Diophantine Approximation by Cubes of Primes and an Almost Prime

A. Kumchev

Abstract

Let $\lambda_1, \dots, \lambda_s$ be non-zero with λ_1/λ_2 irrational and let $\mathcal{S}$ be the set of values attained by the form

$$\lambda_1 x_1^3 + \cdots + \lambda_s x_s^3$$

when x_1 has at most 6 prime divisors and the remaining variables are prime. In the case $s = 4$, we establish that most real numbers are "close" to an element of $\mathcal{S}$. We then prove that if $s = 8$, $\mathcal{S}$ is dense on the real line.

1 Introduction and preliminaries

Let $\lambda_1, \dots, \lambda_s$ be non-zero real numbers with λ_1/λ_2 irrational, and let $\mathbb{P}^s$ denote the set of integer points in $\mathbb{R}^s$ all coordinates of which are prime. We will be concerned with the distribution of the values taken by the form

$$\lambda_1 x_1^3 + \cdots + \lambda_s x_s^3 \tag{1.1}$$

on $\mathbb{P}^s$. The (optimistic) conjecture is that if $s \geq 4$, they are dense on $\mathbb{R}$, but our factual knowledge on the topic is much worse. Back in 1963, W. Schwarz [13] showed that if $s \geq 9$, the values of (1.1) on $\mathbb{P}^s$ are dense, and although sharper quantitative versions of this result have been obtained (see R. C. Vaughan [16] and R. C. Baker and G. Harman [1]), it seems that reducing the minimum value of s is beyond the limit of the present methods. On the other hand, in the similar situation with the classical Waring–Goldbach problem for cubes K. F. Roth [11] showed that if one allows x to take arbitrary integer values, the equation

$$x^3 + p_1^3 + p_2^3 + p_3^3 = n \tag{1.2}$$

is solvable for almost all integer n (in the sense usually adopted in additive number theory). Let $\mathcal{P}_r$ denote the set of integers having at most r prime divisors counted with multiplicities. J. Brüdern [3] proved that if $n \equiv 4 \pmod{18}$ one can restrict the variable x in (1.2) to the set $\mathcal{P}_4$, and K. Kawada [9] replaced $\mathcal{P}_4$ by $\mathcal{P}_3$ via Chen's reversal of rôles. The main goal of the present paper is to carry Brüdern's result to Diophantine inequalities. We shall prove

Theorem 1. *Let $\lambda_1, \dots, \lambda_4$ be non-zero real numbers with λ_1/λ_2 irrational. For $\delta > 0$, let $E(N)$ denote the Lebesgue measure of the set of real numbers $\nu \in [-N, N]$ for which the inequality*

$$|\lambda_1 x_1^3 + \lambda_2 x_2^3 + \lambda_3 x_3^3 + \lambda_4 x_4^3 - \nu| < (\max |x_j|)^{-\delta} \tag{1.3}$$

has no solution in prime $x_2, \dots, x_4$ and $x_1 \in \mathcal{P}_6$. Then, for sufficiently small $\delta > 0$, there exist arbitrarily large values of N such that $E(N) \ll N^{1-\eta}$ for some $\eta > 0$ (depending at most on δ). Furthermore, if λ_1/λ_2 is also algebraic, the assertion is true for all sufficiently large N.

We can deduce from Theorem 1 the following

Theorem 2. *If $\lambda_1, \dots, \lambda_8$ are non-zero and λ_1/λ_2 is irrational, the values taken by the form*

$$\lambda_1 x_1^3 + \cdots + \lambda_8 x_8^3$$

at the points $(x_1, \dots, x_8) \in \mathcal{P}_6 \times \mathbb{P}^7$ are dense on the real line.

The effect of having almost primes with (possibly) more prime factors than in the result on the corresponding "equation problem", although undesirable, is not unexpected. For example, when dealing with the analogue of the binary Goldbach problem, R. C. Vaughan [17] managed to show that reals can be approximated by the values taken by a linear form when one of the variables is prime and the other is in $\mathcal{P}_4$. Later, using a method that draws havily on the fact that only two variables are present, G. Harman [7] succeeded in replacing $\mathcal{P}_4$ by $\mathcal{P}_3$ in Vaughan's result, but even this is weaker than Chen's theorem (all sufficiently large even integers are the sum of a prime and an element of $\mathcal{P}_2$).

Through the rest of this section we use the linear sieve to derive Theorem 1 from Propositions 1 and 2 below. The Propositions are proved in Sections 2–4 and the proof of Theorem 2 is given in Section 5.

Without loss of generality we can assume that λ_1 and ν are positive. Let a/q be a convergent to the continued fraction of λ_1/λ_2 with q being sufficiently large, and choose N so that

$$N^{3/20+6\delta+21\eta} \le q \le N^{1/2-2\delta-9\eta} \tag{1.4}$$

(this can always be done provided that δ and η are sufficiently small). Note that if λ_1/λ_2 is an algebraic irrationality, by Roth's theorem on Diophantine approximation, the denominators of two consecutive convergents to its continued fraction satisfy $q_{m+1} \ll q_m^{1+\varepsilon}$ (hereafter ε denotes a positive number that can be taken arbitrarily small). Hence, in this case *all* sufficiently large N satisfy (1.4) for some q, and so both parts of the theorem will follow, if we show that $E(N) \ll N^{1-\eta}$ whenever N satisfies (1.4).

Define P, P_1, and Q by

$$P^3 = \frac{1}{2}N, \quad 8|\lambda_2|P_1^3 = \frac{1}{4}N, \quad Q = P^{4/5},$$

and let $L = \log N$, $\tau = P^{-\delta} L^{-2}$. Also, for $N < \nu \le 2N$, define $r^*(\nu)$ as the number of solutions of

$$|\lambda_1 m^3 + \lambda_2 p_2^3 + \lambda_3 p_3^3 + \lambda_4 p_4^3 - \nu| < \tau L \tag{1.5}$$

in prime p_2, p_3, p_4 and $m \in \mathcal{P}_6$ subject to

$$P < m \le 2P, \quad P_1 < p_2 \le 2P_1, \quad Q < p_3, p_4 \le 2Q. \tag{1.6}$$

We shall prove that $r^*(\nu) > 0$ for almost all $\nu \le N$ (by the phrase "for almost all $\nu \le N$" we will mean "with the possible exceptions $\nu \in (N, 2N]$ forming a set of Lebesgue measure $O(N^{1-\eta})$").

We first bring in the function $K(x) = e^{-\pi x^2}$ to "smooth" the condition (1.5) in the definition of $r^*(\nu)$. The important properties of $K(x)$ are

$$\widehat{K}(x) := \int_{-\infty}^{\infty} K(\xi)\, e(-x\xi)\, d\xi = K(x), \tag{1.7}$$

and

$$e^{-\pi} \chi_{(-1,1)}(x) \le K(x) \le \chi_{(-1,1)}(x/\rho) + e^{-\pi\rho^2} \qquad (\rho > 0). \tag{1.8}$$

It is also convenient to weight the primes p_2, p_3, p_4 by logarithms. Thus, defining the weights

$$w(\nu; m) = \sum_{\substack{P_1 < p_2 \le 2P_1 \\ Q < p_3, p_4 \le 2Q}} \log p_2 \cdots \log p_4 \, K\left(\frac{|\lambda_1 m^3 + \lambda_2 p_2^3 + \lambda_3 p_3^3 + \lambda_4 p_4^3 - \nu|}{\tau} \right),$$

we have, by (1.8) with $\rho = L$,

$$r^*(\nu) \gg L^{-3} \sum_{\substack{P < m \le 2P \\ m \in \mathcal{P}_6}} w(\nu; m).$$

Hence, it suffices to show that, for example,

$$\sum_{\substack{P < m \le 2P \\ m \in \mathcal{P}_6}} w(\nu; m) \ge 1 \tag{1.9}$$

for almost all $\nu \le N$. Let $r(\nu)$ be the last sum with the condition $m \in \mathcal{P}_6$ omitted and $r_d(\nu)$ be the subsum of $r(\nu)$ with $m \equiv 0 \pmod{d}$. Let also X approximate $r(\nu)$, and for a squarefree d define the remainders $R(d)$ by

$$R(d) := r_d(\nu) - X/d. \tag{1.10}$$

(X will be defined explicitly by (4.3); at this point we will use only the estimate $X \asymp \tau P^{-1} Q^2 \asymp \tau N^{1/5}$.)

To prove (1.9) we will use the weighted linear sieve of G. Greaves [5, 6]. For a particular ν, (1.9) follows from the results in [5], provided that the next two conditions hold

(A1) If $\theta < \frac{1}{5}$, ξ_d *are complex numbers of modulus* ≤ 1, *and* $\mu(d)$ *is the Möbius function, then*

$$\sum_{d \leq P^{\theta}} \mu^2(d)\, \xi_d\, R(d) \ll X L^{-2}.$$

(A2) There exists a $\rho > 0$ *such that*

$$\sum_{p > P^{1/10}} r_{p^2}(\nu) \ll X^{1-\rho}.$$

The upper bound for θ in axiom (A1) is often referred to as the *level of distribution* and is closely related to the number of prime divisors of the almost prime variable. In particular, Brüdern (Lemma 1 in [3]) has $\theta < \frac{1}{3}$ in place of $\theta < \frac{1}{5}$, whence $\mathcal{P}_4$ in place of $\mathcal{P}_6$ in the final result. The reason that we cannot achieve the same level of distribution as Brüdern is that the Davenport–Heilbronn method is much more sensitive to "imparities" among the variables than the classical form of the circle method. In particular, there is no analogue (that the author is aware of) of the Vaughan's result from [18], which was an essential part of the proofs of most mean value estimates used in Brüdern's work. Instead, we are forced to rely on the weaker Lemma 9 in [4], leading to a lower level of distribution.

We shall be able to verify the axioms for almost all $\nu \leq N$ by proving the following results

Proposition 1. *Let* $\theta < \frac{1}{5}$, $D = P^{\theta}$, *and* $R(d)$ *be defined by* (1.10). *Let also* ξ_d *be complex numbers of modulus* ≤ 1. *Then, for any* $A > 0$, *the values of* $\nu \in (N, 2N]$ *for which the estimate*

$$\sum_{d \leq D} \xi_d\, R(d) \ll \tau P^{-1} Q^2 L^{-A}$$

does not hold form a set of Lebesgue measure $O(N^{1-\eta})$.

Proposition 2. *If* $\gamma > 4\delta/3$ *and* η *is sufficiently small,*

$$\int_N^{2N} \left| \sum_{p > P^{\gamma}} r_{p^2}(\nu) \right|^2 d\nu \ll \tau^2 P^{1-4\eta} Q^4.$$

Clearly Proposition 1 establishes axiom (A1) for almost all $\nu \leq N$. Also, we can deduce from Proposition 2 that the set of values of $\nu \in (N, 2N]$ for which

$$\sum_{p > P^{\gamma}} r_{p^2}(\nu) > X^{1-\eta/2}$$

has measure $\ll N^{1-\eta}$, so axiom (A2) is satisfied for almost all $\nu \leq N$ as well. Thus, once we have the propositions the proof of Theorem 1 will be completed.

2 Counting solutions of Diophantine inequalities

This section contains estimates for the number of solutions of some Diophantine inequalities. Also, in its end we prove Proposition 2. The first lemma states some estimates from [4] in the present context.

Lemma 1. *Let λ and μ be fixed real numbers with*

$$1 \ll |\lambda| \ll 1, \qquad 1 \ll |\mu| \ll 1,$$

and let Z be sufficiently large in terms of λ, μ. Denote by S_k the number of solutions of the inequality

$$|\lambda(n_1^3 - n_2^3) + \mu(m_1^3 + \cdots + m_k^3 - m_{k+1}^3 - \cdots - m_{2k}^3)| < 1/2$$

with $Z < n_i \le 2Z$, $Z^{4/5} < m_i \le 2Z^{4/5}$. Then,
(a) $S_2 \ll Z^{13/5+\varepsilon}$;
(b) $S_3 \ll Z^{19/5+\varepsilon}$.

Proof. Define

$$f(x) = \sum_{Z<n\le 2Z} e(xn^3), \quad g(x) = \sum_{Z^{4/5}<n\le 2Z^{4/5}} e(xn^3), \tag{2.1}$$

and consider the integral

$$J_k = \int_{-\infty}^{\infty} |f(\lambda x)|^2 |g(\mu x)|^{2k} \left(\frac{\sin \pi x}{\pi x}\right)^2 dx.$$

By (11.3) and (11.4) in [15], $S_k \le 2J_k$. So, the result follows from Lemmas 8 and 9 of [4] (which contain the corresponding estimates for J_k). □

Lemma 2. *Let λ, μ, and κ be fixed real numbers with*

$$1 \ll |\lambda| \ll 1, \quad 1 \ll |\mu| \ll 1, \quad 1 \ll |\kappa| \ll 1,$$

and let Z be sufficiently large in terms of λ, μ, κ. Denote by $S(\mathcal{W})$ the number of solutions of the inequality

$$|\lambda(n_1^3 - n_2^3) + \mu(w_1^3 - w_2^3) + \kappa(m_1^3 + m_2^3 - m_3^3 - m_4^3)| < 1/2$$

with $Z < n_i \le 2Z$, $Z^{4/5} < m_i \le 2Z^{4/5}$, and $w_i \in \mathcal{W}$ where $\mathcal{W}$ is a set of positive integers $\ll Z$ having cardinality W. Then,

$$S(\mathcal{W}) \ll Z^{69/20+\varepsilon} W^{3/4}.$$

Proof. Let $f(x)$ and $g(x)$ be given by (2.1), and define

$$h(x) = \sum_{w \in \mathcal{W}} e(xw^3).$$

As in the previous proof, it suffices to show that

$$J(\mathcal{W}) = \int_{-\infty}^{\infty} |f(\lambda x)|^2 |h(\mu x)|^2 |g(\kappa x)|^4 \Phi(x)\, dx \ll Z^{69/20+\varepsilon} W^{3/4}$$

where $\Phi(x) = (\sin \pi x/\pi x)^2$. By Hölder's inequality, Lemma 2.5 of [15], and Lemma 9 of [4],

$$\begin{aligned} J(\mathcal{W}) &\ll \left(\int_{-\infty}^{\infty} |f(\lambda x)|^8 \Phi(x) dx \right)^{1/12} \left(\int_{-\infty}^{\infty} |h(\mu x)|^8 \Phi(x)\, dx \right)^{1/4} \times \\ &\qquad \times \left(\int_{-\infty}^{\infty} |f(\lambda x)|^2 |g(\kappa x)|^6 \Phi(x)\, dx \right)^{2/3} \\ &\ll (Z^{5+\varepsilon})^{1/12} (Z^{21/5+\varepsilon})^{2/3} \left(\int_{-\infty}^{\infty} |h(\mu x)|^8 \Phi(x)\, dx \right)^{1/4}. \end{aligned}$$

Thus, the result follows from the inequality

$$\int_{-\infty}^{\infty} |h(\mu x)|^8 \Phi(x)\, dx \ll Z^{2+\varepsilon} W^3,$$

which one can easily derive from

$$\int_0^1 |h(x)|^8 dx \ll Z^{2+\varepsilon} W^3. \tag{2.2}$$

The proof of (2.2) follows the argument on pp. 12–13 of [15]. If b_j is the number of solutions of $w_1^3 - w_2^3 = j$ in $w_i \in \mathcal{W}$, and c_j is the number of solutions of $(w+k)^3 - w^3 = j$ in $w \in \mathcal{W}$, $|k| \ll Z$, one has

$$|h(x)|^2 = \sum_j b_j\, e(-xj) \quad \text{and} \quad |h(x)|^2 \ll \sum_j c_j\, e(xj). \tag{2.3}$$

Hence, by Parseval's identity,

$$\int_0^1 |h(x)|^4 dx \ll \sum_j b_j\, c_j.$$

Since $c_0 \ll W$ and $c_j \ll Z^{\varepsilon}$, $j \neq 0$, we now find

$$\int_0^1 |h(x)|^4 dx \ll W b_0 + Z^{\varepsilon} \sum_j b_j \ll Z^{\varepsilon} W^2. \tag{2.4}$$

For $\sum b_j \ll W^2$ and

$$b_0 = \int_0^1 |h(x)|^2 dx = W.$$

The analogues of (2.3) for the fourth power of h are

$$|h(x)|^4 = \sum_j b_j^* \, e(-xj) \quad \text{and} \quad |h(x)|^4 \ll Z \sum_j c_j^* \, e(xj) \tag{2.5}$$

where now b_j^* is the number of solutions of

$$w_1^3 + w_2^3 - w_3^3 - w_4^3 = j$$

in $w_i \in \mathcal{W}$, and c_j^* is the number of solutions of

$$3k_1k_2(3w + k_1 + k_2) = j$$

in $w \in \mathcal{W}$ and $|k_1|, |k_2| \ll Z$. Again, these satisfy

$$c_0^* \ll ZW, \quad b_0^* \ll Z^\varepsilon W^2, \quad c_j^* \ll Z^\varepsilon \ (j \neq 0), \quad \sum_j b_j^* \ll W^4$$

(the estimate for b_0^* follows from (2.4)). Hence, by (2.5) and Parseval's identity,

$$\begin{aligned} \int_0^1 |h(x)|^8 dx \ll Z \sum_j b_j^* \, c_j^* &= Z b_0^* \, c_0^* + Z \sum_{j \neq 0} b_j^* \, c_j^* \\ &\ll Z^{2+\varepsilon} W^3 + Z^{1+\varepsilon} W^4 \ll Z^{2+\varepsilon} W^3. \end{aligned}$$

The proof of (2.2) is complete. □

We are now in position to prove Proposition 2.

Proof of Proposition 2. Let

$$f(x) = \sum_{p > P^\gamma} \sum_{\substack{P < n \leq 2P \\ n \equiv 0 \pmod{p^2}}} e(xn^3), \quad g(Y; x) = \sum_{Y < p \leq 2Y} (\log p) \, e(xp^3), \tag{2.6}$$

and write

$$F(x) = f(\lambda_1 x) \, g(P_1; \lambda_2 x) \, g(Q; \lambda_3 x) \, g(Q; \lambda_4 x). \tag{2.7}$$

Then, by the Fourier inversion formula,

$$\sum_{p > P^\gamma} r_{p^2}(\nu) = \int_{-\infty}^{\infty} F(x) \, K_\tau(x) \, e(-x\nu) \, dx$$

where $K_\tau(x) = \tau K(\tau x)$.

We first show that

$$\int_N^{2N} \left| \int_{-\infty}^{\infty} F(x)\, K_\tau(x)\, e(-x\nu)\, dx \right|^2 d\nu \ll L \int_{-\infty}^{\infty} |F(x)|^2\, K_\tau(x)^2 dx. \tag{2.8}$$

Let $H = \tau^{-1}L$ and

$$J(\nu) := \int_{-H}^{H} F(x)\, K_\tau(x)\, e(-x\nu)\, dx.$$

Since the contribution of $|x| > H$ to the left side of (2.8) is negligible, we consider

$$\int_N^{2N} |J(\nu)|^2 d\nu = \int_{-H}^{H} F(x)\, K_\tau(x) \int_N^{2N} \overline{J(\nu)\, e(x\nu)}\, d\nu dx.$$

By the Cauchy–Schwarz inequality, the last integral is

$$\leq \mathcal{I}^{1/2} \left(\int_{-H}^{H} \left| \int_N^{2N} J(\nu)\, e(x\nu)\, d\nu \right|^2 dx \right)^{1/2}$$

where $\mathcal{I}$ is the integral in the right side of (2.8). Furthermore,

$$\begin{aligned} &\int_{-H}^{H} \left| \int_N^{2N} J(\nu)\, e(x\nu)\, d\nu \right|^2 dx \\ &= \int_N^{2N} \int_N^{2N} J(\nu_1)\, \overline{J(\nu_2)} \int_{-H}^{H} e(x(\nu_1 - \nu_2))\, dx\, d\nu_1 d\nu_2 \\ &\leq \int_N^{2N} \int_N^{2N} |J(\nu_1)|^2 \left| \int_{-H}^{H} e(x(\nu_1 - \nu_2))\, dx \right| d\nu_1 d\nu_2. \end{aligned}$$

Since for any $\nu_1 \in (N, 2N]$,

$$\int_N^{2N} \left| \int_{-H}^{H} e(x(\nu_1 - \nu_2))\, dx \right| d\nu_2 \ll L,$$

we obtain that

$$\int_{-H}^{H} \left| \int_N^{2N} J(\nu)\, e(x\nu)\, d\nu \right|^2 dx \ll L \int_N^{2N} |J(\nu)|^2 d\nu,$$

and (2.8) follows. Note that in the above argument it sufficed to assume that $F(x)$ is bounded by a fixed power of P, say $F(x) \ll P^{100}$.

Now, by virtue of (2.8), the proposition will follow from the estimate

$$\int_{-\infty}^{\infty} |F(x)|^2 K_\tau(x)\, dx \ll \tau P^{1-5\eta} Q^4.$$

By the inequality between the arithmetic and geometric means and another Fourier inversion, we obtain that this integral is $\ll L^3(T_3+T_4)$ where T_j is the number of solutions of

$$|\lambda_1(w_1^3-w_2^3)+\lambda_2(n_1^3-n_2^3)+\lambda_j(m_1^3+m_2^3-m_3^3-m_4^3)|<1/2$$

in integers $P_1<n_i\le 2P_1$, $Q<m_i\le 2Q$, $P<w_i\le 2P$ with w_i divisible by the square of a prime $>P^\gamma$. Thus, by Lemma 2 with $Z=P$ and $W=P^{1-\gamma}$,

$$T_j \ll P^{69/20+\varepsilon}(P^{1-\gamma})^{3/4} \ll P^{1-3\gamma/4+\varepsilon}Q^4,$$

and the result follows. □

3 Estimates for Weyl sums

Lemma 3. *Assume that $|r\alpha-b|<P^{-3/2}$ where $r\le P^{3/2}$ and $(b,r)=1$. Assume also that a_m are complex numbers of modulus ≤ 1, and define*

$$S_I := \sum_{\substack{M<m\le 2M\\ P<mn\le 2P}} a_m\, e(\alpha(mn)^3). \tag{3.1}$$

Then,

$$S_I \ll M^{1/4}P^{3/4+\varepsilon}+r^{-1/3}P^{1+\varepsilon}\left(1+P^3\,|\alpha-b/r|\right)^{-1/3}.$$

Proof. This is Lemma 6 of J. Brüdern [3]. □

Our next result is a version of Lemma 5 of A. Balog and J. Brüdern [2]. Since the underlying ideas are the same, we give only a brief sketch of the proof.

Lemma 4. *Assume that $|r\alpha-b|<P^{-3/2}$ where $r\le P^{3/2}$ and $(b,r)=1$. Assume also that $M\gg P^{2/3}$, and a_m, b_n are complex numbers of modulus ≤ 1, and define*

$$S_{II} := \sum_{\substack{M<m\le 2M\\ P<mn\le 2P}} a_m\, b_n\, e(\alpha(mn)^3). \tag{3.2}$$

Then,

$$S_{II} \ll (PM)^{1/2}+P^{1+\varepsilon}M^{-1/8}+P^{1+\varepsilon}r^{-1/6}.$$

Proof. By Cauchy's inequality and change of the order of summation and the summation variables, we obtain (cf. (3.13) in [2])

$$|S_{II}|^2 \ll PM+M\sum_{h,z}\left|\sum_m e(\alpha k m^3)\right| \tag{3.3}$$

where $k=\frac{1}{4}h(3z^2+h^2)$, h,z satisfy the conditions

$$h\le 2PM^{-1},\quad PM^{-1}\le z\le 4PM^{-1},\quad h\equiv z \pmod 2,$$

and m runs through a subinterval of $(M, 2M]$. By Lemma 3 in [2], if the sum over m is not $\ll M^{3/4+\varepsilon}$, there exist integers b_1, r_1 with

$$(b_1, r_1) = 1, \quad 1 \le r_1 \le M^{3/4-\varepsilon}, \quad |r_1 \alpha k - b_1| < M^{-9/4-\varepsilon}, \tag{3.4}$$

and the estimate

$$\sum_m e(\alpha k m^3) \ll M^{1+\varepsilon} r_1^{-1/3}$$

holds. However, (3.4) and the assumptions of the lemma imply

$$|kbr_1 - b_1 r| \le rM^{-9/4-\varepsilon} + kr_1 P^{-3/2} \ll P^{3/2} M^{-9/4-\varepsilon} \ll M^{-\varepsilon},$$

so that $r_1 = r/(k, r)$. Therefore, in all the cases,

$$\sum_m e(\alpha k m^3) \ll M^{3/4+\varepsilon} + M^{1+\varepsilon} \left(\frac{(k, r)}{r} \right)^{1/3}. \tag{3.5}$$

Substitution of (3.5) into (3.3) and (a simplified version of) the summation argument leading to (3.17) in [2] complete the proof. □

Lemma 5. *Let $g(\alpha) = g(P; \alpha)$ be given by (2.6). Let also $0 < \rho < \frac{1}{12}$ and assume that $|r\alpha - b| < P^{-3/2}$ where $r \le P^{3/2}$ and $(b, r) = 1$. Then,*

$$g(\alpha) \ll P^{1-\rho+\varepsilon} + P^{1+\varepsilon} r^{-1/6}.$$

Proof. Using Heath-Brown's identity (Lemma 3 in [8]), we can decompose $g(\alpha)$ as the linear combination of $O(L^{10})$ sums of the forms S_I with $M \ll P^{1/2+\rho}$ and S_{II} with $P^{2/3} \ll M \ll P^{1-2\rho}$ (cf. (3.1) and (3.2)). To complete the proof, we estimate the sums of type S_I via Lemma 3 and the sums of type S_{II} via Lemma 4. □

4 Proof of Proposition 1

By the Fourier inversion formula,

$$\sum_{d \le D} \xi_d \, r_d(\nu) = \int_{-\infty}^{\infty} F(x) \, K_\tau(x) \, e(-x\nu) \, dx$$

where $K_\tau(x) = \tau K(\tau x)$ and $F(x)$ is defined by (2.7) with $g(Y; x)$ given by (2.6) and

$$f(x) = \sum_{d \le D} \sum_{P < dn \le 2P} \xi_d \, e(x(dn)^3).$$

Set $\omega = D^{-1} P^{-2-\varepsilon}$, $H = \tau^{-1} L$, and define the sets

$$\mathfrak{M} = (-\omega, \omega), \quad \mathfrak{m} = \{x : \; \omega \le |x| \le H\}, \quad \mathfrak{t} = \{x : \; |x| > H\}.$$

The proposition will follow from the estimates

$$\int_{\mathfrak{M}} F(x)\, K_\tau(x)\, e(-x\nu)\, dx - X \sum_{d \le D} \xi_d/d \ll \tau P^{-1} Q^2 L^{-A}, \tag{4.1}$$

$$\int_N^{2N} \left| \int_{\mathfrak{m}} F(x)\, K_\tau(x)\, e(-x\nu)\, dx \right|^2 d\nu \ll \tau^2 P^{1-4\eta} Q^4, \tag{4.2}$$

$$\int_{\mathfrak{t}} F(x)\, K_\tau(x)\, e(-x\nu)\, dx \ll 1.$$

Observe that the last inequality follows momentarily from the choice of H and the properties of $K(x)$, so we need to consider only (4.1) and (4.2). We also need to finally define X. Let

$$I(Y;x) = \int_Y^{2Y} e(xt^3)\, dt,$$

and $g(x) = g(P_1;x)$, $h(x) = g(Q;x)$, $I_0(x) = I(P;x)$, $I_1(x) = I(P_1;x)$, $I_2(x) = I(Q;x)$. We set

$$X = \int_{-\infty}^{\infty} I_0(\lambda_1 x) I_1(\lambda_2 x)\, I_2(\lambda_3 x)\, I_2(\lambda_4 x)\, K_\tau(x)\, e(-x\nu)\, dx \tag{4.3}$$

and

$$F_1(x) = I_0(\lambda_1 x) I_1(\lambda_2 x)\, I_2(\lambda_3 x)\, I_2(\lambda_4 x) \sum_{d \le D} \xi_d/d.$$

Note that by the choice of P, P_1, Q, we have $X \asymp \tau P^{-1} Q^2$.

We shall show that

$$\int_{\mathfrak{M}} |F(x) - F_1(x)|\, dx \ll P^{-1} Q^2 L^{-A}; \tag{4.4}$$

then (4.1) will follow in view of the estimates

$$I_0(x), I_1(x) \ll P^{-2}|x|^{-1}, \quad I_2(x) \ll Q^{-2}|x|^{-1}.$$

The proof of (4.4) is a standard major arc treatment. We will use the mean value estimates

$$\int_{-\omega}^{\omega} |f(\lambda_1 x)|^2 dx \ll P^{-1} L^4, \quad \int_{-\omega}^{\omega} |g(\lambda_2 x)|^2 dx \ll P^{-1} L^2, \tag{4.5}$$

$$\int_{-\omega}^{\omega} |I(Y;x)|^2 dx \ll P^{-2} Y L, \tag{4.6}$$

$$\int_{-\omega}^{\omega} |g(\lambda_2 x) - I_1(\lambda_2 x)|^2 dx \ll P^{-1} L^{-2A-1}, \tag{4.7}$$

as well as the approximate formulas

$$f(x) = I_0(x) \sum_{d \le D} \xi_d/d + O(D), \tag{4.8}$$

$$h(x) = I_2(x) + O\left(Q \exp(-(\log Q)^{1/5})\right) \tag{4.9}$$

valid if $|x| \ll \omega$. From these, (4.5) and (4.6) are easy, (4.8) follows via Poisson summation (Lemma 4.2 in [15]), and (4.9) can be proven by repeating the argument on pp. 301–303 in [14] (which is based on the approximate formula for $\psi(t)$ and zero-density estimates for $\zeta(s)$). Hence, to complete the proof of (4.4) we need to establish (4.7). Since, for $|x| \ll \omega$,

$$v(x) := \sum_{P_1 < n \le 2P_1} e(xn^3) = I_1(x) + O(1),$$

it suffices to show that

$$\int_{\mathfrak{M}} |g(x) - v(x)|^2 dx \ll P^{-1} L^{-2A-1} \tag{4.10}$$

(where ω should really be $\lambda_2\omega$). Defining

$$b(n) = \begin{cases} \log p & , \ n = p^3, \\ 0 & , \ \text{otherwise}, \end{cases} \qquad c(n) = \begin{cases} 1 & , \ n = m^3, \\ 0 & , \ \text{otherwise}, \end{cases}$$

we can rewrite the left-hand side of (4.10) as

$$\int_{\mathfrak{M}} \left| \sum_{P_1^3 < n \le 8P_1^3} (b(n) - c(n)) e(xn) \right|^2 dx,$$

and, by Lemma 1.9 in [10], this integral is

$$\ll \omega^2 \int_{-\infty}^{\infty} \left| \sum_{\substack{P_1^3 < n \le 8P_1^3 \\ |x-n| < (2\omega)^{-1}}} (b(n) - c(n)) \right|^2 dx. \tag{4.11}$$

Observe that the last sum vanish unless $x \in [P_1^3 - (2\omega)^{-1}, 8P_1^3 + (2\omega)^{-1}]$. We split these values of x into three intervals:

$$\mathcal{I}_1 : \left|x - P_1^3\right| \le (2\omega)^{-1}, \quad \mathcal{I}_2 : \left|x - 8P_1^3\right| \le (2\omega)^{-1},$$

and

$$\mathcal{I}_3 = \left(P_1^3 + (2\omega)^{-1}, 8P_1^3 - (2\omega)^{-1}\right).$$

By the trivial estimate, the contribution of $\mathcal{I}_1$ and $\mathcal{I}_2$ to (4.11) is

$$\ll \omega^{-1} P^{-4+\varepsilon} \ll P^{-1-\varepsilon},$$

and the contribution of $\mathcal{I}_3$ is

$$\begin{aligned} \ll & \omega^2 \int_{\mathcal{I}_3} \Big| \vartheta\left(\sqrt[3]{x + (2\omega)^{-1}}\right) - \vartheta\left(\sqrt[3]{x - (2\omega)^{-1}}\right) \\ & \qquad - \left(\sqrt[3]{x + (2\omega)^{-1}} - \sqrt[3]{x - (2\omega)^{-1}}\right) + O(L) \Big|^2 dx \\ \ll & (\omega P)^2 \int_{P_1}^{2P_1} \max_{0 \le u \le U} |\vartheta(t+u) - \vartheta(t) - u|^2 \, dt + P^{-1-\varepsilon} \end{aligned}$$

where $U \asymp \omega^{-1}P^{-2}$ and $\vartheta(y) = \sum_{p\le y} \log p$. By Lemma 7 in [12], the last integral is $O(U^2PL^{-B})$ whenever $P_1^{1/6+\varepsilon} < U \le P_1$, whence (4.10) follows.

We now turn to (4.2). By (2.8) (see the remark after the end of its proof), it can be obtained from

$$\int_{\mathfrak{m}} |F(x)|^2\, K_\tau(x)\, dx \ll \tau P^{1-4\eta} Q^4 L^{-1},$$

which will follow if we establish that for any $\mu \asymp 1$,

$$\int_{\mathfrak{m}} |f(\lambda_1 x)\, g(\lambda_2 x)\, h(\mu x)^2|^2 K_\tau(x)\, dx \ll \tau P^{1-4\eta-\varepsilon} Q^4. \tag{4.12}$$

By Dirichlet's theorem on Diophantine approximation, for any $x \in \mathfrak{m}$ one can find integers a_1, q_1, a_2, q_2 such that

$$\left|\lambda_i x - \frac{a_i}{q_i}\right| < \frac{1}{q_i P^{3/2}} \qquad , \ i = 1, 2, \tag{4.13}$$

$1 \le q_i \le P^{3/2}$, $(a_i, q_i) = 1$ (and hence, $|a_i| \ll q_i H$). Let

$$\begin{aligned}
\mathfrak{m}_1 &= \left\{x \in \mathfrak{m} :\ q_1 > \tau^{-6} P^{24\eta+24\varepsilon}\right\},\\
\mathfrak{m}_2 &= \left\{x \in \mathfrak{m}\setminus\mathfrak{m}_1 :\ \left|\lambda_1 x - \frac{a_1}{q_1}\right| > \frac{P^{6\eta+6\varepsilon}}{q_1(\tau^{1/2}Q)^3}\right\},\\
\mathfrak{m}_3 &= \mathfrak{m}\setminus(\mathfrak{m}_1\cup\mathfrak{m}_2).
\end{aligned}$$

By Lemma 3,

$$f(\lambda_1 x) \ll D^{1/4} P^{3/4+\varepsilon} + \Phi(x)$$

where

$$\Phi(x) = q_1^{-1/3} P^{\varepsilon} \min\left(P, |\lambda_1 x - a_1/q_1|^{-1/3}\right).$$

Note that, using the restriction on D and choosing δ, η, and ε sufficiently small, we can always ensure that $D^{1/4}P^{3/4+\varepsilon} \ll \tau^{1/2}QP^{-2\eta-\varepsilon}$; also for $x \in \mathfrak{m}_2$, we have $\Phi(x) \ll \tau^{1/2}QP^{-2\eta-\varepsilon}$. Hence,

$$\begin{aligned}
&\int_{\mathfrak{m}_1\cup\mathfrak{m}_2} |f(\lambda_1 x)\, g(\lambda_2 x)\, h(\mu x)^2|^2 K_\tau(x)\, dx\\
&\ll \tau Q^2 P^{-4\eta-2\varepsilon} \int_{-\infty}^{\infty} |g(\lambda_2 x)\, h(\mu x)^2|^2 K_\tau(x)\, dx\\
&\quad + \int_{\mathfrak{m}_1} |\Phi(x)\, g(\lambda_2 x)\, h(\mu x)^2|^2 K_\tau(x)\, dx.
\end{aligned} \tag{4.14}$$

Considering the underlying Diophantine inequality, we can estimate the first term in the right side of (4.14) via Lemma 1(a); the resulting contribution to the final estimate is $\ll \tau P^{1-4\eta-\varepsilon}Q^4$. By Hölder's inequality, the second term is

$$\ll \left(\int_{\mathfrak{m}_1} \Phi(x)^8 K_\tau(x)\, dx\right)^{1/4} \left(\int_{-\infty}^{\infty} |g(\lambda_2 x)\, h(\mu x)|^4 K_\tau(x)\, dx\right)^{1/4} \times$$
$$\times \left(\int_{-\infty}^{\infty} |g(\lambda_2 x)\, h(\mu x)^3|^2 K_\tau(x)\, dx\right)^{1/2}.$$

The first integral is easily seen to be $\ll \tau^4 P^{5-16\eta-16\varepsilon}$, and by considering the underlying inequalities, the second and third integrals can be estimated via Lemmas 2 and 1(b) respectively. Thus, the contribution of the second term in (4.14) is also $\ll \tau P^{1-4\eta-\varepsilon} Q^4$, and so

$$\int_{\mathfrak{m}_1\cup\mathfrak{m}_2} |f(\lambda_1 x)\, g(\lambda_2 x)\, h(\mu x)^2|^2 K_\tau(x)\, dx \ll \tau P^{1-4\eta-\varepsilon} Q^4. \tag{4.15}$$

Now, let $P^{-\rho} = \tau^{3/2} P^{-3/40-6\eta-6\varepsilon}$ and suppose for a moment that

$$g(\lambda_2 x) \ll P^{1-\rho+\varepsilon} \qquad \text{for all } x \in \mathfrak{m}_3. \tag{4.16}$$

Then,

$$\begin{aligned}&\int_{\mathfrak{m}_3} |f(\lambda_1 x)\, g(\lambda_2 x)\, h(\mu x)^2|^2 K_\tau(x)\, dx \\ \ll& P^{2-2\rho} \left(\int_{\mathfrak{m}_3} K_\tau(x)\, dx \right)^{1/4} \left(\int_{-\infty}^{\infty} |f(\lambda_1 x)\, h(\mu x)|^4 K_\tau(x)\, dx \right)^{1/4} \times \\ &\times \left(\int_{-\infty}^{\infty} |f(\lambda_1 x)\, h(\mu x)^3|^2 K_\tau(x)\, dx \right)^{1/2}.\end{aligned}$$

Again, the second and third integrals can be estimated via Lemmas 2 and 1(b), and the first one is

$$\ll \tau |\mathfrak{m}_3| \ll \tau^{-7.5} Q^{-3} P^{30\eta+31\varepsilon}.$$

Thus, by the choice of ρ,

$$\int_{\mathfrak{m}_3} |f(\lambda_1 x)\, g(\lambda_2 x)\, h(\mu x)^2|^2 K_\tau(x)\, dx \ll \tau P^{1-4\eta-\varepsilon} Q^4, \tag{4.17}$$

provided that (4.16) holds.

Assume that (4.16) fails. Then, by Lemma 5, we must have in (4.13)

$$q_1 \le \tau^{-6} P^{24\eta+24\varepsilon}, \qquad q_2 \le \tau^{-9} P^{9/20+36\eta+31\varepsilon},$$

and also $|q_1\lambda_1 x - a_1| < (\tau^{1/2} Q)^{-3} P^{6\eta+6\varepsilon}$ (since $x \in \mathfrak{m}_3$). Hence,

$$\begin{aligned}&|a_2 q_1 (\lambda_1/\lambda_2) - a_1 q_2| \\ =& \left| \frac{a_2/q_2}{\lambda_2 x} q_1 q_2 \left(\lambda_1 x - \frac{a_1}{q_1} \right) - \frac{a_1/q_1}{\lambda_2 x} q_1 q_2 \left(\lambda_2 x - \frac{a_2}{q_2} \right) \right| \\ \ll& q_2 (\tau^{1/2} Q)^{-3} P^{6\eta+6\varepsilon} + q_1 P^{-3/2} \ll \tau^{-6} P^{-3/2+24\eta+24\varepsilon} = o(q^{-1})\end{aligned}$$

and

$$a_2 q_1 \ll q_1 q_2 H \ll \tau^{-16} P^{9/20+60\eta+56\varepsilon} = o(q)\ .$$

But, by Legendre's law of best approximation, if δ, η, and ε are sufficiently small, the last two inequalities cannot hold simultaneously (note that $a_1 a_2 \neq 0$ for $x \in \mathfrak{m}_3$). Therefore, (4.16) is true and (4.12) follows from (4.15), (4.17). □

5 Proof of Theorem 2

We adopt all the notation set in the proof of Theorem 1. Also, let $\rho(\nu)$ denote the number of solutions of the inequality

$$|\lambda_5 p_5^3 + \cdots + \lambda_8 p_8^3 - \nu| < \tau$$

in primes $P < p_5, p_6 \le 2P$, $Q < p_7, p_8 \le 2Q$, and let $N_1 = \mu N$ where μ is a constant sufficiently large in terms of $\lambda_5, \ldots, \lambda_8$. Then, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\operatorname{meas}\{\nu : |\nu| \le N_1,\ \rho(\nu) > 0\} &\ge \left(\int_{-N_1}^{N_1} \rho(\nu)\, d\nu \right)^2 \left(\int_{-N_1}^{N_1} \rho^2(\nu)\, d\nu \right)^{-1} \\ &\gg \tau^2 P^4 Q^4 L^{-8} \left(\int_{-N_1}^{N_1} \rho^2(\nu)\, d\nu \right)^{-1}.\end{aligned}$$

Also, similarly to Proposition 2, we can prove that

$$\int_{-N_1}^{N_1} \rho^2(\nu)\, d\nu \ll \tau P^{1+\varepsilon} Q^4,$$

whence

$$\operatorname{meas}\{\nu : |\nu| \le N_1,\ \rho(\nu) > 0\} \gg \tau N^{1-\varepsilon}. \tag{5.1}$$

Now, if

$$|\lambda_1 m^3 + \lambda_2 p_2^3 + \cdots + \lambda_8 p_8^3 - \nu| < \tau L$$

is not solvable in $m \in \mathcal{P}_6$ and prime $p_2, \ldots, p_8$ as above, the set

$$\{\nu - \lambda_5 p_5^3 - \cdots - \lambda_8 p_8^3 - \theta\tau :\ P < p_5, p_6 \le 2P, Q < p_7, p_8 \le 2Q, |\theta| < 1\}$$

must be contained in the exceptional set considered in Theorem 1, so that its measure is $\ll N^{1-\eta}$. On the other hand, by (5.1), this measure is $\gg N^{1-\delta-2\varepsilon}$. To complete the proof it remains to observe that if δ and ε are sufficiently small, one can choose the number η in Theorem 1 so that $\eta > \delta + 2\varepsilon$. □

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Department of Mathematics
University of South Carolina
Columbia, SC 29208
U.S.A.

koumtche@math.sc.edu