t), \sin nt\rangle}{\langle \sin nt, \sin nt\rangle} \sin nt \\
&= \frac{\langle \mathbf{f}(t), 1\rangle}{2\pi} 1 + \frac{\langle \mathbf{f}(t), \cos t\rangle}{\pi} \cos t + \frac{\langle \mathbf{f}(t), \sin t\rangle}{\pi} \sin t \\
&\quad + \frac{\langle \mathbf{f}(t), \cos 2t\rangle}{\pi} \cos 2t + \frac{\langle \mathbf{f}(t), \sin 2t\rangle}{\pi} \sin 2t + \cdots \\
&\quad + \frac{\langle \mathbf{f}(t), \cos nt\rangle}{\pi} \cos nt + \frac{\langle \mathbf{f}(t), \sin nt\rangle}{\pi} \sin nt.
\end{aligned}$$

The numbers

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{f}(t)\, dt, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathbf{f}(t) \cos nt\, dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathbf{f}(t) \sin nt\, dt,$$

are called the *Fourier coefficients of* $\mathbf{f}(t)$. We can use these numbers to construct an infinite series

$$a_0 + \sum_{n=1}^{\infty} \big(a_n \cos nt + b_n \sin nt\big). \tag{22.5}$$

This series is called the *Fourier series of* $\mathbf{f}(t)$.

It is not immediately clear that the Fourier series of a continuous function converges at all or that its sum (assuming it exists) has anything to do with $\mathbf{f}(t)$. However, it turns but that the following theorem holds

Theorem 44. *If* $\mathbf{f}(t)$ *is a continuous function on* $[-\pi, \pi]$*, then the series* (22.5) *converges and*

$$\mathbf{f}(t) = a_0 + \sum_{n=1}^{\infty} \big(a_n \cos nt + b_n \sin nt\big)$$

for all t *in the range* $-\pi < t < \pi$.

We can use the above theorem to calculate the sum of the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$.

Example 22.3.1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$

Solution. We apply Theorem 44 to the function $\mathbf{f}(t) = t^2$. The theorem states that

$$(*) \qquad t^2 = a_0 + \sum_{n=1}^{\infty} \big(a_n \cos nt + b_n \sin nt\big),$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\,dt, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos nt\,dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \sin nt\,dt.$$

We have

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{1}{\pi}\int_{0}^{\pi} t^2\,dt = \frac{1}{\pi}\frac{\pi^3}{3} = \frac{\pi^2}{3};\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \sin nt\,dt = 0, \qquad \text{because } t^2 \sin nt \text{ is odd};\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos nt\,dt = \frac{2}{\pi}\int_{0}^{\pi} t^2 \cos nt\,dt\\
&= \frac{2}{n\pi}\int_{0}^{\pi} t^2\,d(\sin nt) = \frac{2}{n\pi}\left(\big[t^2 \sin nt\big]_0^{\pi} - \int_0^{\pi} 2t \sin nt\,dt\right)\\
&= \frac{-4}{n\pi}\int_0^{\pi} t \sin nt\,dt = \frac{4}{n^2\pi}\int_0^{\pi} t\,d(\cos nt)\\
&= \frac{4}{n^2\pi}\left(\big[t \cos nt\big]_0^{\pi} - \int_0^{\pi} \cos nt\,dt\right) = \frac{4}{n^2\pi}\big(\pi \cos n\pi - 0\big) = \frac{4(-1)^n}{n^2}.
\end{aligned}$$

Hence, $(*)$ takes the form

$$t^2 = \frac{\pi^2}{3} + 4\left(-\frac{\cos t}{1} + \frac{\cos 2t}{4} - \frac{\cos 3t}{9} + \frac{\cos 4t}{16} - \cdots - \frac{\cos(2k-1)t}{(2k-1)^2} + \frac{\cos 2kt}{(2k)^2} - \cdots\right).$$

Setting $t = 0$, we get

$$(**) \qquad 0 = \frac{\pi^2}{3} - 4\left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots + \frac{1}{(2k-1)^2} - \frac{1}{(2k)^2} + \cdots\right).$$

Now, let S be the sum that we want to calculate and let S' be the infinite sum in $(**)$. Then

$$\begin{aligned}
S' &= \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{(2k-1)^2} + \frac{1}{(2k)^2} + \cdots\right) - 2\left(\frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{(2k)^2} + \cdots\right)\\
&= S - \frac{2}{4}\left(1 + \frac{1}{4} + \cdots + \frac{1}{k^2} + \cdots\right) = \frac{S}{2}.
\end{aligned}$$

Thus, $(**)$ yields

$$0 = \frac{\pi^2}{3} - 2S \quad \Longrightarrow \quad S = \frac{\pi^2}{6}. \qquad \square$$

Exercise. *Apply Theorem 44 to the function* $\mathbf{f}(t) = t^4$ *and use the resulting identity to calculate the sum of the series* $\sum_{n=1}^{\infty} n^{-4}$.

Exercise. *Apply Theorem 44 to the function* $\mathbf{f}(t) = t^3$ *and use the resulting identity to calculate the sum of the series* $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$.

Recall that an $n \times n$ matrix A is not always diagonalizable and that (in general) there is no easy way to say whether it is or isn't. The purpose of this lecture is to describe in further detail the diagonalization of an important class of matrices, which are known to be always diagonalizable.

23.1. **Diagonalization of symmetric matrices.** An $n \times n$ matrix A is called *symmetric* if $A^t = A$. For example, the matrices

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & 8 \\ 0 & 8 & -3 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 8 \\ 5 & 8 & 7 \end{bmatrix}$$

are symmetric, but the matrix

$$\begin{bmatrix} 0 & -3 & 0 \\ 3 & 5 & 1 \\ 0 & 1 & 6 \end{bmatrix}$$

isn't. A symmetric matrix, it turns out, is always diagonalizable:

Theorem 45. *If A is a symmetric matrix, then A is diagonalizable.*

Example 23.1.1. *Diagonalize the matrix*

$$A = \begin{bmatrix} 16 & -4 \\ -4 & 1 \end{bmatrix}.$$

Solution. Since

$$\det(A - \lambda I) = \begin{vmatrix} 16-\lambda & -4 \\ -4 & 1-\lambda \end{vmatrix} = (16-\lambda)(1-\lambda) - 16 = \lambda(\lambda - 17),$$

the eigenvalues of A are $\lambda = 0, 17$. Since

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 16 & -4 & 0 \\ -4 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -0.25 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the solutions of $A\mathbf{x} = \mathbf{0}$ are the vectors $\mathbf{x}$ such that

$$\begin{cases} x_1 = 0.25x_2 \\ x_2 \text{ free} \end{cases} \qquad \Leftrightarrow \qquad \mathbf{x} = x_2 \begin{bmatrix} 0.25 \\ 1 \end{bmatrix}$$

and a basis for the eigenspace for $\lambda = 0$ is $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$. Since

$$\begin{bmatrix} A - 17I & \mathbf{0} \end{bmatrix} = \begin{bmatrix} -1 & -4 & 0 \\ -4 & -16 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the solutions of $(A - 17I)\mathbf{x} = \mathbf{0}$ are the vectors $\mathbf{x}$ such that

$$\begin{cases} x_1 = -4x_2 \\ x_2 \text{ free} \end{cases} \qquad \Leftrightarrow \qquad \mathbf{x} = x_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

and a basis for the eigenspace for $\lambda = 17$ is $\left\{ \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$. Thus,

$$P = \begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 17 \end{bmatrix}.$$

□

Notice that the columns of P are orthogonal. So by scaling, we could make them orthonormal, which would change P into an orthogonal matrix U:

$$U = \begin{bmatrix} 1/\sqrt{17} & -4/\sqrt{17} \\ 4/\sqrt{17} & 1/\sqrt{17} \end{bmatrix}.$$

So actually, A can be factored as UDU^t, where U is orthogonal. If a matrix A can be factored in this way, then we say that A is *orthogonally diagonalizable*.

Example 23.1.2. *Diagonalize the matrix*

$$A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}.$$

Solution. The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 5-\lambda & -4 & -2 \\ -4 & 5-\lambda & 2 \\ -2 & 2 & 2-\lambda \end{vmatrix} \\ &= (5-\lambda)[(5-\lambda)(2-\lambda) - 4] + 4[-4(2-\lambda) + 4] - 2[-8 + 2(5-\lambda)] \\ &= (5-\lambda)(\lambda-1)(\lambda-6) + 16(\lambda-1) + 4(\lambda-1) \\ &= (\lambda-1)[(5-\lambda)(\lambda-6) + 20] \\ &= -(\lambda-1)^2(\lambda-10), \end{aligned}$$

so the eigenvalues of A are $\lambda = 1, 10$. Since

$$\begin{bmatrix} A - I & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 4 & -4 & -2 & 0 \\ -4 & 4 & 2 & 0 \\ -2 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the solutions of $(A - I)\mathbf{x} = \mathbf{0}$ are the vectors $\mathbf{x}$ such that

$$\begin{cases} x_1 = x_2 + 0.5x_3 \\ x_2, x_3 \text{ free} \end{cases} \qquad \Leftrightarrow \qquad \mathbf{x} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0.5 \\ 0 \\ 1 \end{bmatrix}$$

and a basis for the eigenspace for $\lambda = 1$ is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

Since

$$\begin{bmatrix} A-10I & \mathbf{0} \end{bmatrix} = \begin{bmatrix} -5 & -4 & -2 \\ -4 & -5 & 2 \\ -2 & 2 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 4 \\ -4 & -5 & 2 \\ -5 & -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 4 \\ 0 & -9 & 18 \\ 0 & -9 & 18 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the solutions of $(A - 10I)\mathbf{x} = \mathbf{0}$ are the vectors $\mathbf{x}$ such that

$$\begin{cases} x_1 = -2x_3 \\ x_2 = 2x_3 \\ x_3 \text{ free} \end{cases} \qquad \Leftrightarrow \qquad \mathbf{x} = x_3 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

and a basis for the eigenspace for $\lambda = 1$ is

$$\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Thus,

$$P = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}.$$

□

This time, the columns of P are not orthogonal. What is true is that the first two columns of P (the eigenvectors of A corresponding to the eigenvalue $\lambda = 1$) are orthogonal to the last column of P (the eigenvector of A corresponding to the eigenvalue $\lambda = 10$). This is not an accident. The following theorem is true.

Theorem 46. *Let A be symmetric matrix. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A corresponding to different eigenvalues, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.*

Proof. The proof of this theorem hinges on the following property of the inner product:

$$\boxed{\text{If } A \text{ is an } n \times n \text{ matrix and } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \text{ then } (A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A^t\mathbf{y}).}$$

Suppose $\mathbf{v}_1$ corresponds to an eigenvalue λ_1 and $\mathbf{v}_2$ corresponds to a different eigenvalue λ_2. Then using the above fact, the properties of the inner product, and the definition of an eigenvector, we find that

$$\lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2) = (\lambda_1\mathbf{v}_1) \cdot \mathbf{v}_2 = (A\mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot (A^t\mathbf{v}_2) = \mathbf{x}_1 \cdot (A\mathbf{v}_2) = \mathbf{x}_1 \cdot (\lambda_2\mathbf{v}_2) = \lambda_2(\mathbf{v}_1 \cdot \mathbf{v}_2).$$

Since $\lambda_1 \neq \lambda_2$, the left-most and right-most expressions can be equal only if $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. □

Going back to the previous example, we can perform the Gram-Schmidt process on the eigenvectors of A corresponding to the eigenvalue $\lambda = 1$. We obtain the orthogonal vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \\ 2 \end{bmatrix},$$

so

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}$$

is an orthogonal basis for $\mathbb{R}^3$ consisting of eigenvectors of A. Normalizing these vectors, we derive the following orthonormal basis:

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{bmatrix}, \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} \right\}.$$

Hence, A factors as UDU^t, where

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & -1/\sqrt{18} & 2/3 \\ 0 & 4/\sqrt{18} & 1/3 \end{bmatrix}$$

is an orthogonal matrix and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}.$$

Remark 23.1.3. What we just did here works in general. That is, every symmetric matrix is orthogonally diagonalizable. The converse is also true: if a matrix is orthogonallu diagonalizable, then it must be symmetric. The latter is not difficult to see. Suppose that $A = UDU^t$, where D is diagonal. Then

$$A^t = (UDU^t)^t = (U^t)^t D^t U^t = UD^t U^t = UDU^t = A,$$

that is, A is symmetric. In this chain of identities, the second and third follow from the properties of the transpose (recall Proposition 8.3.1) and the fourth is merely the observation that a diagonal matrix satisfies $D^t = D$.

23.2. **The spectrum of a symmetric matrix.** The set of eigenvalues of a matrix A is sometimes called the *spectrum of A*. The following theorem, known as the *spectral theorem*, describes the spectrum of a symmetric matrix and its eigenspaces.

Theorem 47 (Spectral Theorem)**.** *An $n \times n$ symmetric matrix A has the following properties:*

1. *A has n real eigenvalues, counting multiplicities.*
2. *The dimension of the eigenspace for an eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.*
3. *The eigenvectors corresponding to different eigenvalues are orthogonal.*
4. *A is orthogonally diagonalizable.*

Suppose that A is a symmetric matrix and let $A = UDU^t$ be its orthogonal diagonalization (so U is an orthogonal matrix and D is a digonal matrix). Then

$$A = UDU^t = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^t \\ \vdots \\ \mathbf{u}_n^t \end{bmatrix},$$

and a (not particularly difficult, but rather tedious) direct calculation shows that the product on the right side of this identity equals

$$\lambda_1(\mathbf{u}_1\mathbf{u}_1^t) + \lambda_2(\mathbf{u}_2\mathbf{u}_2^t) + \cdots + \lambda_n(\mathbf{u}_n\mathbf{u}_n^t).$$

Thus,

$$A = \lambda_1(\mathbf{u}_1\mathbf{u}_1^t) + \lambda_2(\mathbf{u}_2\mathbf{u}_2^t) + \cdots + \lambda_n(\mathbf{u}_n\mathbf{u}_n^t).$$

This formula for A is known as the *spectral decomposition of A*. It breaks A into a sum of of n terms determined by the spectrum of A. Each of the terms in the spectral decomposition is a symmetric $n \times n$ matrix of rank 1.

Example 23.2.1. *Find the spectral decomposition of the matrix A from Example 23.1.2.*

Solution. We know from earlier in the lecture that the eigenvalues are 1, 1, 10 and that an orthonormal basis of eigenvectors is

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{bmatrix}, \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} \right\}.$$

Thus, the spectral decomposition of A is

$$A = (\mathbf{u}_1 \cdot \mathbf{u}_1^t) + (\mathbf{u}_2 \cdot \mathbf{u}_2^t) + 10(\mathbf{u}_3 \cdot \mathbf{u}_3^t),$$

where

$$\mathbf{u}_1\mathbf{u}_1^t = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{u}_2\mathbf{u}_2^t = \begin{bmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{bmatrix} \begin{bmatrix} 1/\sqrt{18} & -1/\sqrt{18} & 2/9 \end{bmatrix} = \begin{bmatrix} 1/18 & -1/18 & 4/18 \\ -1/18 & 1/18 & -2/9 \\ 4/18 & -4/18 & 8/9 \end{bmatrix},$$

$$\mathbf{u}_3\mathbf{u}_3^t = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} \begin{bmatrix} -2/3 & 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 4/9 & -4/9 & -2/9 \\ -4/9 & 4/9 & 2/9 \\ -2/9 & 2/9 & 1/9 \end{bmatrix}.$$

□

24. Quadratic Forms

In this lecture, we study the only non-linear object in linear algebra: quadratic forms.

24.1. **Definition.** A *quadratic form on* $\mathbb{R}^n$ is a homogeneous quadratic polynomial in n variables, that is, a function $Q : \mathbb{R}^n \to \mathbb{R}$ of the form

$$Q(\mathbf{x}) = Q(x_1, \ldots, x_n) = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j = a_{11}x_1^2 + \cdots + a_{nn}x_n^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n + a_{23}x_2x_3 + \cdots . \tag{24.1}$$

For example,

$$Q(\mathbf{x}) = 2x_1^2 + x_3^2 + 2x_1x_3 + x_2x_3$$

is a quadratic form on $\mathbb{R}^3$ with

$$a_{11} = 2, \quad a_{12} = 0, \quad a_{13} = 2, \quad a_{22} = 0, \quad a_{23} = 1, \quad a_{33} = 1.$$

Another common way of writing a quadratic form Q is

$$Q(\mathbf{x}) = Q(x_1, \ldots, x_n) = \sum_{1 \le i \le n} a_{ii} x_i^2 + \sum_{1 \le i < j \le n} 2a_{ij} x_i x_j = a_{11}x_1^2 + \cdots + a_{nn}x_n^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n + 2a_{23}x_2x_3 + \cdots . \tag{24.2}$$

When this convention is used, the quadratic form

$$Q(\mathbf{x}) = 2x_1^2 + x_3^2 + 2x_1x_3 + x_2x_3$$

corresponds to the coefficients

$$a_{11} = 2, \quad a_{12} = 0, \quad a_{13} = 1, \quad a_{22} = 0, \quad a_{23} = 1/2, \quad a_{33} = 1.$$

It is mostly a matter of taste whether to write quadratic forms as in (24.1) or as in (24.2), but (24.2) has one slight advantage. Consider the quadratic form (24.2). It can be written as a matrix product as follows:

$$Q(\mathbf{x}) = \mathbf{x}^t A \mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

That is, Q has associated with it a symmetric matrix A, whose diagonal entries are the coefficients a_{ii} in (24.2) and whose off-diagonal entries at (i, j)th and (j, i)th positions (with $i < j$) are the coefficients a_{ij} in (24.2). The matrix A is called the *matrix of the quadratic form Q*. Here is an example.

Example 24.1.1. *Find the matrix of the quadratic form*

$$Q(\mathbf{x}) = 2x_1^2 + x_3^2 + 2x_1x_3 + x_2x_3.$$

Solution. The matrix of Q is

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0.5 \\ 1 & 0.5 & 1 \end{bmatrix}.$$

□

24.2. **Change of variables in a quadratic form.** Two of the simplest examples of quadratic forms are

$$Q(\mathbf{x}) = \|\mathbf{x}\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 \qquad \text{and} \qquad Q(x_1, x_2) = x_1^2 - x_2^2.$$

Their most important feature is that they don't have any "mixed terms". Next, we discuss how to transform any quadratic form into a quadratic form without mixed terms.

If $\mathbf{x}$ is a variable vector in $\mathbb{R}^n$, then a *change of variables* is an equation of the form

$$\mathbf{x} = P\mathbf{y} \qquad \text{or (equivalently)} \qquad \mathbf{y} = P^{-1}\mathbf{x}, \tag{24.3}$$

where P is an invertible $n \times n$ matrix. If we make such a change of variables in a quadratic form $Q(\mathbf{x}) = \mathbf{x}^t A\mathbf{x}$, we obtain

$$Q(\mathbf{x}) = \mathbf{x}^t A\mathbf{x} = (P\mathbf{y})^t A(P\mathbf{y}) = (\mathbf{y}^t P^t)A(P\mathbf{y}) = \mathbf{y}^t(P^t AP)\mathbf{y},$$

that is, the quadratic form $Q(\mathbf{x})$ with matrix A gets transformed into the quadratic form $\tilde{Q}(\mathbf{y})$ with matrix $P^t AP$.

We now recall from Lecture #23 that given a symmetric matrix A, we can always find an orthogonal matrix U and a diagonal matrix D such that $A = UDU^t$. Let's make a change of variables (24.3) with $P = U$. Then $Q(\mathbf{x})$ is transformed into the quadratic form $\tilde{Q}(\mathbf{y})$ with matrix $U^t AU = D$. That is, $\tilde{Q}(\mathbf{y})$ is the quadratic form

$$\tilde{Q}(\mathbf{y}) = \mathbf{y}^t D\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2, \tag{24.4}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. This little argument is summarized in the following theorem.

Theorem 48 (Principal axes theorem)**.** *Let A be a symmetric $n \times n$ matrix. Then there is an orthogonal change of variables, $\mathbf{x} = U\mathbf{y}$, that transforms the quadratic form $\mathbf{x}^t A\mathbf{x}$ into a quadratic form $\mathbf{y}^t D\mathbf{y}$ with no cross-product terms (i.e., a form whose matrix D is diagonal).*

Example 24.2.1. *Make a change of variables that transforms the quadratic form*

$$Q(x_1, x_2, x_3) = 5x_1^2 + 5x_2^2 + 2x_3^2 - 8x_1x_2 - 4x_1x_3 + 4x_2x_3$$

into a quadratic form with no cross-product terms.

Solution. The matrix of Q is the matrix A from Example 23.1.2. We know from Lecture #23 (see the paragraph between Theorem 46 and Remark 23.1.3) that $A = U^t DU$, where

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & -1/\sqrt{18} & 2/3 \\ 0 & 4/\sqrt{18} & 1/3 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}.$$

Thus, the change of variables $\mathbf{x} = U\mathbf{y}$ transforms $Q(\mathbf{x})$ into the quadratic form

$$\mathbf{y}^t D\mathbf{y} = y_1^2 + y_2^2 + 10y_3^2.$$

□

24.3. **Classification of quadratic forms.** Now, we will classify the quadratic forms according to the values they can attain.

Definition 24.3.1. A quadratic form Q is:

- *positive definite* if $Q(\mathbf{x}) > 0$ whenever $\mathbf{x} \neq 0$;
- *negative definite* if $Q(\mathbf{x}) < 0$ whenever $\mathbf{x} \neq 0$;
- *positive semidefinite* if $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$;
- *negative semidefinite* if $Q(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \mathbb{R}^n$;
- *indefinite* if $Q(\mathbf{x})$ assumes both positive and negative values.

It might seem at first that the question whether a quadratic form is definite, semidefinite, or indefinite is quite difficult, but in fact, given the spectrum of the matrix of Q, it is quite easy to answer this question.

Theorem 49. *Let Q be a quadratic form on $\mathbb{R}^n$ and let A be the matrix of Q. Then:*

1. *Q is positive definite, if all the eigenvalues of A are positive;*
2. *Q is negative definite, if all the eigenvalues of A are negative;*
3. *Q is positive semidefinite, if all the eigenvalues of A are non-negative;*
4. *Q is negative semidefinite, if all the eigenvalues of A are non-positive;*
5. *Q is indefinite, if A has both positive and negative eigenvalues.*

Each of the five conclusions of the theorem becomes almost obvious when we transform Q into the form (24.4). For example:

- If $\lambda_1, \ldots, \lambda_n$ are all positive, then
$$\lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 > 0$$
unless $y_1 = y_2 = \cdots = y_n = 0$. Thus, Q is positive definite.
- If $\lambda_1 = 0$ and $\lambda_2, \ldots, \lambda_n < 0$, then
$$\lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 = \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \leq 0$$
for any choice of the variables, so Q is negative semidefinite. However, Q is not negative definite, because it can attain a zero value for nonzero vectors $\mathbf{y}$: say, take $y_1 = 1$ and $y_2 = \cdots = y_n = 0$.
- If $\lambda_1 > 0$ and $\lambda_2 < 0$, the value at $\mathbf{y} = \mathbf{e}_1$ is
$$\lambda_1(1)^2 + \lambda_2(0)^2 + \cdots + \lambda_n(0)^2 = \lambda_1 > 0,$$
and the value at $\mathbf{y} = \mathbf{e}_2$ is
$$\lambda_1(0)^2 + \lambda_2(1)^2 + \cdots + \lambda_n(0)^2 = \lambda_2 < 0.$$
Thus, Q is indefinite.

Example 24.3.2. *Determine whether the quadratic form*
$$Q(x_1, x_2, x_3) = 3x_2^2 + 4x_1x_3$$
is definite, semidefinite, or indefinite.

Solution. The matrix of Q is
$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial of A is

$$\begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 3-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} = (3-\lambda)(\lambda^2-4),$$

so the eigenvalues are $3, 2, -2$. Since A has both positive and negative eigenvalues, Q is indefinite. □

Example 24.3.3. *Determine whether the quadratic form*

$$Q(x_1, x_2, x_3) = 7x_1^2 + 5x_2^2 + 9x_3^2 - 8x_1x_2 + 8x_1x_3$$

is definite, semidefinite, or indefinite.

Solution. The matrix of Q is

$$A = \begin{bmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{bmatrix}.$$

The characteristic polynomial of A is

$$\begin{aligned} \begin{vmatrix} 7-\lambda & -4 & 4 \\ -4 & 5-\lambda & 0 \\ 4 & 0 & 9-\lambda \end{vmatrix} &= (9-\lambda)\begin{vmatrix} 7-\lambda & -4 \\ -4 & 5-\lambda \end{vmatrix} + 4\begin{vmatrix} -4 & 5-\lambda \\ 4 & 0 \end{vmatrix} \\ &= (9-\lambda)(7-\lambda)(5-\lambda) - 16(9-\lambda) - 16(5-\lambda) \\ &= (9-\lambda)(7-\lambda)(5-\lambda) - 16(14-2\lambda) \\ &= (9-\lambda)(7-\lambda)(5-\lambda) - 32(7-\lambda) \\ &= (7-\lambda)\big[(9-\lambda)(5-\lambda) - 32\big] \\ &= (7-\lambda)\big(\lambda^2 - 14\lambda + 13\big), \end{aligned}$$

so the eigenvalues are $1, 7, 13$. Since all the eigenvalues of A are positive, Q is positive definite. □

Examples 24.3.2 and 24.3.3 expose the main shortcoming of Theorem 49: it requires the calculation of the eigenvalues of A, which could be difficult (and even impossible) task for large matrices. In fact, this is not such a big issue in numerical computations, where an approximation usually suffices. However, in more theoretical considerations, the need to work with a polynomial equation of a high degree can be a serious obstacle. In such situations, we can determine whether a quadratic form is definite using the following theorem.

Theorem 50 (Sylvester's criterion)**.** *Let $Q(\mathbf{x})$ be a quadratic form on $\mathbb{R}^n$ and let*

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

be its matrix. Then Q is positive definite if and only if the determinants

$$\begin{vmatrix} a_{11} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{23} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad \ldots, \quad \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

are all positive.

Remark 24.3.4. There is a variant of Sylvester's criterion that determines whether a given quadratic form is negative definite, which we omit, as it is not really needed. Indeed, if Q is negative definite, then $-Q$ is positive definite. Thus, if we want to check whether Q is negative definite, all we need to do is apply Sylvester's criterion for positive definiteness to $-Q$.

Remark 24.3.5. The advantage of Sylvester's criterion over Theorem 49 is that it involves only the coefficients of the form. On the other hand, it only applies to positive/negative definiteness.

Example 24.3.6. *Determine whether the quadratic form*

$$Q(x_1, x_2, x_3, x_4) = 4x_1^2 + 4x_2^2 + 4x_3^2 + 4x_4^2 + 2x_1x_2 + 6x_1x_3 + 2x_1x_4 + 2x_2x_3 + 6x_2x_4 + 2x_3x_4$$

is definite, semidefinite, or indefinite.

Solution. The matrix of Q is

$$A = \begin{bmatrix} 4 & 1 & 3 & 1 \\ 1 & 4 & 1 & 3 \\ 3 & 1 & 4 & 1 \\ 1 & 3 & 1 & 4 \end{bmatrix}.$$

The four determinants appearing in Sylvester's criterion are:

$$\begin{vmatrix} 4 \end{vmatrix} = 4, \qquad \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = 15, \qquad \begin{vmatrix} 4 & 1 & 3 \\ 1 & 4 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 26, \qquad \begin{vmatrix} 4 & 1 & 3 & 1 \\ 1 & 4 & 1 & 3 \\ 3 & 1 & 4 & 1 \\ 1 & 3 & 1 & 4 \end{vmatrix} = 45,$$

so the form is positive definite.

Alternatively, we could have computed the eigenvalues (they are $1, 1, 5, 9$), but that would have involved solving a polynomial equation of degree four, and hence, some trickery. □