A Note on the $2k$-th Mean Value of the Hurwitz Zeta Function

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Abstract

Consider the error term in the asymptotic formula

$$\int_0^1 |\zeta_1(1+it,\alpha)|^{2k} d\alpha = A(k) + O\left(|t|^{-\delta(k)} \log|t|\right) .$$

In this note we obtain $\delta(k) \asymp 1/(k^6 \log^2 k)$ which, for large values of k, presents a substantial improvement over the previously known result $\delta(k) \asymp 1/(k^2 2^{k^2})$.

For a complex $s = \sigma + it$ and a real α, $0 < \alpha < 1$, the Hurwitz zeta function is defined by

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s} ,$$

if $\sigma > 1$, and then continued analytically on $\mathbb{C} \setminus \{1\}$ via a functional equation similar to the one for the Riemann zeta function [2, Sections 1.2 and 1.4]. Let

$$\zeta_1(s,\alpha) = \zeta(s,\alpha) - \alpha^{-s}.$$

Recently, Y. Wang [3] proved the asymptotic formula

$$\int_0^1 |\zeta_1(1+it,\alpha)|^{2k} d\alpha = A(k) + O\left(|t|^{-\delta(k)} \log|t|\right) \tag{1}$$

where $A(k)$ and $\delta(k) > 0$ are explicit constants depending only on k. In this note we are concerned with the error term in (1) for large values of k. Via van der Corput's estimates for the arising zeta sums, in [3], $\delta(k)$ of order $1/(k^2 2^{k^2})$ was shown to be admissible in (1). Applying Vinogradov's method (which is the natural approach in this situation), we show that one can take $\delta(k) \asymp 1/(k^6 \log^2 k)$. We establish the following

Theorem. *There exists an absolute constant $c > 0$ such that for any real t, $|t| > t_0$, and for any positive integer k, the asymptotic formula* (1) *holds with*

$$\delta(k) = \frac{c}{k^6 \log^2(2k)} .$$

The constant $A(k)$ and the implied constant depend only on k.

As one would expect, the explicit value for c, although effectively computable, is too small to justify its place in the statement of the Theorem. For $k \geq 3$, certainly $c = 2^{-130}$ will suffice, and this value of c would improve over the result in [3] for such k. For large k a better value of c (say 10^{-7}) can easily be obtained.

Proof of the Theorem. Since the argument is similar to that in [3], we give just a brief sketch. Let

$$\theta(k) = \frac{1}{4k+2}\left(\frac{1}{2k+1} - \delta(k)\right),$$

and set $N = |t|^{\theta(k)}$. By [3, (3.4) and (3.5)],

$$\int_0^1 \left|\sum_{n=1}^{N} \frac{1}{(n+\alpha)^{1+it}}\right|^{2k} d\alpha = A(k) + O(N^{-1}) + O(|S|) \tag{2}$$

with $A(k) = \int_0^1 \zeta_1(2,\alpha)^k d\alpha$ and

$$S = \sum_{n_1,\ldots,n_{2k}=1}^{N}{}^{*} \int_0^1 \frac{(n_1+\alpha)^{-it}\cdots(n_{2k}+\alpha)^{it}}{(n_1+\alpha)\cdots(n_{2k}+\alpha)}\, d\alpha$$

where the sum is only over $2k$-tuples in which $n_1, \ldots, n_k$ do not form a permutation of $n_{k+1}, \ldots, n_{2k}$. Sharpening the last inequality in the proof of [3, Lemma 2], we find that each specific term in S is

$$\ll \frac{\left(|t|^{-1} N^{8k(k+1)}\right)^{1/(2k+1)}}{n_1 \cdots n_{2k}}$$

and, hence,

$$\int_0^1 \left|\sum_{n=1}^{N} \frac{1}{(n+\alpha)^{1+it}}\right|^{2k} d\alpha = A(k) + O\left(N^{-1}\right) + O\left(|t|^{-1/(2k+1)} N^{4k+2}\right).$$

Thus, it suffices to show that

$$\zeta_1(1+it,\alpha) = \sum_{n=1}^{N} \frac{1}{(n+\alpha)^{1+it}} + O\left(|t|^{-\delta(k)}\right).$$

This approximate formula follows from the approximate functional equation for $\zeta_1(s,\alpha)$ [2, Theorem III.2.1] and the esimate

$$\left|\sum_{x<n\leq 2x} (n+\alpha)^{-it}\right| \ll x|t|^{-\delta(k)} \quad (N < x \leq |t|). \tag{3}$$

If $N \leq x \leq |t|^{1/121}$, (2) can be derived from [1, Theorem III.1.3] via Vinogradov's method (see for example [2, Theorem IV.2.1]); if $|t|^{1/121} \leq x \leq |t|$, one can estimate van der Corput's method of exponent pairs. This completes the proof.

References

[1] G. I. Arkhipov, V. N. Chubarikov, A. A. Karatsuba, "Theory of Multiple Exponential Sums", Nauka, Moscow, 1987 (in Russian).

[2] A. A. Karatsuba and S. M. Voronin, "The Riemann Zeta Function", Walter de Gruyter & Co., Berlin, 1992.

[3] Y. Wang, On the $2k$-th mean value of Hurwitz zeta function, *Acta Math. Hungar.* **74** (1997), 301–307.