ON AN EQUATION WITH PRIME NUMBERS

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1. Introduction.

B. I. Segal ([13], [14]) was the first who in 1933 considered additive problems with non-integer degrees. He studied the inequality

$$|x_1^c + x_2^c + \cdots + x_k^c - N| < \varepsilon \tag{1}$$

and the equation

$$[x_1^c] + [x_2^c] + \cdots + [x_k^c] = N \ , \tag{2}$$

where $c > 1$ is not integer, and proved in both cases that there exists $k_0(c)$ such that the corresponding problem has solutions if $k \geq k_0$ and N is sufficiently large. Later Deshouillers [4] and Arhipov and Zhitkov [1] improved Segal's result on (2). One may also mention the papers of Deshouillers [5] and Gritsenko [7], where the equation (2) in two variables was considered.

In 1952 I. I. Piatetski–Shapiro [12] considered (1) with $x_1, \ldots, x_k$ restricted to prime numbers. Let $H(c)$ denote the least k such that the inequality (1) with fixed $\varepsilon > 0$ has solutions in prime numbers for every sufficiently large real N. Piatetski–Shapiro proved that

$$\limsup_{c \to \infty} \frac{H(c)}{c \log c} \leq 4 \ .$$

He also proved that $H(c) \leq 5$ for $1 < c < 3/2$. The theorem of Goldbach–Vinogradov [16] motivates the conjecture that for c close to 1 $H(c) \leq 3$. This was proved by D. I. Tolev [15]. He showed that if $1 < c < 15/14$ and $\varepsilon = N^{-(1/c)(15/14-c)} \log^9 N$ the quantity

$$D(N) \stackrel{\text{def}}{=} \sum_{|p_1^c + p_2^c + p_3^c - N| < \varepsilon} \log p_1 \ \log p_2 \ \log p_3$$

is positive for a sufficiently large N. Recently Y. C. Cai [3] improved the upper bound for c to 13/12.

In [10] Laporta and Tolev considered the corresponding equation of the type (2). For $1 < c < 17/16$ they proved an asymptotic formula for the sum

$$R(N) \stackrel{\text{def}}{=} \sum_{[p_1^c] + [p_2^c] + [p_3^c] = N} \log p_1 \ \log p_2 \ \log p_3 \ .$$

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In the present paper we improve the range for c they obtained.

Theorem 1. *Assume that $1 < c < 12/11$ and $\delta > 0$ is arbitrary small. Then for any sufficiently large integer N the asymptotic formula*

$$R(N) = \frac{\Gamma^3\big(1+\frac{1}{c}\big)}{\Gamma\big(\frac{3}{c}\big)} N^{3/c-1} + O\big(N^{3/c-1} \exp\big(-(\log N)^{1/3-\delta}\big)\big)$$

holds.

We also improve the result from [3]. We obtain an asymptotic formula for the sum $D(N)$. Since the proof is similar to the proof of Theorem 1, we omit it.

Theorem 2. *Assume that $1 < c < 11/10$ and $\delta > 0$ is arbitrary small. Then for any sufficiently large real N and $\varepsilon \ge N^{-(1/c)(11/10-c)+\nu}$ for some $\nu > 0$ the asymptotic formula*

$$D(N) = 2\varepsilon \frac{\Gamma^3\big(1+\frac{1}{c}\big)}{\Gamma\big(\frac{3}{c}\big)} N^{3/c-1} + O\big(\varepsilon N^{3/c-1} \exp\big(-(\log N)^{1/3-\delta}\big)\big)$$

holds.

The range for c in both problems depends on the estimate of an exponential sum over primes. In [10] and [15] Vaughan's identity and the exponent pair $(\frac{1}{2}, \frac{1}{2})$ are used. We derive Theorem 1 from a more precise estimate of this sum (Lemma 5 below). To prove it we use the identity of Heath-Brown [8], van der Corput's method as described in Chapters 2 and 3 of [6] and the estimate of a double exponential sum due to Kolesnik [9].

2. Notation. Since for $1 < c < 17/16$ Theorem 1 is proved in [10], we can assume that $17/16 \le c < 12/11$. In this paper $\eta > 0$ is a fixed small number depending only on c; $P = N^{1/c}$; $\omega = P^{1-c-\eta}$; p, $p_1, \dots$ are primes; $\alpha \in (0,1)$; ε is an arbitrary small positive number, not necessary the same in different appearances. We use $[x]$, $\{x\}$ and $\|x\|$ for the integral part of x, fractional part of x and the distance from x to the nearest integer correspondingly; $\Lambda(n)$ is von Mangoldt's function.

$e(x) = \exp(2\pi i x)$;
$\sigma = \exp\big((\log N)^{1/3-\delta}\big)$;
$f(x) \ll g(x)$ means that $f(x) = \mathrm{O}(g(x))$;
$f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$;
$x \sim X$ means that x runs through a subinterval of $[X, 2X]$;
$f(x_1, \dots, x_n) \tilde{\Delta} g(x_1, \dots, x_n)$ means that

$$\frac{\partial^{j_1+\dots+j_n}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}} f(x_1, \dots, x_n) = \frac{\partial^{j_1+\dots+j_n}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}} g(x_1, \dots, x_n)\big(1 + \mathrm{O}(\Delta)\big)$$

for all n-tuples $(j_1, \dots, j_n)$ for which it makes sense.

We use sums of two types, which we define in the following way:
Type I sums:

$$\sum_{\substack{m\sim M, n\sim L\\ mn\sim X}}\sum a_m\, F(mn)$$

Type II sums:

$$\sum_{\substack{m\sim M, n\sim L\\ mn\sim X}}\sum a_m\, b_n\, F(mn)$$

where the coefficients satisfy the conditions $a_m \ll m^{\varepsilon}$, $b_n \ll n^{\varepsilon}$.

We also denote

$$S(\alpha) = \sum_{p\le P} \log p \cdot e(\alpha[p^c])\, ;$$
$$R_i = \int_{\Omega_i} S^3(\alpha)\, e(-\alpha N)\, d\alpha \qquad (i = 1, 2)$$

where $\Omega_1 = (-\omega, \omega)$ and $\Omega_2 = (\omega, 1-\omega)$.

3. Some preliminary results.

Lemma 1. *Let D be a subdomain of the rectangle $\{(x, y) \mid X \le x \le 2X\, , Y \le y \le 2Y\}$ $(X \ge Y)$ such that any line parallel to any coordinate axis intersects it in $O(1)$ line segments. Let α, β be real numbers, $\alpha\beta \ne 0$, $\alpha + \beta \ne 1$, $\alpha + \beta \ne 2$, and let $f(x, y)$ be a real sufficiently many times differentiable function such that $f(x, y)\tilde{\Delta} Ax^{\alpha}y^{\beta}$ throughout D. Denoting $N = XY$, $F = AX^{\alpha}Y^{\beta}$, we have*

$$\Big|\sum_{(x,y)\in\mathcal{D}} e(f(x, y))\Big| \ll (NF)^{\varepsilon}\big(F^{1/3}N^{1/2} + NY^{-1/2} + N^{5/6} + NF^{-1/4} + NF^{-1/8}X^{-1/8}$$
$$+\Delta^{2/5}F^{1/5}N^{9/10}X^{-2/5} + \Delta^{1/4}NX^{-1/4}\big)\, .$$

Proof: This is a version of Theorem 1 of [9]. The proof may be found in [11].

Lemma 2. *Let $3 < U < V < Z < X$ and suppose that $Z - \frac{1}{2} \in \mathbb{N}$, $X \ge 64Z^2U$, $Z \ge 4U^2$, $V^3 \ge 32N$. Assume further that $F(n)$ is a complex valued function such that $|F(n)| \le 1$. Then the sum*

$$\sum_{n\sim X} \Lambda(n)\, F(n)$$

may be decomposed into $O(\log^{10} X)$ sums, each either of type I *with $L > Z$, or of type* II *with $U < L < V$.*

Proof: This is Lemma 3 of [8].

Lemma 3. *Let x not be an integer, $\alpha \in (0,1)$, $H \geq 3$. Then we have*

$$e(-\alpha\{x\}) = \sum_{|h|\leq H} c_h(\alpha)\, e(hx) + O\Big(\min\Big(1, \frac{1}{H\|x\|}\Big)\Big)$$

where $c_h(\alpha) = \dfrac{1-e(-\alpha)}{2\pi i(h+\alpha)}$.

Proof: See Lemma 12 of [2].

In the following lemma we estimate the number $\mathrm{N}(\Delta)$ of quadruples (h_1, h_2, n_1, n_2) for which $h_1, h_2 \sim H$, $n_1, n_2 \sim N$ and

$$\big|(h_1+\alpha)n_1^c - (h_2+\alpha)n_2^c\big| \leq \Delta\ .$$

Lemma 4. *Suppose that $c \neq 0$, $\alpha \in (0,1)$, $\Delta > 0$, $H \geq 3$ and N is large. Then we have*

$$\mathrm{N}(\Delta) \ll \Delta H N^{2-c} + H^{3/2} N \log(2HN)\ .$$

Proof: We follow the approach of D. R. Heath-Brown from [8]. We define the quantity

$$\begin{aligned}\mathrm{N}(\Delta; a, b) = \ &\#\big\{(h_1, h_2, n_1, n_2)\mid h_1, h_2 \sim H\,,\ (h_1,h_2) = a\,,\ n_1, n_2 \sim N \\ &\qquad (n_1,n_2) = b\,,\ \big|(h_1+\alpha)n_1^c - (h_2+\alpha)n_2^c\big| \leq \Delta\big\}\end{aligned}$$

which we are going to estimate. If $h_1, h_2 \sim H$, $n_1, n_2 \sim N$ and $|(h_1+\alpha)n_1^c - (h_2+\alpha)n_2^c| \leq \Delta$ we have

$$\Big|\Big(\frac{n_1}{n_2}\Big)^c - \frac{h_2+\alpha}{h_1+\alpha}\Big| \ll \frac{\Delta}{HN^c}\quad,\quad \Big|\frac{h_2}{h_1} - \frac{h_2+\alpha}{h_1+\alpha}\Big| \ll \frac{1}{H}\ ,$$

hence

$$\Big|\frac{h_2}{h_1} - \Big(\frac{n_1}{n_2}\Big)^c\Big| \ll \frac{1}{H} + \frac{\Delta}{HN^c}\ . \tag{3}$$

We also have

$$\Big|\frac{n_1}{n_2} - \Big(\frac{h_2+\alpha}{h_1+\alpha}\Big)^{1/c}\Big| \ll \frac{\Delta}{HN^c}\ . \tag{4}$$

From (3) and (4), arguing as on pp.256–257 of [8], we obtain

$$\mathrm{N}(\Delta; a, b) \ll \frac{\Delta}{HN^c} \cdot \frac{H^2N^2}{a^2b^2} + \min\Big(\frac{H^2}{a^2}\ ,\ \frac{N^2}{b^2} + \frac{HN^2}{a^2b^2}\Big)\ .$$

Since

$$\mathrm{N}(\Delta) \leq \sum_{a\leq 2H}\sum_{b\leq 2N} \mathrm{N}(\Delta; a, b)\ ,$$

the proof of the lemma is completed.

4. The main lemma.

Lemma 5. *Suppose that $X > P^{9/10}$, $H = \sigma X^{c-1}$ and $c_h(\alpha)$ are complex numbers such that $|c_h(\alpha)| \ll (1+|h|)^{-1}$. Then, uniformly with respect to $\alpha \in (\omega, 1-\omega)$, we have*

$$T(\alpha) = \sum_{|h|\le H} c_h(\alpha) \sum_{n\sim X} \Lambda(n)\, e((h+\alpha)n^c) \ll X^{2-c-\rho}$$

for some sufficiently small $\rho > 0$, depending only on c.

Proof: We use Lemma 2 with $F(n) = e((h+\alpha)n^c)$ to reduce the esimation of $T(\alpha)$ to the estimation of sums

$$T_i(\alpha) = \sum_{|h|\le H} c_h(\alpha) \sum\nolimits_i \qquad (i = 1, 2)$$

where $\sum_1$, $\sum_2$ are type I and type II sums, correspondingly. We choose the parameters U, V, Z as follows:

$$U = X^{2c-2+2\rho}/256 \quad , \quad V = 4X^{1/3}$$

and

$$Z = \begin{cases} \left[X^{(16c-16)/3+3\rho}\right] + 1/2 & \text{,if } 17/16 \le c < 14/13\ ; \\ \left[X^{(13c-13)/3+3\rho}\right] + 1/2 & \text{,if } 14/13 \le c < 13/12\ ; \\ \left[X^{(20c-21)/2+5\rho}\right] + 1/2 & \text{,if } 13/12 \le c < 12/11\ . \end{cases}$$

Let us consider $T_2(\alpha)$. We have

$$T_2(\alpha) \ll \max_{\omega\le\lambda\le 2} \left|T_2^{(1)}(\lambda)\right| + \log X \max_{2\le J\le H} \left|T_2^{(2)}(\alpha; J)\right| \tag{5}$$

where

$$T_2^{(1)}(\lambda) = \sum_{m\sim M} \sum_{n\sim L} a_m\, b_n\, e(\lambda(mn)^c)$$
$$T_2^{(2)}(\alpha; J) = \sum_{h\sim J} c_h(\alpha) \sum_{m\sim M} \sum_{n\sim L} a_m\, b_n\, e((h+\alpha)(mn)^c)\ .$$

First we esimate $T_2^{(2)}(\alpha; J)$. We obtain

$$T_2^{(2)}(\alpha; J) \ll \frac{X^\varepsilon}{J} \sum_{m\sim M} \sum_{q\le Q} \Big| \sum_{(h,n)\in\mathcal{I}_q} d(h,n)\, e((h+\alpha)(mn)^c) \Big|$$

where $|d(h,n)| \le 1$, $Q > 1$ is a parameter to be defined later and for $q \le Q$

$$\mathrm{I}_q = \{(h,n) \mid h\sim J\, ,\ n\sim L\, ,\ 5(q-1)JL^c < Q(h+\alpha)n^c \le 5qJL^c\}\ .$$

So, using the Cauchy inequality, we get

$$\left|T_2^{(2)}(\alpha; J)\right|^2 \ll \frac{X^\varepsilon MQ}{J^2} \sum\sum_{\substack{h_1,h_2\sim J\\ n_1,n_2\sim L\\ |\lambda|\le 5JL^c/Q}} \Big| \sum_{m\sim M} e(\lambda m^c) \Big|$$

where $\lambda = (h_1+\alpha)n_1^c - (h_2+\alpha)n_2^c$. We estimate the innermost sum trivially if $|\lambda| \le M^{-c}$, and using the exponent pair $(\frac{13}{40}, \frac{11}{20})$ otherwise. From Lemma 4 we obtain now

$$\begin{aligned}
\left|T_2^{(2)}(\alpha;J)\right|^2 &\ll \frac{X^\varepsilon MQ}{J^2}\Big(M\,\mathrm{N}(M^{-c})+ \\
&\qquad + \max_{M^{-c}\le\Delta\le 5JL^c/Q}\left(\Delta^{13/40}M^{(9+13c)/40}+\Delta^{-1}M^{1-c}\right)\mathrm{N}(\Delta)\Big) \\
&\ll X^\varepsilon\big(J^{-1/2}M^2LQ + J^{13/40}M^{(49+13c)/40}L^{(80+13c)/40}Q^{-13/40}+ \\
&\qquad +J^{-1}M^{2-c}L^{2-c}Q + J^{-7/40}M^{(49+13c)/40}L^{(40+13c)/40}Q^{27/40}\big)\,.
\end{aligned}$$

We choose Q via Lemma 2.4 of [6] and the conditions on J, M and L imply

$$\max_{2\le J\le H}\left|T_2^{(2)}(\alpha;J)\right| \ll X^{2-c-\rho+\varepsilon}\,. \tag{6}$$

Let us estimate now $T_2^{(1)}(\lambda)$. Using the Cauchy inequality and Lemma 2.5 of [6] we get

$$\left|T_2^{(1)}(\lambda)\right|^2 \ll X^\varepsilon\left(\frac{M^2L^2}{Q}+\frac{ML}{Q}\sum_{q\le Q}\sum_{n\sim L}\Big|\sum_{m\sim M}e(\lambda((n+q)^c-n^c)m^c)\Big|\right)$$

where $Q \ll L$ is a positive integer. We apply the exponent pair $(\frac{13}{40}, \frac{11}{20})$ to the innermost sum and choose Q via Lemma 2.4 of [6] to obtain

$$\begin{aligned}
\left|T_2^{(1)}(\lambda)\right|^2 &\ll X^\varepsilon\Big(M^2L+\lambda^{13/40}M^{(49+13c)/40}L^{(67+13c)/40} \\
&\qquad +\lambda^{13/53}M^{(75+13c)/53}L^{(93+13c)/53}\Big)
\end{aligned}$$

and using the conditions on M, L and λ:

$$\max_{\omega\le\lambda\le 2}\left|T_2^{(1)}(\lambda)\right| \ll X^{2-c-\rho+\varepsilon}\,. \tag{7}$$

The needed estimate for $T_2(\alpha)$ follows from (5)–(7).

Let us consider now $T_1(\alpha)$. We have

$$T_1(\alpha) \ll X^\varepsilon \max_{|\lambda|\in(\omega,H+1)}\sum_{m\sim M}\Big|\sum_{n\sim L}e(\lambda(mn)^c)\Big|\,. \tag{8}$$

If $L \ge X^{(57c-49)/23+3\rho}$ we esimate the sum over n using the exponent pair $(\frac{8}{41}, \frac{26}{41})$ and we obtain

$$\left|T_1(\alpha)\right| \ll X^{2-c-\rho+\varepsilon}\,. \tag{9}$$

Otherwise we first use the Cauchy inequality and Lemma 2.5 of [6] to the sum in the right-hand side of (8) and obtain

$$|T_1|^2 \ll X^\varepsilon\left(\frac{M^2L^2}{Q}+\frac{ML}{Q}\sum_{q\sim J}\sum_{n\sim L}\sum_{m\sim M}e(f(m,n,q))\right)$$

where $f(m,n,q)=\lambda((n+q)^c-n^c)m^c$, $J\le Q/2$ and $Q\ll L$ is parameter to be chosen later. Then we apply Poisson summation formula (Lemma 3.6 of [6]) to the sums over m and n successively and Abel's transformation:

$$\begin{aligned}
&\sum_q\sum_{m,n} e(f(m,n,q))\\
&=\sum_{q,n}\sum_{\mu}\Big(\frac{\partial^2 f(m_\mu,n,q)}{\partial m^2}\Big)^{-1/2} e(1/8+f(m_\mu,n,q)-\mu m_\mu)\\
&\qquad\qquad\qquad\qquad\qquad\qquad +\mathrm{O}\big(MLJF^{-1/2}+LJ\log X\big)
\end{aligned}$$

$$\begin{aligned}
&\ll MF^{-1/2}\Big|\sum_{q,\mu}\sum_{n} e(f_1(\mu,q,n))\Big| + XJF^{-1/2}+LJ\log X\\
&\ll MF^{-1/2}\Big|\sum_{q,\mu}\sum_{\nu}\Big(\frac{\partial^2 f_1(\mu,q,n_\nu)}{\partial n^2}\Big)^{-1/2} e(1/8+f_1(\mu,q,n_\nu)-\nu n_\nu)\Big|\\
&\qquad\qquad +MF^{-1/2}JFM^{-1}\big(LF^{-1/2}+\log X\big)+XJF^{-1/2}+LJ\log X\\
&\ll MLF^{-1}\Big|\sum_{q,\mu,\nu} e(g(\mu,\nu,q))\Big| + F^{1/2}J\log X+LJ\log X+XJF^{-1/2}
\end{aligned}$$

where $F=\lambda JM^cL^{c-1}$, $f_1(\mu,q,n)=f(m_\mu,n,q)-\mu m_\mu$,

$$g(\mu,\nu,q)=f_1(\mu,q,n_\nu)-\nu n_\nu\tilde{\Delta}c_0(\lambda q)^{1/(2-2c)}\nu^{1/2}\mu^{c/(2c-2)}\asymp F\ ,$$

c_0—a constant depending only on c, $\Delta=J/L$, $\nu\asymp FL^{-1}$, $\mu\asymp FM^{-1}$.
Hence

$$\begin{aligned}
X^{-\varepsilon}|T_1|^2 &\ll X^2Q^{-1}+X^2F^{-1}Q^{-1}\sum_{q\sim J}\Big|\sum_{\mu\asymp FM^{-1}}\ \sum_{\nu\asymp FL^{-1}} e(g(\mu,\nu,q))\Big|\\
&\qquad\qquad +X^2F^{-1/2}+XL+XF^{1/2}\ . \qquad (10)
\end{aligned}$$

If $X^{1/2}\le L<X^{(57c-49)/23+3\rho}$ we estimate the sum over μ,ν in (10) using Lemma 1 with $X=FM^{-1}$, $Y=FL^{-1}$ and $f(x,y)=g(\mu,\nu,q)$. We get

$$\begin{aligned}
X^{-\varepsilon}|T_1|^2 &\ll X^2Q^{-1}+F^{1/3}X^{3/2}+XF^{1/2}L^{1/2}+X^{7/6}F^{2/3}+X^{3/2}F^{3/5}J^{2/5}L^{-4/5}\\
&\qquad +XF^{3/4}M^{1/8}+J^{1/4}X^{5/4}F^{3/4}L^{-1/2}+X^2F^{-1/2}+XL\ .
\end{aligned}$$

Now we substitute the expression for F in the last estimate and choose Q via Lemma 2.4 of [6]. We obtain (9).

If $Z\le L<X^{1/2}$ we interchange roles of μ and ν and we prove again that the estimate (9) holds.

This completes the proof of the lemma.

5. Proof of Theorem 1: It is easy to see that

$$R(N) = \int_0^1 S^3(\alpha)\, e(-\alpha N)\, d\alpha = R_1 + R_2 \ .$$

The integral R_1 is studied by Laporta and Tolev in [10], pp.928–929. They proved that if $1 < c < 17/16$

$$R_1 = \frac{\Gamma^3\big(1+\frac{1}{c}\big)}{\Gamma\big(\frac{3}{c}\big)} N^{3/c-1} + \mathrm{O}\big(\sigma^{-1} N^{3/c-1}\big)$$

but the same argument shows that this asymptotic formula holds for $1 < c < 3/2$. Hence the theorem follows from the estimate

$$R_2 \ll \sigma^{-1} P^{3-c} \ . \tag{11}$$

It is not difficult to prove that

$$R_2 \ll P \log P \max_{\alpha \in \Omega_2} |S(\alpha)| \ .$$

To prove (11) it remains to show that

$$\max_{\alpha \in \Omega_2} |S(\alpha)| \ll \sigma^{-1} P^{2-c} \ .$$

We have

$$S(\alpha) = \sum_{n \le P} \Lambda(n)\, e(\alpha n^c)\, e(-\alpha \{n^c\}) + \mathrm{O}(P^{1/2}) \ .$$

So, it is sufficient to obtain that for X satisfying $P^{9/10} < X \le P$

$$S_1(\alpha) = \sum_{n \sim X} \Lambda(n)\, e(\alpha n^c)\, e(-\alpha \{n^c\}) \ll \sigma^{-1} X^{2-c} \ .$$

Using Lemma 3 with $x = n^c$ and $H = \sigma X^{c-1}$ we obtain

$$S_1(\alpha) = \sum_{|h| \le H} c_h(\alpha) \sum_{n \sim X} \Lambda(n)\, e((h+\alpha) n^c) + \mathrm{O}\Big(\log X \sum_{n \sim X} \min\Big(1 \, , \, \frac{1}{H \|n^c\|}\Big)\Big) \ .$$

The estimation of the error term in the above equality is standard (see [8], pp.245–246). Hence (11) follows from Lemma 5.

The proof of Theorem 1 is copmleted.

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