A Diophantine Inequality Involving Prime Powers

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1 Introduction

In 1952 I. I. Piatetski-Shapiro [8] studied the inequality

$$|p_1^c + p_2^c + \cdots + p_s^c - N| < \varepsilon \tag{1.1}$$

where $c > 1$ is not an integer, ε is a fixed small positive number, and $p_1, \ldots, p_s$ are primes. He established the existence of an $H(c)$, depending only on c, such that for all sufficiently large real N, (1.1) has solution whenever $s \geq H(c)$. He proved that

$$\limsup_{c \to \infty} \frac{H(c)}{c \log c} \leq 4,$$

and also that $H(c) \leq 5$, if $1 < c < 3/2$. On the other hand, the Vinogradov–Goldbach theorem [11] suggests that at least for c close to 1, one should expect $H(c) \leq 3$. The first result in this direction was obtained by D. I. Tolev [10], who showed that the inequality

$$|p_1^c + p_2^c + p_3^c - N| < \varepsilon \tag{1.2}$$

with $\varepsilon = N^{-(1/c)(15/14-c)} \log^9 N$ is solvable in primes p_1, p_2, p_3, provided that $1 < c < 15/14$ and N is sufficiently large. Later this result was improved by Y. C. Cai [1] who replaced $15/14$ by $13/12$, and by T. Nedeva and the author [5] who obtained the range $1 < c < 11/10$. In this paper we give a

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further improvement, changing the upper bound for c to $61/55$ (this constant can be improved somewhat, but not substantially; even 1.11 seems to be out of the scope of the method). For comparison

$$\begin{aligned} 15/14 &= 1.071428\ldots, & 13/12 &= 1.083333\ldots, \\ 11/10 &= 1.100000\ldots, & 61/55 &= 1.109090\ldots. \end{aligned}$$

The results in [1] and [5] are based on the version of the circle method used by Tolev enhanced by sharper exponential sum estimates. This approach, however, does not allow much further improvement, as a closer look at [5] shows. So, we combine it with Harman's sieve [3, 4], which allows more flexible use of the available arithmetical information. The essence is to apply Vinogradov's method [11] to the arithmetical problem itself rather than to the exponential sums arising from the analytic part of the argument. This allows us to discard some awkward cases that needed to be treated before. As a result we obtain a lower bound for the number of solutions instead of the asymptotic formula given by the previous approach. We prove

Theorem 1. *Let c be fixed with $1 < c < 61/55$ and $\delta > 0$ be a fixed number sufficiently small in terms of c. Let also N be a sufficiently large real number, and $\varepsilon \geq N^{-(1/c)(61/55-c+\delta)}$. Then the number $R(N)$ of the solutions of (1.2) satisfies*

$$R(N) \gg \frac{\varepsilon N^{3/c-1}}{\log^3 N}. \tag{1.3}$$

The implied constant depends only on c.

A natural question to ask is what the "ideal" result should be. We give a probabilistic argument, which suggests that one should expect $H(c) \leq 3$ at least for $1 < c < 3/2$. The theorem we prove is as follows

Theorem 2. *Let for $1 < c < 3/2$,*

$$\varepsilon_0(c) = N^{-(1/c)(3/2-c)} \log^{10} N.$$

Then for almost all (in the sense of Lebesgue measure) values of $c \in (1, \frac{3}{2})$, the inequality (1.2) is solvable for $\varepsilon \geq \varepsilon_0(c)$ and sufficiently large values of N.

Notation. Throughout the paper p, q, r, indexed or not, always denote primes; d, k, l, m, n denote integers. We choose $X = \frac{1}{4}N^{1/c}$; in Sections 2–5, $\eta = \delta^2$ where δ is the number from the statement of Theorem 1. Also, $m \sim M$ means that m runs through the interval $(M, 2M]$ and $e(x) = e^{2\pi i x}$; $\varphi(y)$ is a function having $r = [\log X]$ continuous derivatives and the following properties

1) $\varphi(y) = 1$, for $|y| \leq 9\varepsilon/10$,
2) $\varphi(y) = 0$, for $|y| \geq \varepsilon$,
3) $0 < \varphi(y) < 1$, for $9\varepsilon/10 < |y| < \varepsilon$,
4) its Fourier transform

$$\Phi(x) = \int_{-\infty}^{\infty} \varphi(y) e(-xy)\, dy$$

satisfies the inequality

$$|\Phi(x)| \leq \min\left(2\varepsilon, \frac{1}{\pi|x|}, \frac{1}{\pi|x|}\left(\frac{5r}{\pi\varepsilon|x|}\right)^r\right). \tag{1.4}$$

One can construct it using Lemma 1 of [8].

Finally, throughout the proof of Theorem 1 we assume, as we can, that $\varepsilon = N^{-(1/c)(61/55-c+\delta)}$. Similarly, in Section 6 $\varepsilon = \varepsilon_0(c)$.

2 The Sieve Method

We write

$$P(z) = \prod_{p<z} p$$

and, as usually, for any sequence of integers $\mathcal{E}$ weighted by the numbers $w(n)$, $n \in \mathcal{E}$, we set

$$S(\mathcal{E}, z) = \sum_{\substack{n \in \mathcal{E} \\ (n, P(z))=1}} w(n),$$

and denote by $\mathcal{E}_d$ the subsequence of elements $n \in \mathcal{E}$ with $n \equiv 0 \pmod{d}$. So, if we define $\mathcal{A}$ to be the sequence of the integers $n \in (X, 2X]$ weighted by

$$w(n) = \sum_{p_1, p_2 \sim X} \varphi(p_1^c + p_2^c + n^c - N),$$

we will have

$$R(N) \geq \sum_{p_1,p_2,p_3\sim X} \varphi(p_1^c + p_2^c + p_3^c - N) = S(\mathcal{A}, (3X)^{1/2}).$$

Hence, it suffices to show that

$$S(\mathcal{A}, (3X)^{1/2}) \gg \frac{\varepsilon X^{3-c}}{\log^3 X}. \tag{2.1}$$

We prove (2.1) using the Buchstab identity

$$S(\mathcal{E}, z_1) = S(\mathcal{E}, z_2) - \sum_{z_2 \leq p < z_1} S(\mathcal{E}_p, p) \tag{2.2}$$

and asymptotic formulas of the form

$$\sum_{m\sim M} a(m)\, S(\mathcal{A}_m, z(m)) = \lambda \sum_{m\sim M} a(m)\, S(\mathcal{B}_m, z(m)) + \text{ error terms} \tag{2.3}$$

where $\mathcal{B}$ is the set of the integers in $(X, 2X]$, and λ, M, and $z(m)$ are appropriately chosen. The idea is to use (2.2) to represent $S(\mathcal{A}, (3X)^{1/2})$ as the linear combination of sums of the form appearing in the left-hand side of (2.3) so that we are able to give asymptotic formulas for all sums having a negative contribution to $S(\mathcal{A}, (3X)^{1/2})$, as well as for most ones with a positive contribution. If it happens that the positive sums prevail, discarding the remaining positive terms, we get a positive lower bound.

Throughout the rest of this section we set up the decomposition. We set $A = X^{89/825}$, $B = X^{12/55}$, $C = X^{844/3025}$, $D = X^{56/165}$, and $F = X^{123/275}$. Applying (2.2), we find

$$\begin{aligned} S(\mathcal{A}, (3X)^{1/2}) &= S(\mathcal{A}, A) - \sum_{A \leq p < B} S(\mathcal{A}_p, p) \\ &\quad - \sum_{B \leq p \leq C} S(\mathcal{A}_p, p) - \sum_{C < p < D} S(\mathcal{A}_p, p) \\ &\quad - \sum_{D \leq p \leq F} S(\mathcal{A}_p, p) - \sum_{F < p < \sqrt{3X}} S(\mathcal{A}_p, p) \\ &= S_1 - S_2 - S_3 - S_4 - S_5 - S_6 \quad , \text{ say.} \end{aligned}$$

We give further decomposition for S_2 and S_4. Another application of (2.2) gives

$$\begin{aligned} S_2 = & \sum_{A\le p<B} S(\mathcal{A}_p, A) - \sum_{\substack{A\le q<p<B \\ pq<B}} S(\mathcal{A}_{pq}, q) \\ & - \sum_{\substack{A\le q<p<B \\ B\le pq\le C}} S(\mathcal{A}_{pq}, q) - \sum_{\substack{A\le q<p<B \\ C<pq<D}} S(\mathcal{A}_{pq}, q) \\ & - \sum_{\substack{A\le q<p<B \\ pq\ge D}} S(\mathcal{A}_{pq}, q) \\ = & S_7 - S_8 - S_9 - S_{10} - S_{11} \quad , \text{ say.} \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} S_4 = & \sum_{C<p<D} S(\mathcal{A}_p, A) - \sum_{\substack{C<p<D \\ A\le q<B}} S(\mathcal{A}_{pq}, q) \\ & - \sum_{\substack{C<p<D \\ B\le q\le C}} S(\mathcal{A}_{pq}, q) - \sum_{C<q<p<D} S(\mathcal{A}_{pq}, q) \\ = & S_{12} - S_{13} - S_{14} - S_{15} \quad , \text{ say.} \end{aligned}$$

Now, we deal with S_6. It counts numbers of the form pq, namely

$$S_6 = \sum_{\substack{pq\sim X \\ F<p\le q}} w(pq) \ .$$

It turns out to be more convinient to switch the sifting process from the product pq to one of the primes p_1, p_2 from the definition of $w(n)$, say p_2. In order to do so, we write S_6 as $S(\mathcal{A}^*, (3X)^{1/2})$ where $\mathcal{A}^*$ is the set of integers in $(X, 2X]$ weighted by

$$w^*(n) = \sum_{p_1\sim X} \sum_{\substack{pq\sim X \\ F<p\le q}} \varphi(p_1^c + n^c + (pq)^c - N) \ .$$

Let S_i^* denote a sum similar to S_i in which $\mathcal{A}$ has been replaced by $\mathcal{A}^*$. We decompose $S(\mathcal{A}^*, (3X)^{1/2})$ following the same lines without decomposing S_4^* and S_6^*.

Putting all together and using that S_8, S_{10}, S_4^*, and S_6^* have positive contributions to $S(\mathcal{A},(3X)^{1/2}$, we obtain

(2.4)
$$\begin{aligned} S(\mathcal{A},(3X)^{1/2}) \geq S_1 - S_3 - S_5 - S_7 + S_9 + S_{11} - S_{12} + S_{13} + S_{14} + S_{15} \\ - S_1^* + S_3^* + S_5^* + S_7^* - S_8^* - S_9^* - S_{10}^* - S_{11}^*. \end{aligned}$$

We will be able to find asymptotic formulas for all of these except for S_{13}, S_{15}, S_8^*, and S_{10}^*. Also, we will find asymptotic formulas for parts of S_{13} and S_{15} as well as admissible upper bounds for S_8^* and S_{10}^*.

3 Exponential Sums

In this section we prove the exponential sum estimates we will need to get the asymptotic formulas (2.3).

Lemma 1. *Let α and β be real, $\alpha\beta(\alpha-1)(\beta-1)(\alpha-2)(\beta-2) \neq 0$, $X > 0$, $M, N \geq 1$, $|a(m)| \leq 1$, $|b(n)| \leq 1$. Then*

$$\begin{aligned} &(XMN)^{-\eta} \left| \sum_{m\sim M} \sum_{n\sim N} a(m)\, b(n)\, e\left(X \frac{m^\alpha n^\beta}{M^\alpha N^\beta}\right) \right| \\ \ll &(X^4 M^{31} N^{34})^{1/42} + (X^6 M^{53} N^{51})^{1/66} + (X^6 M^{46} N^{41})^{1/56} \\ &+ (X^2 M^{38} N^{29})^{1/40} + (X M^9 N^6)^{1/10} + (X^2 M^7 N^6)^{1/10} \\ &+ (X M^6 N^6)^{1/8} + M^{1/2} N + M N^{1/2} + X^{-1/4} M N. \end{aligned}$$

Proof. This is Theorem 9 of [9]. □

In the following lemma and its applications $f(x,y) \sim_\Delta g(x,y)$ means that
$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} f(x,y) = \frac{\partial^{i+j}}{\partial x^i \partial y^j} g(x,y)(1 + O(\Delta))$$
for all pairs (i,j) for which this makes sense.

Lemma 2. *Let $\mathcal{D}$ be a subdomain of the rectangle*

$$\{(x,y) :\ M < x \leq 2M,\ N < y \leq 2N\},$$

$(M \geq N)$, such that any line parallel to any coordinate axis intersects it in $O(1)$ line segments. Let α, β be real numbers, $\alpha\beta(\alpha+\beta-1)(\alpha+\beta-2) \neq 0$,

and let $f(m,n)$ be a real sufficiently many times differentiable function such that $f(m,n) \sim_\Delta Am^\alpha n^\beta$ throughout $\mathcal{D}$. Denoting $X = MN$, $F = AM^\alpha N^\beta$, we have

$$(XF)^{-\eta}\left|\sum_{(m,n)\in\mathcal{D}} e(f(m,n))\right| \ll (F^2X^3)^{1/6} + XN^{-1/2} + X^{5/6} + X(\Delta M^{-1})^{1/4} + X(FM)^{-1/8} + (\Delta^4F^2X^9M^{-4})^{1/10} + XF^{-1/4}.$$

Proof. This is a version of Kolesnik's AB-Theorem. For the proof see [6]. □

Lemma 3. *Assume that x is a real number with $X^{1/2-c+\eta} < |x| < X^{1-c-\eta}$, and that $a(m)$, $b(k)$ are complex numbers of modulus ≤ 1. Assume further that $MK \asymp X$ and*

$$X^\eta \ll K \ll X^{1/2}.$$

Then

$$\sum_{m\sim M}\sum_{k\sim K} a(m)\, b(k)\, e(x(mk)^c) \ll X^{1-\eta/3}.$$

Proof. Denote the given sum by U. We first apply Cauchy's inequality and Weyl's lemma to the sum over k to get

$$|U|^2 \ll \frac{X^2}{Q} + \frac{X}{Q}\sum_{q\le Q}\sum_{k\sim K}\left|\sum_{m\sim M} e(f(m))\right|. \tag{3.1}$$

Here $f(m) = x((k+q)^c - k^c)m^c$ and $Q = X^{2\eta/3}$. Under the assumptions of the lemma we have

$$|f'(m)| \asymp |x|qX^{c-1} \ll X^{-\eta/3} \le \frac{1}{2},$$

so by the Kusmin–Landau inequality

$$|U|^2 \ll |x|^{-1}X^{2-c-\eta/3}K + X^{2-2\eta/3} \ll X^{2-2\eta/3}.$$

Clearly this proves the lemma. □

Lemma 4. *Assume that x is a real number with $X^{1/2-c+\eta} < |x| < X^{1-c-\eta}$, and that $a(m)$ are complex numbers of modulus ≤ 1. Assume further that $MK \asymp X$ and*

$$K \gg X^{1/3+\eta}.$$

Then

$$\sum_{m\sim M}\sum_{k\sim K} a(m)\, e(x(mk)^c) \ll X^{1-\eta/3}.$$

Proof. Denote the given sum by U. The exponent pair $(\frac{1}{6}, \frac{2}{3})$ gives

$$\begin{aligned}|U| &\ll M\left((|x|X^cK^{-1})^{1/6}K^{2/3} + (|x|X^cK^{-1})^{-1}\right)\\ &\ll X\left(X^{1/6}K^{-1/2} + X^{-1/2}\right) \ll X^{1-\eta/2}.\end{aligned}$$

□

Lemma 5. *Let $1 < c < 61/55$. Assume that x is a real number with $X^{1-c-\eta} < |x| < X^{61/55-c}$, and that $a(m)$, $b(k)$ are complex numbers of modulus ≤ 1. Assume further that $MK \asymp X$ and*

$$X^{56/165} \ll K \ll X^{123/275}. \tag{3.2}$$

Then

$$\sum_{m\sim M}\sum_{k\sim K} a(m)\, b(k)\, e(x(mk)^c) \ll X^{49/55+\eta}.$$

Proof. This follows immediately from Lemma 1 with $(m, n) = (k, m)$. □

Lemma 6. *Let $1 < c < 61/55$. Assume that x is a real number with $X^{1-c-\eta} < |x| < X^{61/55-c}$, and that $a(m)$, $b(k)$ are complex numbers of modulus ≤ 1. Assume further that $MK \asymp X$ and*

$$X^{12/55} \ll K \ll X^{844/3025}. \tag{3.3}$$

Then

$$\sum_{m\sim M}\sum_{k\sim K} a(m)\, b(k)\, e(x(mk)^c) \ll X^{49/55+\eta}.$$

Proof. Denote the given sum by U. We start as in Lemma 3, but we choose a different Q, namely $Q = X^{12/55}$, and estimate the sum over m in (3.1) using an exponent pair (κ, λ) rather than the Kusmin–Landau inequality. Thus,

$$|U|^2 \ll (|x|Q)^{\kappa} X^{1+\lambda+\kappa(c-1)} K^{1-\lambda} + X^{98/55} \ll X^{98/55}$$

provided that

$$X^{12/55} \ll K \ll X^{1-6(2+3\kappa)/55(1-\lambda)}.$$

Choosing $(\kappa, \lambda) = BA^2BABABA^2B(0,1) = (\frac{81}{242}, \frac{6}{11})$, we obtain the result. $\square$

Lemma 7. *Let $1 < c < 61/55$. Assume that x is a real number with $X^{1-c-\eta} < |x| < X^{61/55-c}$, and that $a(m)$ are complex numbers of modulus ≤ 1. Assume further that $MK \asymp X$ and*

$$K \gg X^{53/110}. \tag{3.4}$$

Then

$$\sum_{m\sim M}\sum_{k\sim K} a(m)\, e(x(mk)^c) \ll X^{49/55+\eta}.$$

Proof. If $K \gg X^{97/165}$, the argument of Lemma 4 proves the desired estimate. If this is not the case, we follow the argument on pp. 123–124 of [5]. Denote the given sum by U. Using Cauchy's inequality and Weyl's lemma with $Q = X^{12/55}$, we obtain

$$|U|^2 \ll \frac{X}{Q}\sum_{q\leq Q}\sum_{m\sim M}\sum_{k\cong K} e(x((k+q)^c - k^c)m^c) + X^{98/55}. \tag{3.5}$$

(Here $k \cong K$ means that k runs through a subinterval of $(K, 2K]$ which end points may depend on m and q.) Denote the sum over (m,k) by $U_1(q)$. Applying the Poisson formula and partial summation to m and k successively we find

$$|U_1(q)| \ll MKF^{-1}\left|\sum_{\mu,\nu} e(f(\mu,\nu))\right| + E$$

where $F = |x|qX^cK^{-1}$, $\mu \cong FM^{-1}$, $\nu \cong FK^{-1}$,

$$f(\mu,\nu) \sim_{q/K} c_0(xq)^{1/(2-2c)}\nu^{1/2}\mu^{c/(2c-2)} \asymp F$$

(here c_0 is a constant depending only on c), and

$$E \ll X^\eta\left(XF^{-1/2} + FX^{-1/2}\right).$$

Substituting the estimate for $U_1(q)$ in (3.5), we get

$$|U|^2 \ll \frac{X^2}{Q}\sum_{q\leq Q} F^{-1}\left|\sum_{\mu,\nu} e(f(\mu,\nu))\right| + X^{98/55}. \tag{3.6}$$

Finally, we estimate the sum over (μ,ν) in (3.6) using Lemma 2 with $(m,n)=(\mu,\nu)$, if $K \geq X^{1/2}$, and with $(m,n)=(\nu,\mu)$, otherwise. After some calculations the result follows. $\square$

4 Asymptotic Formulas

In this section $\Phi(x)$ is the Fourier transform of the function $\varphi(y)$ defined in Section 1. If $\tau = X^{1-c-\eta}$, and

$$I_0(x) = \int_X^{2X} \frac{e(xt^c)}{\log t}\, dt$$

we define

$$W_0(n) = \int_{-\tau}^{\tau} I_0^2(x)\, \Phi(x)\, e((n^c - N)x)\, dx. \tag{4.1}$$

Also, while dealing with $\mathcal{A}^*$, we will use

$$I_1(x) = \sum_{X^{123/275} < p \leq \sqrt{3X}} \frac{1}{p} \int_X^{2X} \frac{e(xt^c)}{\log pt}\, dt$$

and

$$W_1(n) = \int_{-\tau}^{\tau} I_0(x)\, I_1(x)\, \Phi(x)\, e((n^c - N)x)\, dx. \tag{4.2}$$

Finally, X^σ denotes a function of the form $e^{-a(\log X)^{1/4}}$ with some unspecified constant $a > 0$; in particular, we may write $X^{-\sigma}(\log X)^A \ll X^{-\sigma}$ instead of

$$e^{-a(\log X)^{1/4}}(\log X)^A \ll e^{-\frac{1}{2}a(\log X)^{1/4}}.$$

Lemma 8. *Let $\nu > 0$ be fixed, $x \geq x_0(\nu)$, $x^\nu \leq z \leq x$. Let also $\omega(x)$ be the continuous solution of the differential-difference equation*

$$\begin{cases} \omega(x) = 1/x & , \text{ if } 1 \leq x \leq 2, \\ (x\,\omega(x))' = \omega(x-1) & , \text{ if } x > 2. \end{cases}$$

Then for any $u \in (x, 2x]$, we have

$$\sum_{\substack{x < n \leq u \\ (n, P(z)) = 1}} 1 = \omega\left(\frac{\log x}{\log z}\right) \cdot \frac{u - x}{\log z} + O\left(\frac{x}{\log^2 x}\right).$$

Proof. This follows easily from Lemma 2 of [2]. □

Lemma 9. *Let $1 < c < 61/55$. Assume that $a(m)$, $b(k)$ are complex numbers of modulus ≤ 1. Assume also that $MK \asymp X$ with K satisfying one of the inequalities* (3.2) *or* (3.3). *Then*

$$\begin{aligned}\sum_{m\sim M}\sum_{k\sim K} a(m)\, b(k)\, w(mk) \\ = \sum_{m\sim M}\sum_{k\sim K} a(m)\, b(k)\, W_0(mk) + O\left(\varepsilon X^{3-c-\sigma}\right).\end{aligned} \tag{4.3}$$

Moreover, if K satisfies (3.4),

$$\begin{aligned}\sum_{m\sim M}\sum_{k\sim K} a(m)\, w(mk) \\ = \sum_{m\sim M}\sum_{k\sim K} a(m)\, W_0(mk) + O\left(\varepsilon X^{3-c-\sigma}\right).\end{aligned} \tag{4.4}$$

Proof. We consider in detail only (4.3) under the assumption (3.2), since the changes needed in the other cases are obvious. Let $D(X)$ denote the left-hand side of (4.3). By the Fourier inversion formula,

$$D(X) = \int_{-\infty}^{\infty} S^2(x)\, U(x)\, \Phi(x)\, e(-Nx)\, dx \tag{4.5}$$

where

$$\begin{aligned} S(x) &= \sum_{p\sim X} e(xp^c), \\ U(x) &= \sum_{m\sim M}\sum_{k\sim K} a(m)\, b(k)\, e(x(mk)^c). \end{aligned}$$

We set $\tau = X^{1-c-\eta}$ and $H = \varepsilon^{-1}\log^2 X$, and define the sets

$$\begin{aligned} E_1 &= \{x\in\mathbb{R}:\ |x| < \tau\}, \\ E_2 &= \{x\in\mathbb{R}:\ \tau \leq |x| \leq H\}, \\ E_3 &= \{x\in\mathbb{R}:\ |x| > H\}. \end{aligned}$$

From (1.4) and the trivial estimates for $S(x)$ and $U(x)$ we find

$$\begin{aligned} D_3(X) &= \int_{E_3} S^2(x)\, U(x)\, \Phi(x)\, e(-Nx)\, dx \\ &\ll X^3\left(\frac{r}{2\pi\Delta}\right)^r \int_H^{\infty} x^{-(r+1)} dx \ll 1. \end{aligned} \tag{4.6}$$

Now, for $x \in E_2$ and K satisfying (3.2), Lemma 5 provides the estimate

$$\max_{x \in E_2} |U(x)| \ll X^{49/55+\eta}.$$

(If K satisfies (3.3) or (3.4), we refer to Lemmas 6 or 7, respectively.) Also it is easy to prove (see for example Lemma 7 of [10]) that for any integer n,

$$\int_n^{n+1} |S(x)|^2 \, dx \ll X. \tag{4.7}$$

Using he last two inequalities and (1.4), we obtain

$$\begin{aligned} D_2(X) &= \int_{E_2} S^2(x) \, U(x) \, \Phi(x) \, e(-Nx) \, dx \\ &\ll \varepsilon X^{49/55+\eta} \int_0^1 |S(x)|^2 \, dx + X^{49/55+\eta} \sum_{n \le H} \frac{1}{n} \int_n^{n+1} |S(x)|^2 \, dx \\ &\ll \varepsilon X^{104/55+\eta} + X^{104/55+\eta} \log X \ll \varepsilon X^{3-c-\eta}. \end{aligned} \tag{4.8}$$

Finally, consider

$$D_1(X) = \int_{E_1} S^2(x) \, U(x) \, \Phi(x) \, e(-Nx) \, dx.$$

For $x \in E_1$, the argument on pp. 301–303 of [10] establishes the asymptotic formula

$$S(x) = I_0(x) + O\left(X^{1-\sigma}\right).$$

Also, following the argument of Lemma 7 of [10], we have the estimate

$$\int_{-\tau}^{\tau} |U(x)|^2 dx \ll X^{2-c} \log^4 X,$$

and similar (and even better) upper bounds for the corresponding means of $S(x)$ and $I_0(x)$. Hence,

$$\begin{aligned} D_1(X) &= \int_{-\tau}^{\tau} I_0^2(x) \, U(x) \, \Phi(x) \, e(-Nx) \, dx + O\left(\varepsilon X^{3-c-\sigma}\right) \\ &= \sum_{m \sim M} \sum_{k \sim K} a(m) \, b(k) \, W_0(mk) + O\left(\varepsilon X^{3-c-\sigma}\right). \end{aligned} \tag{4.9}$$

The lemma follows from (4.5)–(4.9). □

The next result is the $\mathcal{A}^*$-version of Lemma 9.

Lemma 10. *Let* $1 < c < 61/55$. *Assume that* $a(m)$, $b(k)$ *are complex numbers of modulus* ≤ 1. *Assume also that* $MK \asymp X$ *with* K *satisfying one of the inequalities* (3.2) *or* (3.3). *Then*

$$\begin{aligned} &\sum_{m\sim M}\sum_{k\sim K} a(m)\, b(k)\, w^*(mk) \\ &= \sum_{m\sim M}\sum_{k\sim K} a(m)\, b(k)\, W_1(mk) + O\left(\varepsilon X^{3-c-\sigma}\right). \end{aligned} \tag{4.10}$$

Moreover, if K *satisfies* (3.4),

$$\begin{aligned} &\sum_{m\sim M}\sum_{k\sim K} a(m)\, w^*(mk) \\ &= \sum_{m\sim M}\sum_{k\sim K} a(m)\, W_1(mk) + O\left(\varepsilon X^{3-c-\sigma}\right). \end{aligned} \tag{4.11}$$

Proof. The argument is similar to the one used in the proof of Lemma 9. If the sum under consideration is $D^*(X)$, we have

$$D^*(X) = \int_{-\infty}^{\infty} S(x)\, S_1(x)\, U(x)\, \Phi(x)\, e(-Nx)\, dx$$

where $S(x)$ and $U(x)$ are the same as before, and

$$S_1(x) = \sum_{\substack{pq\sim X \\ X^{123/275} < p \leq q}} e(x(pq)^c).$$

The proof of the inequality

$$\int\limits_{|x|>\tau} S(x)\, S_1(x)\, U(x)\, \Phi(x)\, e(-Nx)\, dx \ll \varepsilon X^{3-c-\eta}$$

repeats verbatim the estimates of $D_2(X)$ and $D_3(X)$ from Lemma 9, so we can concentrate on

$$\int_{-\tau}^{\tau} S(x)\, S_1(x)\, U(x)\, \Phi(x)\, e(-Nx)\, dx. \tag{4.12}$$

As in the proof of Lemma 9, we can replace $S(x)$ by $I_0(x)$. Thus, the integral (4.12) equals

$$\int_{-\tau}^{\tau} I_0(x)\, S_1(x)\, U(x)\, \Phi(x)\, e(-Nx)\, dx + O\left(\varepsilon X^{3-c-\sigma}\right). \tag{4.13}$$

However, we cannot do the same with $S_1(x)$. The reason is that the approach from [10] establishes the asymptotic formula

$$S_1(x) = I_1(x) + O\left(X^{1-\sigma}\right)$$

only for $|x| \le \tau_1 = X^{152/275-c-\eta}$. Fortunately, we can go around this by showing that the values of x with $\tau_1 < |x| < \tau$ do not contribute much to (4.13). For, this portion of the integral is

$$\begin{aligned} &\ll \varepsilon \max_{\tau_1<|x|<\tau} |U(x)| \left(\int_{-\tau}^{\tau} |S_1(x)|^2\, dx + \int_{-\tau}^{\tau} |I_0(x)|^2\, dx \right) \\ &\ll \max_{\tau_1<|x|<\tau} |U(x)| \varepsilon X^{2-c} \log X \ll \varepsilon X^{3-c-\eta/4} \end{aligned}$$

by virtue of Lemma 3. Hence, the integral (4.12) equals

$$\begin{aligned} &\int_{-\tau_1}^{\tau_1} I_0(x)\, I_1(x)\, U(x)\, \Phi(x)\, e(-Nx)\, dx + O\left(\varepsilon X^{3-c-\sigma}\right) \\ = &\int_{-\tau}^{\tau} I_0(x)\, I_1(x)\, U(x)\, \Phi(x)\, e(-Nx)\, dx + O\left(\varepsilon X^{3-c-\sigma}\right). \end{aligned}$$

Clearly the above discussion proves (4.10). The proof of (4.11) is similar with Lemma 4 replacing Lemma 3. □

Lemma 11. *Let $1 < c < 61/55$, $MK \asymp X$, and K satisfies one of the inequalities* (3.2) *or* (3.3)*. Let I, J be integers and $\mathcal{I}_i$, $\mathcal{J}_j$ are intervals for $1 \le i \le I$, $1 \le j \le J$. Write*

$$a(m,k) = \sum_{\substack{rp_1\cdots p_I = k \\ p_1<p_2<\cdots<p_I \\ p_i \in \mathcal{I}_i}} c(k) \sum_{\substack{lq_1\cdots q_J = m \\ q_1<q_2<\cdots<q_J \\ q_j \in \mathcal{J}_j}} d(m)$$

with $|c(k)|$, $|d(m)| \le 1$ and $p_1, \ldots, p_I$ and $q_1, \ldots, q_J$ satisfying $O(1)$ joint conditions of the form

$$p_u \le q_v \quad \textit{or} \quad q_v \le p_u$$

or

$$\prod_{u\in\mathcal{U}} p_u \prod_{v\in\mathcal{V}} q_v \le H \quad \textit{or} \quad \prod_{u\in\mathcal{U}} p_u \ge \prod_{v\in\mathcal{V}} q_v$$

or similar (for given $\mathcal{U} \subset \{1,\dots,I\}$, $\mathcal{V} \subset \{1,\dots,J\}$, $H \le X$). Then

$$\begin{aligned}(4.14)\qquad &\sum_{m\sim M}\sum_{k\sim K} a(m,k)\, w(mk)\\ &= \sum_{m\sim M}\sum_{k\sim K} a(m,k)\, W_0(mk) + O\left(\varepsilon X^{3-c-\sigma}\right).\end{aligned}$$

Furthermore, the result still holds if we replace $w(n)$ by $w^*(n)$ and $W_0(n)$ by $W_1(n)$.

Proof. We follow the approach from Lemma 1 of [4]: we first remove the dependencies between the variables m and k, and then refer to (4.3). Each joint condition can be removed via Perron's formula

$$\frac{1}{\pi}\int_{-T}^{T} e^{i\gamma t}\,\frac{\sin\beta t}{t}\,dt = \begin{cases} 1 + O\left(T^{-1}(\beta-|\gamma|)^{-1}\right) & \text{, if } |\gamma| \le \beta,\\ O\left(T^{-1}(|\gamma|-\beta)^{-1}\right) & \text{, if } |\gamma| > \beta,\end{cases}$$

at the cost of an extra $\log X$ factor in the error term. For instance, if we have a single condition $p_u < q_v$, we take $\gamma = \log p_u$, $\beta = \log(q_v + \frac{1}{2})$, and $T = X^2$, and get

$$\begin{aligned}&\sum_{m\sim M}\sum_{k\sim K} a(m,k)\, w(mk)\\ =&\frac{1}{\pi}\int_{-T}^{T}\sum_{m\sim M}\sum_{k\sim K} a(m,k,t)\, w(mk)\,\frac{dt}{t} + O\left(\varepsilon X^{3-c-\eta}\right)\end{aligned}$$

where $a(m,k,t)$ is defined as $a(m,k)$ but with the factors p_u^{it}, $\sin(t(\log(q_v + \frac{1}{2})))$ included, and the condition $p_u < q_v$ removed. Hence, we can rewrite $a(m,k,t)$ as $a^*(m,t)\, b^*(k,t)$, and then refer to (4.3) to obtain

$$\begin{aligned}&\frac{1}{\pi}\int_{-T}^{T}\sum_{m\sim M}\sum_{k\sim K} a^*(m,t)\, b^*(k,t)\, W_0(mk)\,\frac{dt}{t} + O\left(\varepsilon X^{3-c-\sigma}\log X\right)\\ =& \sum_{m\sim M}\sum_{k\sim K} a(m,k)\, W_0(mk) + O\left(\varepsilon X^{3-c-\sigma}\right)\end{aligned}$$

after another application of Perron's formula. Obviously, if we have more joint conditions, say A, we will end up with an A-tuple integral and a $(\log X)^A$ factor in the error term. □

Before going further we define the integrals

$$\begin{aligned} I(x) &= \int_X^{2X} e(xt^c)\, dt, \\ J_0(X) &= \int_{-\infty}^{\infty} I_0^2(x)\, I(x)\, \Phi(x)\, e(-Nx)\, dx, \\ J_1(X) &= \int_{-\infty}^{\infty} I_0(x)\, I_1(x)\, I(x)\, \Phi(x)\, e(-Nx)\, dx. \end{aligned}$$

Lemma 12. *Let $1 < c < 61/55$ and $u \geq 1$, and for some K satisfying one of the conditions* (3.2) *or* (3.3) *there exists a $\mathcal{D} \subset \{1, \ldots, u\}$ with*

$$\prod_{j \in \mathcal{D}} p_j \sim K.$$

Then

$$\sum_{p_1, \ldots, p_u} S(\mathcal{A}_{p_1 \cdots p_u}, p_1) = J_0(X) \sum_{p_1, \ldots, p_u} \frac{S(\mathcal{B}_{p_1 \cdots p_u}, p_1)}{p_1 \cdots p_u} + O\left(\varepsilon X^{3-c} (\log X)^{-10/3}\right). \tag{4.15}$$

Here the summation is over primes $p_1, \ldots, p_u \geq X^{89/825}$ satisfying $p_j > p_1$, together with $O(1)$ further conditions of the type

$$p_j \leq p_l \quad \text{or} \quad Q \leq \prod_{j \in \mathcal{F}} p_j \leq R$$

for some $\mathcal{F} \subset \{1, \ldots, u\}$ and $R \leq X$. Also,

$$\sum_{p_1, \ldots, p_u} S(\mathcal{A}^*_{p_1 \cdots p_u}, p_1) = J_1(X) \sum_{p_1, \ldots, p_u} \frac{S(\mathcal{B}_{p_1 \cdots p_u}, p_1)}{p_1 \cdots p_u} + O\left(\varepsilon X^{3-c} (\log X)^{-10/3}\right). \tag{4.16}$$

The result still holds if, instead of K, X/K satisfies (3.2) *or* (3.3).

Proof. The left-hand side of (4.15) equals

$$\sum_{p_1, \ldots, p_u} \sum_{\substack{n \sim X/p_1 \cdots p_u \\ (n, P(p_1)) = 1}} w(p_1 \cdots p_u n).$$

Setting

$$k = \prod_{j \in \mathcal{D}} p_j \quad \text{and} \quad m = \left(\prod_{j \notin \mathcal{D}} p_j \right) \cdot n$$

we can represent the last sum in the form appearing in the left side of (4.14). So, it equals

$$\int_{-\tau}^{\tau} I_0^2(x)\, U(x)\, \Phi(x)\, e(-Nx)\, dx + O\left(\varepsilon X^{3-c-\sigma}\right)$$

where

$$U(x) = \sum_{p_1,\ldots,p_u} \sum_{\substack{n \sim X/p_1 \cdots p_u \\ (n, P(p_1))=1}} e(x(p_1 \cdots p_u n)^c).$$

Since, by the first derivative estimate for trigonometric integrals (see Lemma 1 on p. 47 of [7]),

$$|I_0(x)| \ll \frac{1}{|x| X^{c-1} \log X},$$

and

$$|U(x)| \ll \frac{X}{\log X},$$

the values of x with $|x| \geq \tau_1 = X^{-c}(\log X)^{1/3}$ contribute to the last integral at most $O\left(\varepsilon X^{3-c}(\log X)^{-10/3}\right)$. On the other hand, if $|x| < \tau_1$, Lemma 8 and partial summation imply that

$$U(x) = I(x) \sum_{p_1,\ldots,p_u} \frac{S(\mathcal{B}_{p_1 \cdots p_u}, p_1)}{p_1 \cdots p_u} + O\left(X(\log X)^{-5/3}\right).$$

Combining the above estimates completes the proof. □

Lemma 13. *Let $1 < c < 61/55$ and $M \leq X^{123/275}$. Suppose further that $a(m)$ are real numbers such that $a(m) \ll 1$ and $a(m) = 0$ unless all prime divisors of m are $\geq X^{89/825}$. Then we have*

$$\sum_{m \sim M} a(m)\, S(\mathcal{A}_m, X^{89/825}) = J_0(X) \sum_{m \sim M} \frac{a(m)}{m}\, S(\mathcal{B}_m, X^{89/825}) + O\left(\varepsilon X^{3-c}(\log X)^{-10/3}\right). \tag{4.17}$$

Also

$$\sum_{m\sim M} a(m)\, S(\mathcal{A}_m^*, X^{89/825}) = J_1(X) \sum_{m\sim M} \frac{a(m)}{m}\, S(\mathcal{B}_m, X^{89/825}) + O\left(\varepsilon X^{3-c} (\log X)^{-10/3}\right). \tag{4.18}$$

Proof. We shall use the Eratosthenes–Legendre sieve, which states that

$$\sum_{\substack{n\le x\\ (n,P(z))=1}} f(n) = \sum_{\substack{nd\le x\\ d\mid P(z)}} \mu(d)\, f(nd) \tag{4.19}$$

where $\mu(d)$ is the Möbius function.

We choose $z = X^{89/825}$ and, applying (4.19) to $S(\mathcal{A}_m, z)$, we find

$$\sum_{m\sim M} a(m)\, S(\mathcal{A}_m, z) = \sum_{m\sim M} \sum_{\substack{nd\sim X/m\\ d\mid P(z)}} a(m)\, \mu(d)\, w(mnd). \tag{4.20}$$

Now we proceed to show that

$$\begin{aligned} &\sum_{m\sim M} \sum_{\substack{nd\sim X/m\\ d\mid P(z)}} a(m)\, \mu(d)\, w(mnd) \\ &= \sum_{m\sim M} \sum_{\substack{nd\sim X/m\\ d\mid P(z)}} a(m)\, \mu(d)\, W_0(mnd) + O\left(\varepsilon X^{3-c-\sigma}\right) \\ &= \sum_{m\sim M} \sum_{\substack{n\sim X/m\\ (n,P(z))=1}} a(m)\, W_0(mn) + O\left(\varepsilon X^{3-c-\sigma}\right) \end{aligned} \tag{4.21}$$

after another application of (4.19).

If $M \ge X^{56/165}$, we produce a new variable $k = dn$ and derive (4.21) from (4.3). Now suppose that $M \le X^{56/165}$. Then we divide the sum in the left-hand side of (4.21) into two parts: $\sum_1$ in which $md \le X^{123/275}$, and $\sum_2$ in which $md > X^{123/275}$. To obtain asymptotic formula for $\sum_1$ we combine m and d into a new variable k and refer to (4.4). Then we turn our attention to $\sum_2$. It can be written in the form

$$-\sum_{m\sim M} \sum_{p<z} \sum_{\substack{npd\sim X/m\\ d\mid P(p)\\ mpd>X^{123/275}}} a(m)\, \mu(d)\, w(mnpd).$$

Let $\sum_3$ be the part of this sum with $md \le X^{123/275} < mpd$, and $\sum_4$ the part with $md > X^{123/275}$. Introducing the variables $k = md$ and $l = pn$, we can put $\sum_3$ in the form appearing in the left-hand side of (4.14) and then refer to Lemma 11 to get the desired asymptotic formula (since $mpd > X^{123/275}$ and $p < z$, we must have $md > X^{56/165}$). Again, $\sum_4$ can be rewritten as

$$\sum_{m\sim M} \sum_{p_1<p_2<z} \sum_{\substack{np_1p_2d\sim X/m \\ d|P(p_1) \\ mp_1d>X^{123/275}}} a(m)\,\mu(d)\,w(mnp_1p_2d);$$

then we can use Lemma 11 to find an asymptotic formula for the part of the last sum with $md < X^{123/275}$, and can proceed further with the rest of it. We can continue in this fashion, obtaining at each step a sum to which Lemma 11 applies and a sum for which further decomposition can be given. Since every integer $\le 2X$ has at most $O(\log X)$ prime divisors, after $\ll \log X$ steps, this procedure will stop and we will be left with a sum which does not require further decomposition (Lemma 11 applies to all of it). Combining the asymptotic formulas for all the occurring sums, we complete the proof of (4.21).

So, using (4.20) and (4.21), we get

$$\sum_{m\sim M} a(m)\,S(\mathcal{A}_m, z) = \sum_{m\sim M} \sum_{\substack{n\sim X/m \\ (n,P(z))=1}} a(m)\,W_0(mn) + O\left(\varepsilon X^{3-c-\sigma}\right). \tag{4.22}$$

Also, using the approach from the proof of the previous lemma, we have

$$\begin{aligned} &\sum_{m\sim M} \sum_{\substack{n\sim X/m \\ (n,P(z))=1}} a(m)\,W_0(mn) \\ &= J_0(X) \sum_{m\sim M} \frac{a(m)}{m}\,S(\mathcal{B}_m, z) + O\left(\varepsilon X^{3-c}(\log X)^{-10/3}\right). \end{aligned} \tag{4.23}$$

The result follows from (4.22) and (4.23). □

5 Proof of Theorem 1

We start with (2.4). We can estimate S_1, S_7, S_{12}, S_1^*, and S_7^* using Lemma 13. Consider, for example, S_{12}. By (4.17), it equals (we use the values of

A, B, C, D, and F from Section 2)

$$J_0(X) \sum_{C<p<D} \frac{1}{p} \sum_{\substack{pn\sim X \\ (n,P(A))=1}} 1 + O\left(\varepsilon X^{3-c} (\log X)^{-10/3}\right).$$

Using Lemma 8 and partial summation, we obtain that the sum over p and n is

$$\begin{aligned} &\frac{825}{89 \log X} \int_{\frac{844}{3025}}^{\frac{56}{165}} \omega\left(\frac{1-x}{89/825}\right) \frac{dx}{x} + O\left((\log X)^{-2}\right) \\ \le &\frac{1.0200}{\log X} + O\left((\log X)^{-2}\right). \end{aligned}$$

We also have

$$\begin{aligned} J_0(X) &= \frac{1}{\log^2 X} \int_{-\infty}^{\infty} I^3(x)\, \Phi(x)\, e(-Nx)\, dx + O\left(\varepsilon X^{3-c} (\log X)^{-3}\right) \\ &=: \frac{J(X)}{\log^2 X} + O\left(\varepsilon X^{3-c} (\log X)^{-3}\right). \end{aligned}$$

Hence,

$$S_{12} \le 1.0200 \frac{J(X)}{\log^3 X} + O\left(\varepsilon X^{3-c} (\log X)^{-10/3}\right).$$

We can evaluate the rest four quantities similarly. Introducing the notation $S_{12} \lesssim 1.0200$ as a shortcut for the last inequality, we can state the corresponding estimates as

$$S_1 \gtrsim 5.2039, \quad S_7 \lesssim 3.6666, \qquad S_1^* \lesssim 1.1017, \quad S_7^* \gtrsim 0.7762,$$

where we have used that

$$\begin{aligned} J_1(X) &= \frac{J(X)}{\log^2 X} \int_{123/275}^{1/2} \frac{dx}{x(1-x)} + O\left(\varepsilon X^{3-c} (\log X)^{-3}\right) \\ &= \frac{J(X)}{\log^2 X} \ln\left(\frac{152}{123}\right) + O\left(\varepsilon X^{3-c} (\log X)^{-3}\right). \end{aligned}$$

(This follows easily from the Prime Number Theorem.) Further, we obviously have

$$S_8^* \le \sum_{\substack{A\le q<p<B \\ pq<B}} S(\mathcal{A}_{pq}^*, A) \quad \text{and} \quad S_{10}^* \le \sum_{\substack{A\le q<p<B \\ C<pq<D}} S(\mathcal{A}_{pq}^*, A),$$

and these sums can also be estimated via Lemma 13. Hence,

$$S_8^* \lesssim 0.0002 \quad \text{and} \quad S_{10}^* \lesssim 0.1290.$$

Now consider S_9^*. By Lemma 12, it equals

$$J_1(X) \sum_{\substack{A \le q < p < B \\ B \le pq \le C}} \frac{1}{pq} \sum_{\substack{pqn \sim X \\ (n, P(q))=1}} 1 + O\left(\varepsilon X^{3-c} (\log X)^{-10/3}\right).$$

As before, using Lemma 8 and partial summation, we can evaluate the sum over p, q, and n. We have

$$S_9^* \lesssim 0.0619,$$

and similarly,

$$\begin{aligned} S_3 &\lesssim 0.5632, \quad S_5 \lesssim 0.4544, \qquad & S_9 &\gtrsim 0.2923, \quad S_{11} \gtrsim 0.2369, \\ S_{14} &\gtrsim 0.1108, \quad S_3^* \gtrsim 0.1192, \qquad & S_5^* &\gtrsim 0.0961, \quad S_{11}^* \lesssim 0.0502. \end{aligned}$$

Also, we can use Lemma 13 to evaluate the parts of S_{13} with $D \le pq \le F$ and $X/F \le pq \le X/D$, and the part of S_{15} with $X/F \le pq \le X/D$. Hence, we have

$$S_{13} \gtrsim 0.2157 \quad \text{and} \quad S_{15} \gtrsim 0.0480.$$

Substituting all these estimates in (2.4), we find

$$S(\mathcal{A}, (2X)^{1/2}) \ge 0.0519 \frac{J(X)}{\log^3 X} + O\left(\varepsilon X^{3-c} (\log X)^{-10/3}\right).$$

Since, by Lemma 6 of [10], $J(X) \gg \varepsilon X^{3-c}$, this inequality establishes (2.1) and completes the proof of the theorem.

6 Proof of Theorem 2

Fix $\rho > 0$ and consider the set

$$\mathcal{E}_\rho(X) = \left\{ c \in (1+\rho, 3/2 - \rho) : \ |R^*(N,c) - J^*(N,c)| \ge X^{3/2} \log^2 X \right\}$$

where

$$R^*(N,c) = \sum_{p_1, p_2, p_3 \sim X} \varphi(p_1^c + p_2^c + p_3^c - N),$$

$$J^*(N,c) = \int_{-\infty}^{\infty} I_0^3(x)\, \Phi(x)\, e(-Nx)\, dx.$$

Note that by Lemma 6 of [10],

$$J^*(N,c) \gg X^{3/2} \log^7 X,$$

so we can deduce Theorem 2 by showing that the Lebesgue measure of $\mathcal{E}_\rho(X)$ is $O(\log^{-1} X)$ (hereafter $X \geq X_0(\rho)$ and the implied constants depend at most on ρ). This follows from the estimate

$$\int_{1+\rho}^{3/2-\rho} (R^*(N,c) - J^*(N,c))^2 \, dc \ll X^3 \log^3 X$$

by an application of Chebyshev's inequality. Hence, it suffices to show that

$$\int_a^b (R^*(N,c) - J^*(N,c))^2 \, dc \ll X^3 \log^2 X \tag{6.1}$$

whenever $1 + \rho \leq a \leq 3/2 - \rho$ and $b = a + \log^{-1} X$.

The framework for the proof of (6.1) is the same as for the proof of Lemma 9. We have

$$R^*(N,c) = \int_{-\infty}^{\infty} S^3(x,c) \, \Phi(x) \, e(-Nx) \, dx$$

where $S(x,c) = S(x)$ is the exponential sum used in Section 4. Upon setting $\eta = \rho^2$, we can define (at least formally) the parameters τ and H, and the sets E_1, E_2, and E_3 as in Lemma 9. Then, $R^*(N,c)$ can be written as the sum of three integrals corresponding to E_1, E_2, and E_3, respectively. One can easily check that similarly to the estimate of $D_3(X)$ in the proof of Lemma 9 we have

$$\int_{E_3} \left| S^3(x) \, \Phi(x) \right| \, dx \ll 1$$

uniformly with respect to $c \in (1, \frac{3}{2})$. Also, the argument used to evaluate $D_1(X)$ in that proof (see also pp. 305–306 in [10]) shows that uniformly in c

$$\begin{aligned} \int_{E_1} S^3(x) \, \Phi(x) \, e(-Nx) \, dx &= J^*(N,c) + O\left(\varepsilon X^{3-c} (\log X)^{-20}\right) \\ &= J^*(N,c) + O\left(X^{3/2} (\log X)^{-10}\right). \end{aligned}$$

Thus, (6.1) will follow, if we prove that

$$D(X) := \int_a^b \left(\int_{E_2} \left| S^3(x,c)\,\Phi(x) \right| \, dx \right)^2 dc \ll X^3 \log^2 X. \tag{6.2}$$

By (1.4) and Cauchy's inequality, we obtain that

$$D(X) \ll \int_a^b \varepsilon^2(c) \left(\int_{E_2} |S(x,c)|^2 \, dx \right) \left(\int_{E_2} |S(x,c)|^4 \, dx \right) dc.$$

Since the estimate (4.7) is uniform in n and $c > 1$, we find that

$$\int_{E_2} |S(x,c)|^2 \, dx \ll XH(c) = \varepsilon^{-1}(c) X \log^2 X,$$

and hence,

$$\begin{aligned} D(X) &\ll X \log^2 X \int_a^b \varepsilon(c) \left(\int_{E_2} |S(x,c)|^4 \, dx \right) dc \\ &\ll \varepsilon(b) X \log^2 X \int_{\tau(b)}^{H(a)} \int_a^b |S(x,c)|^4 \, dc \, dx \\ &\ll \varepsilon(b) X \log^2 X \int_{\tau(b)}^{H(a)} |S(x,c_0)|^2 \int_a^b |S(x,c)|^2 \, dc \, dx \end{aligned} \tag{6.3}$$

for some $c_0 \in [a,b]$.

Next, we need an estimate for the inner integral. The one we use is

$$\int_a^b |S(x,c)|^2 \, dc \ll \frac{X}{\log^2 X} + \frac{X^{2-a}}{|x| \log X}.$$

(It can be proved via the approach from Lemma 7 of [10]; note that the integration variable is c.) This estimate, (4.7), and (6.3) imply

$$D(X) \ll \varepsilon(b) X \log^2 X \left\{ \frac{H(a)X^2}{\log^2 X} + \frac{X^{2-a}}{\tau(b) \log X} + X^{3-a} \right\}.$$

Observe that $X^b = eX^a$, and hence, for example, $H(a) \asymp H(b)$. Therefore, the last inequality establishes (6.2) and completes the proof of Theorem 2.

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