The k-free Divisor Problem

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Abstract

Let $d^{(k)}(n)$ be the number of k-free divisors of n, and let $D^{(k)}(x)$ be the counting function of $d^{(k)}(n)$. We improve on the known estimates for the error term in the asymptotic formula for $D^{(3)}(x)$ under the assumption of the Riemann Hypothesis. We also obtain an unconditional asymptotic formula for $D^{(k)}(x+y) - D^{(k)}(x)$, $k = 2, 3$, for small y.

1991 *Mathematics Subject Classification.* 11N25, 11N37.
Key words. k-free numbers, divisor problems.

1 Introduction

An integer n is called k-free if for each prime p, p^k does not divide n. Let $d^{(k)}(n)$ denote the number of k-free divisors of the positive integer n, and let

$$D^{(k)}(x) = \sum_{n \le x} d^{(k)}(n).$$

The expected asymptotic formula for $D^{(k)}(x)$ is

$$D^{(k)}(x) = \frac{1}{\zeta(k)} x \log x + \left(\frac{2\gamma - 1}{\zeta(k)} - \frac{k\zeta'(k)}{\zeta^2(k)} \right) x + \Delta^{(k)}(x). \tag{1}$$

Here γ is Euler's constant, $\zeta(s)$ is the Riemann zeta function, and $\Delta^{(k)}(x)$ denotes the error term. The study of $D^{(k)}(x)$ started in 1874 when Mertens [8] proved that

$$\Delta^{(2)}(x) \ll x^{1/2} \log x.$$

Then, in 1932, O. Perron [11] considered the general case and established the estimates

$$\Delta^{(k)}(x) \ll \begin{cases} x^{1/2} & \text{if } k = 2, \\ x^{1/3} & \text{if } k = 3, \\ x^{33/100} & \text{if } k \ge 4. \end{cases}$$

In the case $k \ge 4$, a simple summation argument shows that any result of the form

$$\sum_{n \le x} d(n) = x \log x + (2\gamma - 1)x + O(x^\alpha), \tag{2}$$

where $d(n)$ stands for the number of (unrestricted) divisors of n, implies

$$\Delta^{(k)}(x) \ll x^{\alpha} \log x.$$

Since it is unlikely that one is able to obtain in (1) a better error term than in (2), this observation settles the case $k \geq 4$, making it a trivial corollary of the classical problem (2). (Note that, by an Ω-theorem of G. H. Hardy, one cannot do better than $\frac{1}{4}$ in (2).)

The situation in the case $k \leq 3$ is quite different. Although some improvement over Perron's results is possible (see [12, 13] for details), one cannot prove

$$\Delta^{(k)}(x) \ll x^{1/k-\delta} \qquad (k = 2, 3)$$

for any fixed $\delta > 0$ unless some substantial progress is made in the study of the zerofree region for $\zeta(s)$. It is reasonable, however, to try to improve the estimates for $\Delta^{(2)}(x)$ and $\Delta^{(3)}(x)$ by assuming the Riemann Hypothesis. Such results were initially given in [10, 12, 13]. In the case $k = 2$, R. C. Baker [1, 2] showed recently that following the approach from a well-known work of H. L. Montgomery and R. C. Vaughan [9] one can improve upon the previous results. He proved that if the Riemann Hypothesis is true,

$$\Delta^{(2)}(x) \ll x^{4/11+\varepsilon}$$

for any $\varepsilon > 0$. Our first result is a similar theorem in the case $k = 3$.

Theorem 1. *If the Riemann Hypothesis is true, one has*

$$\Delta^{(3)}(x) \ll x^{27/85+\varepsilon}$$

for any $\varepsilon > 0$.

The best published result is due to W. G. Nowak and M. Schmeier [10]. They derive the exponent $15/46 = 0.326086\ldots$ from a more general result, which currently gives $25/77 = 0.324675\ldots$ For comparison, $27/85 = 0.317647\ldots$ and the current exponent in (2) due to M. N. Huxley [7] is $23/73 = 0.315068\ldots$

Another goal of the present paper is to obtain some unconditional non-trivial information about $D^{(k)}(x)$. As the history of other similar problems suggests, one should be able to get unconditionally a "short interval" result, i.e., an asymptotic formula for $D^{(k)}(x+y) - D^{(k)}(x)$ for some y that is substantially less than $x^{1/k}$. Indeed, an argument similar to the one given by D. R. Heath-Brown [6] allows us to establish the following

Theorem 2. *Let* $k \geq 2$ *and let* $D(x)$ *denote the left-hand side of* (2). *Assume that* $\theta_0 \in (\frac{1}{4}, 1]$ *has the property that for any fixed* $\theta > \theta_0$, *one can find an* $\varepsilon = \varepsilon(\theta) > 0$ *so that the asymptotic formula*

$$D(x+y) - D(x) = y\left(\log x + 2\gamma + O\left(x^{-\varepsilon}\right)\right) \tag{3}$$

holds uniformly for y satisfying $x^\theta < y \le x$. Then for any fixed $\theta > \theta_0$, one can also find a $\delta = \delta(\theta) > 0$ so that

$$D^{(k)}(x+y) - D^{(k)}(x) = y\left(\frac{1}{\zeta(k)}\log x + \frac{2\gamma}{\zeta(k)} - \frac{k\zeta'(k)}{\zeta^2(k)} + O\left(x^{-\delta}\right)\right) \qquad (4)$$

holds uniformly for y satisfying $x^\theta < y \le x$.

The restriction $\theta_0 > \frac{1}{4}$ can be weakened, and this would be beneficial if we had at our disposal (3) for intervals shorter than $x^{1/4}$. However, it seems that the only method to date for satisfying the assumption (3) is by referring to (2), and hence, it is not likely that we will be able to make use of the stronger version in the near future.

By referring to M. N. Huxley's result [7] on the error term in Dirichlet's divisor problem, one can satisfy (3) with $\theta_0 = 23/73$, and therefore, obtain

Corollary 1. *The asymptotic formula* (4) *holds uniformly in $y \in (x^\theta, x]$, for any $\theta > 23/73$.*

Also, since the proof of Theorem 2 is elementary, by referring to I. M. Vinogradov's elementary approach [5, Chapter 8], one can satisfy (3) with $\theta_0 = 1/3$, and therefore, obtain an elementary proof of the following

Corollary 2. *The asymptotic formula* (4) *holds uniformly in $y \in (x^\theta, x]$, for any $\theta > 1/3$.*

Note that Corollary 1 is of interest only for $k = 2$ and 3, and Corollary 2 only for $k = 2$. In these cases, however, we obtain results which do not follow directly from (1) even under the Riemann Hypothesis.

2 Proof of Theorem 1

Let $\varepsilon < 0.001$ be given, and let $\eta = \varepsilon^2$. Let also $a_k(n)$ be the characteristic function of the k-free numbers, so that

$$a_k(n) = \sum_{d^k|n} \mu(d)$$

and one has

$$D^{(3)}(x) = \sum_{mn \le x} a_3(n) = \sum_{mnd^3 \le x} \mu(d).$$

We split this sum into two parts: $D_1^{(3)}(x)$ being the sum over $mnd^3 \le x$ with $d \le y$, and $D_2^{(3)}(x)$ the sum over $mnd^3 \le x$ with $d > y$; here $1 \le y \le x^{1/3}$ is a parameter to be chosen later.

For a complex $s = \sigma + it$, define the functions

$$f_1(s) = \sum_{n \le y} \mu(n) n^{-s} \quad \text{and} \quad f_2(s) = \zeta^{-1}(s) - f_1(s).$$

Referring to the asymptotic formula for $D(x)$,

$$D(x) = \operatorname{Res}(\zeta^2(s)x^s/s, 1) + \Delta(x),$$

we get

$$\begin{aligned} D_1^{(3)}(x) &= \sum_{m \le y} \mu(m)\, D\left(\frac{x}{m^3}\right) \\ &= \sum_{m \le y} \mu(m) \left(\operatorname{Res}\left(\frac{\zeta^2(s)x^s}{sm^{3s}}, 1\right) + \Delta\left(\frac{x}{m^3}\right)\right) \\ &= \operatorname{Res}\left(\zeta^2(s) f_1(3s) x^s/s, 1\right) + \sum_{m \le y} \mu(m)\, \Delta\left(\frac{x}{m^3}\right). \end{aligned} \tag{5}$$

Now, consider $D_2^{(3)}(x)$. We have

$$D_2^{(3)}(x) = \sum_{\substack{nm^3 \le x \\ m > y}} d(n)\, \mu(m),$$

so, using Perron's formula [14, Lemma 3.19], we find

$$D_2^{(3)}(x) = \frac{1}{2\pi i} \int_{2-ix^2}^{2+ix^2} \zeta^2(s)\, f_2(3s)\, \frac{x^s}{s}\, ds + O(1). \tag{6}$$

On assuming the Riemann Hypothesis, we can integrate the function $F(s) = \zeta^2(s) f_2(3s) x^s/s$ along the boundary of the rectangle having its vertices at the points $2 \pm ix^2$, $\frac{1}{2} + \eta \pm ix^2$. Since the only singularity of $F(s)$ inside this contour is the pole at $s = 1$, we deduce from (6) that

$$D_2^{(3)}(x) = \operatorname{Res}\left(\zeta^2(s) f_2(3s) x^s/s, 1\right) + O(|I_1| + |I_2| + |I_3| + 1), \tag{7}$$

where

$$I_{1,2} = \int_{\frac{1}{2}+\eta\pm ix^2}^{2\pm ix^2} F(s)\, ds \qquad \text{and} \qquad I_3 = \int_{\frac{1}{2}+\eta-ix^2}^{\frac{1}{2}+\eta+ix^2} F(s)\, ds.$$

We now refer to Theorems 14.2 and 14.25 from [14]; according to them if the Riemann Hypothesis is assumed, for $\sigma \ge 1/2 + \eta$, $|s - 1| \ge \eta$, we have

$$f_2(s) \ll y^{1/2-\sigma+\eta}(|t| + 1)^\eta \quad \text{and} \quad \zeta(s) \ll (|t| + 1)^\eta.$$

Using these estimates, we easily get

$$|I_1|, |I_2| \ll 1 \qquad \text{and} \qquad |I_3| \ll x^{1/2+7\eta} y^{-1}. \tag{8}$$

Substituting (8) into (7), we obtain

$$D_2^{(3)}(x) = \operatorname{Res}\left(\zeta^2(s) f_2(3s) x^s/s, 1\right) + O(x^{1/2+7\eta} y^{-1} + 1),$$

and hence, by (5),

$$D^{(3)}(x) = \mathrm{Res}\left(\frac{\zeta^2(s)\,x^s}{s\zeta(3s)}, 1\right) + \sum_{m\le y} \mu(m)\,\Delta\left(\frac{x}{m^3}\right) + O(x^{1/2+7\eta}y^{-1}).$$

At this point we choose $y = x^{31/170}$, so the second error term is admissible and it suffices to show that for any M, $1 \le M \le y$,

$$\sum_{M<m\le 2M} \mu(m)\,\Delta\left(\frac{x}{m^3}\right) \ll x^{27/85+\varepsilon/2}. \tag{9}$$

If $M \le x^{4/85}$, by M. N. Huxley's result [7], we have

$$\begin{aligned}\sum_{M<m\le 2M} \mu(m)\,\Delta\left(\frac{x}{m^3}\right) &\ll \sum_{M<m\le 2M} \left(\frac{x}{m^3}\right)^{23/73+\eta} \\ &\ll x^{23/73+\eta} M^{4/73} \ll x^{27/85+\varepsilon/2}.\end{aligned}$$

When M is larger, we will prove (9) using exponential sums. We first refer to the approximate formula for $\Delta(u)$ [14, (12.4.4)]

$$\begin{aligned}\Delta(u) =& \frac{u^{1/4}}{\pi\sqrt{2}} \sum_{n=1}^{K} \frac{d(n)}{n^{3/4}} \cos(4\pi\sqrt{nu} - \pi/4) \\ &+ O\left(u^{-1/4} + (T^2u^{-1})^{\eta} + u^{1+\eta}T^{-1}\right)\end{aligned} \tag{10}$$

where K is an integer and

$$T^2 = 4\pi^2 u(K + 1/2).$$

Applying (10) with $u = x/m^3$ and $K = x^{31/85}M^{-1}$, we find that

$$\begin{aligned}&\sum_{M<m\le 2M} \mu(m)\,\Delta\left(\frac{x}{m^3}\right) \\ \ll& x^{1/4}\log x \left| \sum_{M<m\le 2M}\ \sum_{N<n\le 2N} \frac{\mu(m)}{m^{3/4}}\,\frac{d(n)}{n^{3/4}}\, e\left(2\sqrt{\frac{nx}{m^3}}\right)\right| + x^{27/85+\eta} \\ \ll& \frac{x^{1/4+\eta}}{(MN)^{3/4}} \left| \sum_{M<m\le 2M}\ \sum_{N<n\le 2N} a_m\, b_n\, e\left(2\sqrt{\frac{nx}{m^3}}\right)\right| + x^{27/85+\eta}\end{aligned} \tag{11}$$

where the coefficients a_m and b_n are $\ll 1$ and

$$x^{4/85} \le M \le x^{31/170}, \quad 1 \le N \le x^{31/85}M^{-1}.$$

Observe that if $MN \le x^{23/85}$, the trivial estimate for the last exponential sum establishes (9), so we need to consider only the values of M, N satisfying

$$x^{4/85} \le M \le x^{31/170}, \quad x^{23/85} \le MN \le x^{31/85}. \tag{12}$$

Applying [1, Theorem 2] with $(M, M_1, M_2) = (N, 1, M)$, $X = \sqrt{xN/M^3}$, and $(\kappa, \lambda) = (\frac{2}{7}, \frac{4}{7})$, we get

$$\begin{aligned}\left| \sum_{M<m\le 2M} \sum_{N<n\le 2N} a_m\, b_n\, e\left(2\sqrt{\frac{nx}{m^3}}\right)\right| \\ \ll x^{\eta}(MN)^{3/4}\left((NM^{-1})^{1/4} + x^{1/18}(NM^{-1})^{1/36}\right).\end{aligned} \tag{13}$$

If $NM^{-1} \ll x^{1/4}$, the last expression is $\ll x^{1/16+\eta}(MN)^{3/4}$, and (9) follows from (11) and (13). Otherwise, the term $x^{1/18}(NM^{-1})^{1/36}$ on the right-hand side of (13) is superfluous, and since the conditions (12) imply

$$(NM^{-1})^{1/4} \ll ((MN)M^{-2})^{1/4} \ll x^{23/340},$$

again, we can derive (9) from (11) and (13). □

3 Proof of Theorem 2

We start with the identity

$$D^{(k)}(x+y) - D^{(k)}(x) = \sum_{x<mn\le x+y} a_k(n) = \sum_{x<mnd^k\le x+y} \mu(d). \tag{14}$$

Let δ satisfy

$$0 < \delta < \min\left(\frac{\varepsilon}{2}, \frac{1}{2}\left(\theta - \frac{1}{4}\right)\right), \tag{15}$$

and let $U(x)$ and $V(x)$ be the parts of the last sum in (14) with $d \le x^{3\delta}$ and $d > x^{3\delta}$, respectively. We have

$$U(x) = \sum_{d\le x^{3\delta}} \mu(d)\left(D\left(\frac{x+y}{d^k}\right) - D\left(\frac{x}{d^k}\right)\right),$$

and hence, on invoking assumption (3) and using (15), we obtain

$$\begin{aligned} U(x) &= y \sum_{d\le x^{3\delta}} \frac{\mu(d)}{d^k}\left(\log\left(\frac{x}{d^k}\right) + 2\gamma + O(x^{-\varepsilon/2})\right) \\ &= y\left((\log x + 2\gamma)\sum_{d=1}^{\infty}\frac{\mu(d)}{d^k} - k\sum_{d=1}^{\infty}\frac{\mu(d)\log d}{d^k} + O(x^{-\varepsilon/2}) + O(x^{-\delta})\right) \\ &= y\left(\frac{1}{\zeta(k)}\log x + \frac{2\gamma}{\zeta(k)} - \frac{k\zeta'(k)}{\zeta^2(k)} + O(x^{-\delta})\right). \end{aligned}$$

Thus, by (14), it suffices to show that

$$|V(x)| \ll yx^{-\delta}. \tag{16}$$

We find

$$|V(x)| \le \sum_{\substack{x<mnd^k\le x+y\\ d>x^{3\delta}}} 1 = \sum_{\substack{x<nd^k\le x+y\\ d>x^{3\delta}}} d(n)$$
$$\ll x^\delta \sum_{d>x^{3\delta}} \left(\left[\frac{x+y}{d^k}\right] - \left[\frac{x}{d^k}\right]\right). \tag{17}$$

Estimating the part of the last sum with $d \le yx^{-2\delta}$ trivially, one has

$$\sum_{x^{3\delta}<d\le yx^{-2\delta}} \left(\left[\frac{x+y}{d^k}\right] - \left[\frac{x}{d^k}\right]\right) \le \sum_{x^{3\delta}<d\le yx^{-2\delta}} \left(\frac{y}{d^k}+1\right) \ll yx^{-2\delta}. \tag{18}$$

In order to estimate the remaining part of the sum in the right-hand side of (17) we refer to some estimates of M. Filaseta and O. Trifonov [3, 4]. Defining

$$S_k(A,B) = \#\{n:\ A<n\le B,\ \{xn^{-k}\} > 1-yn^{-k}\}\ ,$$

we can write the sum under consideration as $S_k(yx^{-2\delta}, \sqrt[k]{2x})$. In the case $k \ge 3$, we refer to [4, Theorem 6]. This is a general result on the number of integer points close to a smooth curve which gives

$$S_k(yx^{-2\delta}, \sqrt[3]{2x}) \ll x^{1/6}\log x.$$

Clearly, this completes the proof of (16) for $k \ge 3$. In the case $k = 2$, this general theorem does not apply, but the idea behind it does. We cite the estimate

$$S_2(M, 2M) \ll x^{1/3}M^{-1/3} \tag{19}$$

from [3] (cf. (4) on p. 217). There the authors state it only for $M \ge y\sqrt{\log x}$, but in fact their argument proves this inequality for $yx^{-2\delta} < M \le \sqrt{x}$, provided that δ is small. Hence,

$$S_2(yx^{-2\delta}, \sqrt{2x}) \ll x^{(1+2\delta)/3}y^{-1/3}\ .$$

This estimate, (17), and (18) imply

$$|V(x)| \ll yx^{-\delta} + x^{(1+5\delta)/3}y^{-1/3}\ .$$

Thus, (16) follows from the choice of δ. □

References

[1] Baker R. C. (1994) The square-free divisor problem. Quart. J. Math. Oxford Ser. (2) **45**: 269–277.

[2] Baker R. C. (1996) The square-free divisor problem II. Quart. J. Math. Oxford Ser. (2) **47**: 133–146.

[3] Filaseta M., Trifonov O. (1992) On gaps between squarefree numbers II. J. London Math. Soc. (2) **45**: 215–221.

[4] Filaseta M., Trifonov O. (1996) The distribution of fractional parts with applications to gap results in number theory. Proc. London Math. Soc. (3) **73**: 241–278.

[5] Gelfond A. O., Linnik Y. V. (1966) Elementary Methods in the Analytic Theory of Numbers. Cambridge, Massachusetts: MIT Press.

[6] Heath-Brown D. R. (1991) Square-full numbers in short intervals. Math. Proc. Cambridge Philos. Soc. **110**: 1–3.

[7] Huxley M. N. (1993) Exponential sums and lattice points II. Proc. London Math. Soc. (3) **66**: 279–301.

[8] Mertens F. (1874) Über einige asymptotische gesetze der zahlentheorie. J. Reine Angew. Math. **77**: 289–338.

[9] Montgomery H. L., Vaughan R. C. (1981) The distribution of squarefree numbers. In Recent Progress in Analytic Number Theory, Vol. I, pp. 247–256. London New York: Academic Press.

[10] Nowak W. G., Schmeier M. (1988) Conditional asymptotic formulae for a class of arithmetical functions. Proc. Amer. Math. Soc. **103**: 713–717.

[11] Perron O. (1932) Über einen asymptotischen ausdruck. Acta Math. **59**: 89–97.

[12] Saffari B. (1968) Sur le nombre de diviseurs r-libres d'un entier, et sur le points à coordonnées entières dans certaines régions du plan. C. R. Acad. Sci. Paris Sér. **A-B 266**: A601–A603.

[13] Siva Rama Prasad V., Suryanarayana D. (1970/71) The number of k-free divisors of an integer. Acta Arith. **17**: 345–354.

[14] Titchmarsh E. C. (1986) The Theory of the Riemann Zeta-function, 2nd edn. (revised by D. R. Heath-Brown). Oxford: Oxford University Press.

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