ON A TWO-DIMENSIONAL EXPONENTIAL SUM

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1. Introduction

The classical Weyl sum

$$f_k(\boldsymbol{\alpha}) = \sum_{1 \leqslant x \leqslant N} e(\alpha_k x^k + \cdots + \alpha_1 x), \tag{1}$$

has been the subject of intense investigations for more than a century, going back to Weyl's pioneering work [8] on uniform distribution. Here, N is a large parameter, $\boldsymbol{\alpha} \in \mathbb{R}^k$, $k \geqslant 2$, and $e(x) = e^{2\pi i x}$. In general, there are two regimes to the behavior of such sums. When all the coefficients $\alpha_2, \dots, \alpha_k$ are close to rationals with small denominators, one can establish an asymptotic formula for $f_k(\boldsymbol{\alpha})$; otherwise, one has a non-trivial upper bound on $f_k(\boldsymbol{\alpha})$. Borrowing terminology from the Hardy–Littlewood circle method, we will refer to these two regimes as $\boldsymbol{\alpha}$ being on a major arc or on a minor arc, respectively.

When $\alpha_1 = \cdots = \alpha_{k-1} = 0$, Vaughan [5] obtained a major arc approximation that was strong enough to give a unified treatment of the quadratic and cubic Weyl sums across both regimes. More recently, Vaughan [7] extended that work to the full quadratic Weyl sum $f_2(\boldsymbol{\alpha})$, while Brüdern and Robert [4] and Brandes, Parsell, Poulias, Shakan, and Vaughan [3] obtained variants in the case $\alpha_2 = \cdots = \alpha_{k-1} = 0$, $k \geqslant 3$.

In this paper, we are interested in the two-dimensional quadratic variant of (1),

$$F(\boldsymbol{\alpha}; \boldsymbol{\theta}) = \sum_{1 \leqslant x \leqslant N} \sum_{1 \leqslant y \leqslant N} e(\alpha_1 x^2 + 2\alpha_2 xy + \alpha_3 y^2 + \theta_1 x + \theta_2 y).$$

For this sum, $\boldsymbol{\alpha}$ is on a major arc if there exist integers a_1, a_2, a_3 and q_1, q_2, q_3 such that

$$|q_i \alpha_i - a_i| \leqslant Q N^{-2}, \quad 1 \leqslant q_i \leqslant Q, \quad (a_i, q_i) = 1, \tag{2}$$

for some parameter $Q \leqslant N$. When $\boldsymbol{\alpha}$ is on a minor arc, a simple argument (see Proposition 1 below) yields the bound

$$F(\boldsymbol{\alpha}; \boldsymbol{\theta}) \ll_\varepsilon N^{2+\varepsilon} Q^{-1/2} \tag{3}$$

for any fixed $\varepsilon > 0$. On the other hand, when $\boldsymbol{\alpha}$ is on the major arc defined by conditions (2), a different routine argument (see Proposition 2) leads to an approximation of the form

$$F(\boldsymbol{\alpha}; \boldsymbol{\theta}) = F^*(\boldsymbol{\alpha}; \boldsymbol{\theta}) + O(qN(1 + N^2|\boldsymbol{\beta}|)), \tag{4}$$

where

$$q = \operatorname{lcm}[q_1, q_2, q_3], \quad \beta_i = \alpha_i - \frac{a_i}{q_i}, \tag{5}$$

and $F^*(\boldsymbol{\alpha}; \boldsymbol{\theta})$ is a local approximant that satisfies the inequality

$$F^*(\boldsymbol{\alpha}; \boldsymbol{\theta}) \ll \frac{N^2 \log N}{(q + qN^2|\boldsymbol{\beta}|)^{1/2}}. \tag{6}$$

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The exponential sum $F(\boldsymbol{\alpha};\boldsymbol{\theta})$ appears in a recent work [1] by Anderson, Palsson and the author on discrete maximal functions. Motivated by the applications in [1], we will be concerned with the error term in (4), which can be as large as $O(NQ^3)$ under conditions (2). As one wants that error term to be smaller than the trivial bound for $F(\boldsymbol{\alpha};\boldsymbol{\theta})$, this restricts the choice of Q to $Q \leqslant N^{1/3}$. Indeed, this error term is weaker than the minor arc bound (3) when $Q \geqslant N^{2/7}$. Our main result establishes an approximation of the form (4) with an error term of size $O(NQ^{1/2}\log N)$, provided that $Q \leqslant \eta N^{1/2}$ for a sufficiently small absolute constant $\eta > 0$.

Given $q \in \mathbb{N}$, $\mathbf{a} \in \mathbb{Z}^3$, $\mathbf{m} \in \mathbb{Z}^2$, $\boldsymbol{\beta} \in \mathbb{R}^3$, and $\boldsymbol{\theta} \in \mathbb{R}^2$, we define

$$S(q;\mathbf{a},\mathbf{m}) = \frac{1}{q^2}\sum_{x=1}^{q}\sum_{y=1}^{q} e_q\big(a_1x^2 + 2a_2xy + a_3y^2 + m_1x + m_2y\big),$$

$$I(N;\boldsymbol{\beta},\boldsymbol{\theta}) = \int_0^N\int_0^N e\big(\beta_1x^2 + 2\beta_2xy + \beta_3y^2 + \theta_1x + \theta_2y\big)\,dxdy,$$

where $e_q(x) = e(x/q)$. Also, for vectors $\mathbf{a},\mathbf{q} \in \mathbb{Z}^3$ with $(a_i,q_i) = 1$ for $i = 1,2,3$, we write

$$b_i = \frac{a_iq}{q_i}, \quad Q_i = \operatorname{lcm}[q_i, q_{i+1}], \tag{7}$$

where $q = \operatorname{lcm}[q_1,q_2,q_3]$, as in (5). With this notation, we can state our theorem as follows.

Theorem 1. *Let $1 \leqslant Q \leqslant 0.1N^{1/2}$, $\boldsymbol{\alpha} \in \mathbb{R}^3$, and $\boldsymbol{\theta} \in \mathbb{R}^2$. Suppose that $\mathbf{a},\mathbf{q} \in \mathbb{Z}^3$ satisfy (2) and that $\mathbf{m} \in \mathbb{Z}^2$ is such that*

$$\left|\theta_i - \frac{m_i}{Q_i}\right| \leqslant \frac{1}{2Q_i}.$$

Then

$$F(\boldsymbol{\alpha};\boldsymbol{\theta}) = H(\mathbf{q};\mathbf{a},\mathbf{m})I(N;\boldsymbol{\beta};\boldsymbol{\delta}) + O\big(NQ^{1/2}\log N\big), \tag{8}$$

where q and $\boldsymbol{\beta}$ are defined by (5), $\delta_i = \theta_i - m_i/Q_i$, and

$$H(\mathbf{q};\mathbf{a},\mathbf{m}) = \frac{1}{q^2}\sum_{h=1}^{q}\sum_{k=1}^{q} e_q(2b_2q_1q_3hk + \tilde{m}_1q_1h + \tilde{m}_2q_3k)S(q;\mathbf{b},\mathbf{n}_{h,k}),$$

with $\tilde{m}_i = m_iq/Q_i$ and $\mathbf{n}_{h,k} = (\tilde{m}_1 + 2b_2q_3k, \tilde{m}_2 + 2b_2q_1h)$.

The main term in (8) is not exactly what one may expect—that would be the main term in (11) below (see also Proposition 2). While this is somewhat disappointing, it is not unprecedented—in [3], the authors of that paper also obtain a major arc approximation to $f_k(\alpha_1,\alpha_k)$ that attains a sharp error term at the expense of including additional main terms (albeit of a slightly different nature). The main strength of our theorem is that the main term satisfies (6), while the error term is much smaller than that in (4). This allows us to combine the above result and the basic bounds in Propositions 1 and 2 below to prove the following theorem.

Theorem 2. *Let $1 \leqslant Q \leqslant N^{1/2}$, $\boldsymbol{\alpha} \in \mathbb{R}^3$, and $\boldsymbol{\theta} \in \mathbb{R}^2$. Then either*

$$F(\boldsymbol{\alpha};\boldsymbol{\theta}) \ll N^2Q^{-1/2}\log N, \tag{9}$$

or there exist $q \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{Z}^3$ such that

$$|q\alpha_i - a_i| \leqslant QN^{-2}, \quad 1 \leqslant q \leqslant Q, \quad (q,a_1,a_2,a_3) = 1, \tag{10}$$

and

$$F(\boldsymbol{\alpha};\boldsymbol{\theta}) = S(q;\mathbf{a},\mathbf{m})I(N;\boldsymbol{\beta};\boldsymbol{\delta}) + O(NQ), \tag{11}$$

where $\boldsymbol{\beta}$ *is defined by* (5) *and* $\mathbf{m} \in \mathbb{Z}^2$ *is chosen so that* $\boldsymbol{\delta} = \boldsymbol{\theta} - q^{-1}\mathbf{m}$ *satisfies* $|\boldsymbol{\delta}| \leqslant (2q)^{-1}$.

For $r \in \mathbb{N}$, define

$$\mathcal{V}_r = \big\{(\mathbf{m},\mathbf{n}) \in \mathbb{Z}^{2d} : |\mathbf{m}|^2 = |\mathbf{n}|^2 = |\mathbf{m}-\mathbf{n}|^2 = r\big\},$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$. In [1], a special case of Theorem 2 (see Lemma 7 in [1]) was used to study the properties of the maximal operator

$$T^*(f,g)(\mathbf{x}) = \sup_{r\in\mathbb{N}} r^{3-d} \sum_{(\mathbf{m},\mathbf{n})\in\mathcal{V}_r} |f(\mathbf{x}-\mathbf{m})g(\mathbf{x}-\mathbf{n})|,$$

where $f \in \ell^p(\mathbb{Z}^d)$ and $g \in \ell^q(\mathbb{Z}^d)$. In particular, the main result of that paper establishes that T^* is bounded from $\ell^p(\mathbb{Z}^d) \times \ell^\infty(\mathbb{Z}^d)$ to $\ell^p(\mathbb{Z}^d)$ whenever $d \geqslant 9$ and $p > \max\big(\frac{32}{d+8}, \frac{d+4}{d-2}\big)$. The lower bound for p in this theorem depends on the saving over the trivial bound in the minor arc bound (3). Theorem 2 with $Q = N^{1/2}$ yields the bound

$$F(\boldsymbol{\alpha};\boldsymbol{\theta}) \ll_\varepsilon N^{7/4+\varepsilon} \tag{12}$$

outside the set of major arcs defined by conditions (10). By comparison, if we relied only on the basic approximation (4), we would be restricted to $Q \leqslant N^{2/7}$, leading to a weaker range for p in the above result.

In [1], we proved also an asymptotic formula for the Fourier multiplier of the underlying bilinear operator,

$$\widehat{T}_r(\boldsymbol{\xi},\boldsymbol{\eta}) = r^{3-d} \sum_{(\mathbf{m},\mathbf{n})\in\mathcal{V}_r} e(\boldsymbol{\xi}\cdot\mathbf{m} + \boldsymbol{\eta}\cdot\mathbf{n}).$$

The preliminary version of Theorem 2 that was used in [1] was not sufficiently general to apply in the proof of that result (Theorem 2 in [1]), and so the bound for the error term in the asymptotic formula is not as sharp as one may hope. Using Theorem 2 as stated above, one may improve on that result. We omit the complete statement of the improvement, since the asymptotic formula for $\widehat{T}_r(\boldsymbol{\xi},\boldsymbol{\eta})$ is too technical to include here for the mere purpose of this brief remark. However, for the benefit of the interested reader, we note that one only need to apply (12) above in place of (6.1) in [1].

Remark. The condition $Q \leqslant 0.1N^{1/2}$ in the statement of Theorem 1 can be replaced by $Q \leqslant \eta N^{1/2}$ for a slightly larger $\eta \in (0, \frac{1}{4})$. The above choice, while not optimal, is relatively close to the optimal one and also allows us to justify easily several claims of the form

$$q_i|\beta_j| \leqslant Q^2N^{-1} \leqslant 0.01,$$

where $1 \leqslant q_i, q_j \leqslant Q$ and $q_j|\beta_j| \leqslant QN^{-2}$. Such estimates are implicit in several places in the proof of Theorem 1 (for example, see (26)).

2. Preliminaries

To begin, we state and prove the main minor arc bound for $F(\boldsymbol{\alpha};\boldsymbol{\theta})$. Although its proof is standard, we include it for completeness. Here and through the remainder of the paper, we write $L = \log N$. We also abbreviate $\gcd(a,b,\dots)$ and $\operatorname{lcm}[a,b,\dots]$ as $(a,b,\dots)$ and $[a,b,\dots]$, respectively.

Proposition 1. *Let $1 \leqslant Q \leqslant N$ and suppose that $\boldsymbol{\alpha} \in \mathbb{R}^3$ has no rational approximation of the form (2). Then, for all $\boldsymbol{\theta} \in \mathbb{R}^2$, one has*

$$|F(\boldsymbol{\alpha};\boldsymbol{\theta})| \ll N^2 Q^{-1/2} L. \tag{13}$$

Proof. By Dirichlet's theorem on Diophantine approximation, there exist rationals a_i/q_i, $i = 1, 2, 3$, such that

$$|q_i\alpha_i - a_i| \leqslant QN^{-2}, \quad 1 \leqslant q_i \leqslant N^2 Q^{-1}, \quad (a_i, q_i) = 1. \tag{14}$$

By our assumption that $\boldsymbol{\alpha}$ does not have a rational approximation satisfying (2), we must have $q_i > Q$ for at least one index i, and by symmetry, we may assume that $i = 1$ or 2.

By Cauchy's inequality,

$$\begin{aligned}
|F(\boldsymbol{\alpha};\boldsymbol{\theta})|^2 &\leqslant N \sum_{y \leqslant N} \bigg| \sum_{x \leqslant N} e\big(\alpha_1 x^2 + 2\alpha_2 xy + \theta_1 x\big) \bigg|^2 \\
&\leqslant N \sum_{y \leqslant N} \sum_{|h| \leqslant N} \sum_{x \in I(h)} e\big(\alpha_1 h(2x+h) + 2\alpha_2 hy + \theta_1 h\big) \\
&\leqslant N \sum_{|k| \leqslant 2N} \bigg| \sum_{x \in I(k/2)} e(\alpha_1 kx) \bigg| \cdot \bigg| \sum_{y \leqslant N} e(\alpha_2 ky) \bigg| \\
&\ll N^3 + N \sum_{k \leqslant 4N} \prod_{j=1}^{2} \min\big(N, \|\alpha_j k\|^{-1}\big),
\end{aligned}$$

where $I(h)$ is a subinterval of $[1, N]$ that depends on h and $\|x\|$ denotes the distance from x to the nearest integer. Recalling that (14) holds with $q_i > Q$ for $i = 1$ or 2, we can now apply Lemma 2.2 in Vaughan [6] to deduce that

$$|F(\boldsymbol{\alpha};\boldsymbol{\theta})|^2 \ll N^4\big(q_i^{-1} + N^{-1} + q_i N^{-2}\big)L,$$

and (13) follows. □

The next proposition establishes the basic major arc approximation of type (4). Note that unless $\boldsymbol{\alpha}$ satisfies conditions (10) for some $Q \leqslant N$, the error term is weaker than the trivial bound for $F(\boldsymbol{\alpha};\boldsymbol{\theta})$. However, when we pair this result with Proposition 1, we apply this approximation on the major arcs defined by (2). In that context, we have $q = [q_1, q_2, q_3]$ and $\mathbf{a} = \mathbf{b}$, where $b_i = a_i q/q_i$; in particular, the error term is $O(NQ^3)$.

Proposition 2. *Let $q \in \mathbb{N}$, $\mathbf{a} \in \mathbb{Z}^3$, $\mathbf{m} \in \mathbb{Z}^2$, $\boldsymbol{\alpha} \in \mathbb{R}^3$, and $\boldsymbol{\theta} \in \mathbb{R}^2$ satisfy*

$$(q, a_1, a_2, a_3) = 1, \quad |q\theta_i - m_i| \leqslant \tfrac{1}{2}.$$

Then

$$F(\boldsymbol{\alpha};\boldsymbol{\theta}) = S(q;\mathbf{a},\mathbf{m})I(N;\boldsymbol{\beta};\boldsymbol{\delta}) + O\big(qN(1 + N^2|\boldsymbol{\beta}|)\big), \tag{15}$$

where $\boldsymbol{\beta} = \boldsymbol{\alpha} - q^{-1}\mathbf{a}$, $\boldsymbol{\delta} = \boldsymbol{\theta} - q^{-1}\mathbf{m}$.

Proof. The result follows by partial summation from the asymptotic formula

$$\sum_{\substack{X < n \leqslant Y \\ n \equiv a \ (\mathrm{mod}\, q)}} e(\theta n) = \frac{1}{q}\int_X^Y e(\theta x)\,\mathrm{d}x + O(1), \tag{16}$$

where $a, q \in \mathbb{N}$ and $|\theta| \leqslant (2q)^{-1}$.

Let
$$F_{r,s}(\boldsymbol{\alpha};\boldsymbol{\theta}) = \sum_{\substack{x\leqslant N\\ x\equiv r \ (\mathrm{mod}\, q)}} \sum_{\substack{y\leqslant N\\ y\equiv s \ (\mathrm{mod}\, q)}} e\big(\alpha_1x^2 + 2\alpha_2xy + \alpha_3y^2 + \theta_1x + \theta_2y\big).$$
By splitting the terms in $F(\boldsymbol{\alpha};\boldsymbol{\theta})$ according to their residues modulo q, we get
$$\begin{aligned} F(\boldsymbol{\alpha};\boldsymbol{\theta}) &= \sum_{r,s=1}^{q} F_{r,s}(\boldsymbol{\alpha};\boldsymbol{\theta}) \\ &= \sum_{r,s=1}^{q} e_q\big(a_1r^2 + 2a_2rs + a_3s^2 + m_1r + m_2s\big)F_{r,s}(\boldsymbol{\beta};\boldsymbol{\delta}). \end{aligned} \tag{17}$$
We apply partial summation and (16) successively to the sums over x and y to get
$$F_{r,s}(\boldsymbol{\beta};\boldsymbol{\delta}) = q^{-2}I(N;\boldsymbol{\beta},\boldsymbol{\delta}) + O\big(q^{-1}N(1 + N^2|\boldsymbol{\beta}|)\big),$$
and the proposition follows from (17). □

The next two lemmas provide basic bounds that we will use to bound the main term in (15). In particular, together they establish that if $(q, a_1, a_2, a_3) = 1$, one has
$$S(q;\mathbf{a},\mathbf{m})I(N;\boldsymbol{\beta},\boldsymbol{\delta}) \ll \frac{N^2L}{(q + qN^2|\boldsymbol{\beta}|)^{1/2}}, \tag{18}$$
uniformly in $\mathbf{m}$ and $\boldsymbol{\delta}$.

Lemma 1. *Suppose that $(q, a_1, a_2, a_3) = 1$. Then*
$$|S(q;\mathbf{a},\mathbf{m})| \ll q^{-1}(q, a_1a_3 - a_2^2)^{1/2}.$$

Proof. This is Lemma 1 in [1], so we provide only a brief sketch. After squaring out, one can use the orthogonality of the additive characters on $\mathbb{Z}_q$ to show that
$$q^4|S(q;\mathbf{a},\mathbf{m})|^2 \leqslant q^2\nu(q;2\mathbf{a}), \tag{19}$$
where $\nu(q;\mathbf{a})$ denote the number of solutions $(h,k) \in \mathbb{Z}_q^2$ of the pair of congruences
$$a_1h + a_2k \equiv a_2h + a_3k \equiv 0 \pmod q.$$
The lemma follows from (19) and the bound
$$\nu(q;\mathbf{a}) \leqslant (q, a_1a_3 - a_2^2),$$
which can be proved using the basic properties of linear congruences. □

We remark that while in this paper we need only the bound $S(q;\mathbf{a},\mathbf{m}) \ll q^{-1/2}$, the full strength of Lemma 1 plays an important role in [1]. In that work, we apply the above bound to averages of $S(q;\mathbf{a},\mathbf{m})$ over $\mathbf{a}$, and are able to make use of the following inequality (see Lemma 2 in [1]): if $s \geqslant 1$, then
$$\sum_{\substack{1\leqslant a_1,a_2,a_3\leqslant q\\ (q,a_1,a_2,a_3)=1}} (q, a_1a_3 - a_2^2)^s \leqslant \tau(q)^2q^{s+2}.$$

Lemma 2. *One has*
$$|I(N;\boldsymbol{\beta},\boldsymbol{\theta})| \ll N^2L(1 + N^2|\boldsymbol{\beta}| + N|\boldsymbol{\theta}|)^{-1/2}.$$

Proof. This is a special case of Theorem 1.5 in [2]. □

Lemma 3. *Let $\boldsymbol{\alpha} \in \mathbb{R}^2$, $q \in \mathbb{N}$, and $\mathbf{a} \in \mathbb{Z}^2$ satisfy*

$$(q, a_2) = 1, \quad |q\alpha_1 - a_1| \leqslant \tfrac{1}{2}.$$

Then

$$f_2(\boldsymbol{\alpha}) = S_1(q; \mathbf{a}) I_1(N; \boldsymbol{\beta}) + O\big((q + qN^2|\beta_2|)^{1/2}\big),$$

where

$$S_1(q; \mathbf{a}) = \frac{1}{q} \sum_{x=1}^{q} e_q(a_2 x^2 + a_1 x), \quad I_1(N; \boldsymbol{\beta}) = \int_0^N e(\beta_2 x^2 + \beta_1 x)\, dx,$$

and $\boldsymbol{\beta} = \boldsymbol{\alpha} - q^{-1}\mathbf{a}$.

Proof. This Theorem 8 in [7]. □

Recall also the well-known bound for the Gauss sum:

$$S_1(q; \mathbf{a}) \ll q^{-1/2} \tag{20}$$

when $(a_2, q) = 1$, and the first-derivative estimate for trigonometric integrals: if $\phi : [a, b] \to \mathbb{R}$ is monotone and $\inf\limits_{a \leqslant x \leqslant b} |\phi'(x)| \geqslant \lambda$, then

$$\int_a^b e(\phi(x))\, dx \ll \lambda^{-1}. \tag{21}$$

3. Proof of Theorem 1

The basic idea of the proof is to apply Lemma 3 successively to each of the summations in $F(\boldsymbol{\alpha}; \boldsymbol{\theta})$. We recall (7) and also define

$$q_{ij} = \frac{q_i}{(q_i, q_j)}.$$

for $1 \leqslant i, j \leqslant 3$.

Fix $u \in \mathbb{Z}$ and $y \in [1, N]$ with

$$-\tfrac{1}{2} q_{21} < u \leqslant \tfrac{1}{2} q_{21}, \qquad 2a_2 q_{12} y + m_1 \equiv u \pmod{q_{21}}. \tag{22}$$

We note that when $q_{21} = 1$ (i.e., when $Q_1 = q_1$), these conditions reduce to setting $u = 0$, and when $q_{21} = 2$ to selecting $u \in \{0, 1\}$ based on the parity of m_1. In all other cases, conditions (22) specify the residue class of y modulo $q_{21}/(2, q_{21})$. We also choose the unique integer k_y with

$$\frac{-1}{2q_1} \leqslant \gamma_y := 2\alpha_2 y + \theta_1 - \frac{k_y}{q_1} < \frac{1}{2q_1}.$$

Lemma 3 with $q = q_1$ then gives

$$f_2(\alpha_1, 2\alpha_2 y + \theta_1) = S_1(q_1; a_1, k_y) I_1(N; \beta_1, \gamma_y) + O(Q^{1/2}). \tag{23}$$

Suppose first that $u \neq 0$. We have

$$\begin{aligned} \inf_{x \in [0,N]} |2\beta_1 x + \gamma_y| &\geqslant \left| \frac{2a_2 q_{12} y + m_1}{Q_1} - \frac{k_y}{q_1} \right| - |\delta_1| - 2(|\beta_1| + |\beta_2|)N \\ &\geqslant \left| \frac{2a_2 q_{12} y + m_1 - u}{Q_1} - \frac{k_y}{q_1} + \frac{u}{Q_1} \right| - \frac{0.6}{Q_1}. \end{aligned}$$

By (22), the fraction

$$w = \frac{2a_2 q_{12} y + m_1 - u}{Q_1} - \frac{k_y}{q_1}$$

is an integer multiple of $1/q_1$. So, either $w = 0$ and

$$\inf_{x \in [0,N]} \left| 2\beta_1 x + \gamma_y \right| \geqslant \frac{|u| - 0.6}{Q_1} > \frac{|u|}{3Q_1},$$

or $|w| \geqslant 1/q_1$ and

$$\inf_{x \in [0,N]} \left| 2\beta_1 x + \gamma_y \right| \geqslant \frac{1}{q_1} - \frac{|u| + 0.6}{Q_1} > \frac{|u|}{3Q_1}.$$

Thus, by (21),

$$I(N; \beta_1, \gamma_y) \ll Q_1 |u|^{-1},$$

and (20) and (23) give

$$f_2(\alpha_1, 2\alpha_2 y + \theta_1) \ll q_1^{1/2} q_{21} |u|^{-1} + Q^{1/2}. \tag{24}$$

Summing this inequality over all pairs u, y, with $u \neq 0$, that satisfy (22), we find that the total contribution to $F(\boldsymbol{\alpha}; \boldsymbol{\theta})$ from such pairs is

$$\ll \sum_{1 \leqslant |u| \leqslant \frac{1}{2} q_{21}} (N/q_{21}) q_1^{1/2} q_{21} |u|^{-1} + NQ^{1/2} \ll NQ^{1/2} L. \tag{25}$$

When $u = 0$, we relate k_y, γ_y to m_1, δ_1. We have

$$\left| \frac{2a_2 y}{q_2} + \frac{m_1}{Q_1} - \frac{k_y}{q_1} \right| \leqslant |\gamma_y| + |\delta_1| + 2|\beta_2| N < \frac{1}{2q_1} + \frac{0.6}{Q_1}. \tag{26}$$

Since the left side of this inequality is a multiple of $1/q_1$, we find that in this case

$$\frac{k_y}{q_1} = \frac{2a_2 y}{q_2} + \frac{m_1}{Q_1} + \frac{e_y}{q_1}, \quad \gamma_y = \delta_1 + 2\beta_2 y - \frac{e_y}{q_1},$$

with $e_y \in \{0, \pm 1\}$. In particular, when $e_y = 0$, we may rewrite (23) as

$$\begin{aligned} f_2(\alpha_1, 2\alpha_2 y + \theta_1) &= S_1(Q_1; a_1 q_{21}, 2a_2 q_{12} y + m_1) I_1(N; \beta_1, \gamma_y^*) + O(Q^{1/2}), \\ &= S_1(q; b_1, 2b_2 y + \tilde{m}_1) I_1(N; \beta_1, \gamma_y^*) + O(Q^{1/2}), \end{aligned} \tag{27}$$

where $\gamma_y^* = 2\beta_2 y + \delta_1$.

On the other hand, it is clear from (26) that the case $e_y = \pm 1$ may occur only when $Q_1 = q_1$ and $0.4 q_1^{-1} \leqslant |\delta_1| \leqslant 0.5 q_1^{-1}$. Under these assumptions, we have

$$\inf_{x \in [0,N]} \left| 2\beta_1 x + \gamma_y \right| \geqslant \frac{1}{q_1} - |\delta_1| - 2(|\beta_1| + |\beta_2|) N > \frac{1}{3q_1}$$

for all $x \in [0, N]$. Thus, by (21),

$$I_1(N; \beta_1, \gamma_y) \ll q_1,$$

and (20) and (23) yield

$$f_2(\alpha_1, 2\alpha_2 y + \theta_1) \ll Q^{1/2}.$$

Moreover, since in this case

$$\inf_{x \in [0,N]} \left| 2\beta_1 x + \gamma_y^* \right| \geqslant |\delta_1| - 2(|\beta_1| + |\beta_2|) N > \frac{1}{3q_1},$$

we have also

$$I_1(N;\beta_1,\gamma_y^*) \ll q_1.$$

We conclude that (27) holds also when $e_y = \pm 1$.

Having established (27) for all y with $2a_2q_{12}y \equiv -m_1 \pmod{q_{21}}$ and recalling the bound (25) for the contribution of the remaining $y \in [1,N]$, we obtain

$$F(\boldsymbol{\alpha};\boldsymbol{\theta}) = \int_0^N G(\alpha_3;\lambda_x)e(\beta_1x^2+\delta_1x)\,dx + O(NQ^{1/2}L), \tag{28}$$

where $\lambda_x = \lambda(0,x)$, $\lambda(r,x) = 2a_2r/q_2 + 2\beta_2x + \theta_2$, and

$$\begin{aligned} G(\alpha;\lambda_x) &= \frac{1}{q}\sum_{r=1}^{q} e_q(b_1r^2+\tilde{m}_1r) \sum_{\substack{1\leqslant y\leqslant N \\ 2a_2q_{12}y\equiv -m_1 \pmod{q_{21}}}} e(\alpha y^2 + \lambda(r,x)y) \\ &= \frac{1}{qq_{21}}\sum_{h=1}^{q_{21}} e_{q_{21}}(m_1h)\sum_{r=1}^{q} e_q(b_1r^2+\tilde{m}_1r)f_2(\alpha,\lambda(r+q_1h,x)). \end{aligned} \tag{29}$$

Next, we approximate the sum $G(\alpha_3,\lambda_x)$ in (28). For a fixed $r \in \mathbb{Z}$ and $x \in [0,N]$, we choose $k_{r,x}$ with

$$\frac{-1}{2q_3} \leqslant \gamma_{r,x} := \lambda(r,x) - \frac{k_{r,x}}{q_3} < \frac{1}{2q_3}.$$

We also choose an integer $u = u_r$ with

$$-\tfrac{1}{2}q_{23} < u \leqslant \tfrac{1}{2}q_{23}, \quad 2a_2q_{32}r + m_2 \equiv u \pmod{q_{23}}. \tag{30}$$

Similarly to (23), we have

$$f_2(\alpha_3,\lambda(r,x)) = S_1(q_3;a_3,k_{r,x})I_1(N;\beta_3,\gamma_{r,x}) + O(Q^{1/2}). \tag{31}$$

When $u \neq 0$, we have

$$\begin{aligned} \inf_{y\in[0,N]} |2\beta_3y+\gamma_{r,x}| &\geqslant \left|\frac{2a_2q_{32}r+m_2}{Q_2} - \frac{k_{r,x}}{q_3}\right| - |\delta_2| - 2(|\beta_2|+|\beta_3|)N \\ &\geqslant \left|\frac{2a_2q_{32}r+m_2-u}{Q_2} - \frac{k_{r,x}}{q_2} + \frac{u}{Q_2}\right| - \frac{0.6}{Q_2} > \frac{|u|}{3Q_2}. \end{aligned}$$

Hence, similarly to (24), we get

$$f_2(\alpha_3,\lambda(r,x)) \ll q_3^{1/2}q_{23}|u|^{-1} + Q^{1/2}.$$

Therefore, the total contribution to the right side of (29) from terms subject to

$$2a_2q_{32}(r+q_1h) \equiv u - m_2 \pmod{q_{23}} \tag{32}$$

with $u \neq 0$, is bounded by

$$\ll \sum_{1\leqslant|u|\leqslant\frac{1}{2}q_{23}} \frac{1}{qq_{21}}\sum_{h=1}^{q_{21}}\sideset{}{^{(u)}}\sum_{r=1}^{q} q_3^{1/2}q_{23}|u|^{-1} + Q^{1/2} \ll Q^{1/2}L,$$

where the notation $\sum^{(u)}$ indicates that r satisfies the congruence (32) for the specified value of u. Thus,

$$G(\alpha;\lambda_x) = \frac{1}{qq_{21}} \sum_{h=1}^{q_{21}} \sum_{r=1}^{q}{}^{(0)} e_q\left(b_1r^2 + \tilde{m}_1(r+q_1h)\right) f_2(\alpha, \lambda(r+q_1h, x)) + O(Q^{1/2}L). \tag{33}$$

On the other hand, we have

$$\left|\frac{2a_2q_{32}r + m_2}{Q_2} - \frac{k_{r,x}}{q_3}\right| \leqslant |\gamma_{r,x}| + |\delta_2| + 2|\beta_2|N < \frac{1}{2q_3} + \frac{0.6}{Q_2},$$

so when r satisfies (30) with $u = 0$ we get

$$\frac{k_{r,x}}{q_3} = \frac{2a_2r}{q_2} + \frac{m_2}{Q_2} + \frac{e_r}{q_3}, \quad \gamma_{r,x} = \delta_2 + 2\beta_2x - \frac{e_r}{q_3},$$

with $e_r \in \{0, \pm 1\}$. When $e_r = 0$, (31) becomes

$$f_2(\alpha_3, \lambda(r,x)) = S_1(q; b_3, 2b_2r + \tilde{m}_2)I_1(N; \beta_3, 2\beta_2x + \delta_2) + O(Q^{1/2}). \tag{34}$$

As before, the alternative case $e_r = \pm 1$ may occur only if $Q_2 = q_3$ and $0.4q_3^{-1} \leqslant |\delta_2| \leqslant 0.5q_3^{-1}$. Again, under these conditions, we retain (34) because both sides are $O(Q^{1/2})$. Combining (33) and (34) (with $r + q_1h$ in place of r), we deduce that

$$G(\alpha_3, \lambda_x) = T(\mathbf{q}; \mathbf{a}, \mathbf{m})I_1(N; \beta_3, 2\beta_2x + \delta_2) + O(Q^{1/2}L), \tag{35}$$

where

$$\begin{aligned}
T(\mathbf{q}; \mathbf{a}, \mathbf{m}) &= \frac{1}{qq_{21}} \sum_{h=1}^{q_{21}} e_{q_{21}}(m_1h) \sum_{r=1}^{q}{}^{(0)} e_q\left(b_1r^2 + \tilde{m}_1r\right) S_1(q; b_3, 2b_2(r+q_1h) + \tilde{m}_2) \\
&= \frac{1}{q_{21}q_{23}} \sum_{h=1}^{q_{21}} \sum_{k=1}^{q_{23}} e_{q_{21}}(m_1h)e_{q_{23}}(m_2k)e_{q_2}(2a_2q_1q_3hk) \\
&\qquad \times \frac{1}{q^2} \sum_{r=1}^{q} \sum_{s=1}^{q} e_q\left(b_1r^2 + 2b_2rs + b_3s^2 + n_{1,k}r + n_{2,h}s\right) \\
&= H(\mathbf{q}; \mathbf{a}, \mathbf{m}),
\end{aligned} \tag{36}$$

with $n_{1,k} = \tilde{m}_1 + 2b_2q_3k$ and $n_{2,h} = \tilde{m}_2 + 2b_2q_1h$.

The theorem is a direct consequence of (28), (35), and (36). □

4. Proof of Theorem 2

Let $Q_0 = 0.1N^{1/2}$. By Dirichlet's theorem on Diophantine approximation, there exist rationals b_i/q_i, $i = 1, 2, 3$, such that

$$|q_i\alpha_i - b_i| \leqslant Q_0N^{-2}, \quad 1 \leqslant q_i \leqslant N^2Q_0^{-1}, \quad (b_i, q_i) = 1. \tag{37}$$

Proposition 1 yields (9) unless $1 \leqslant q_1, q_2, q_3 \leqslant Q_0$. Let $q = [q_1, q_2, q_3]$ and define $a_i = b_iq/q_i$. Under conditions (37) with $q_1, q_2, q_3 \leqslant Q_0$, Theorem 1 and (18) give

$$F(\boldsymbol{\alpha}; \boldsymbol{\theta}) \ll \frac{N^2L}{(q + N^2|q\boldsymbol{\alpha} - \mathbf{a}|)^{1/2}} + N^{5/4}L.$$

The right side of this inequality is $O(NQ^{-1/2}L)$ unless

$$1 \leqslant q \leqslant Q, \quad |q\alpha_i - a_i| \leqslant QN^{-2}.$$

Since (37) and the definition of the a_i's ensure that $(q, a_1, a_2, a_3) = 1$, we conclude that (9) may only fail if $\boldsymbol{\alpha}$ has a rational approximation of the form (10). Under those conditions, (11) follows from Proposition 2. $\square$

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