CORRIGENDUM TO "ON THE PIATETSKI-SHAPIRO–VINOGRADOV THEOREM" [JTNB 9 (1997), 11–23]

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1. Introduction

Recently, the author was alerted by the authors of [2] that there is a seemingly unfixable gap in the proof of Lemma 2 in [3]. As that lemma is a crucial ingredient in the proof of the main proposition and all the theorems in [3], this casts doubt on all the results in [3]. The purpose of this note is to show how some small adjustments to the original proof of [3, Lemma 2] close the gap and reclaim the results of [3].

2. The compromised lemma

We retain all the notation from [3]. Lemma 2 of that paper establishes the bound

$$S_I := \Phi(H) \sum_{h \sim H} \Bigg| \sum_{m \sim M} \sum_{n \sim N} a(m) e\big(\alpha mn + h(mn+u)^\gamma\big) \Bigg| \ll x^{1-\delta-\varepsilon},$$

under the assumptions that[1]

$$16(1-\gamma) + 19\delta < 2 \tag{12o}$$

and

$$N \geqslant \max(x^{36(1-\gamma)+42\delta-4+\varepsilon}, x^{2(1-\gamma)+4\delta+\varepsilon}). \tag{13o}$$

The proof of that lemma goes through several steps and chooses several optimization parameters. In [2, §3.2], Guo *et al.* note that the last of those chosen parameters,

$$Q_2 = \big[x^{20(1-\gamma)+24\delta-2+\varepsilon} N^{-1}\big] + 1, \tag{27o}$$

may fail the requirement $Q_2 \leqslant FM^{-1} = qx^\gamma H(MN)^{-1}$ for some values of q and H. This is indeed the case: the choice (27o) was made based on the worst-case scenario for q, H, and when q, H are not close to that case, the last inequality may fail. We can avoid this issue by modifying some of the original choices so that all comparisons involve the same values of q, H.

Date: June 2025.

[1]Numbered displays from the original article [3] are displayed here with their original equation numbers, followed by the letter 'o'; new versions of old numbered displays will be listed with their old equation numbers, followed by the letter 'n'.

3. Corrected proof

To describe the required changes, we follow the original proof of Lemma 2 until the application of [1, Theorem 2.9] on page 20. We apply that result as before, but we note that the resulting bound is admissible not only when

$$N \geqslant x^{20(1-\gamma)+24\delta-2+\varepsilon},$$

but whenever $N \geqslant x^{4\gamma-6+\varepsilon} q^{10} H^4$. Thus, one may replace the assumption (20) in [3] with

$$N \leqslant x^{4\gamma-6+\varepsilon} q^{10} H^4. \tag{20n}$$

We then follow the proof of the lemma until the choice

$$Q_1 = \left[x^{6(1-\gamma)+7\delta-1+\varepsilon} N\right] + 1, \tag{22o}$$

which we replace by

$$Q_1 = \left[x^{\gamma-2+\varepsilon} q^3 H N\right] + 1. \tag{22n}$$

We remark that

$$x^{\gamma-2+\varepsilon} q^3 H \leqslant x^{6(1-\gamma)+7\delta-1+\varepsilon} < 1,$$

under (12o), so this new choice of Q_1 is smaller. However, it is easy to check that, even with this modification, the contribution of the first term on the right side of [3, (21)] to the right side of [3, (19)] is still acceptable.

We then follow the proof of Lemma 2 until the estimation of the sum T in [3, (25)]. At that point, we apply [1, Lemma 2.5] as before, but we alter the choice of Q_2 from (27o) above to

$$Q_2 = \left[x^{4\gamma-6+\varepsilon} q^{10} H^4 N^{-1}\right] + 1. \tag{27n}$$

It is again easy to check that the contribution from the first term on the right side of [3, (25)] to the upper bound for S_I is acceptable. However, with this choice, the condition $Q_2 \leqslant FM^{-1}$ reduces to

$$N \gg x^{3\gamma-5+\varepsilon} q^9 H^3,$$

and this holds because

$$x^{3\gamma-5} q^9 H^3 \ll x^{3\gamma-5} Q^9 H^3 \ll x^{18(1-\gamma)+21\delta-2+\varepsilon} \ll x^{2(1-\gamma)+2\delta} \ll N x^{-2\delta},$$

on using (12o) and (13o). From there, the proof can proceed as before.

Acknowledgment. The author would like to thank Professor Victor Guo and his collaborators for noticing the above issue and alerting him to their preprint [2].

References

[1] S. W. Graham and G. A. Kolesnik, *Van der Corput's Method of Exponential Sums*, LMS Lecture Notes **126**, Cambridge University Press, 1991.

[2] L. Guo, V. Z. Guo, and L. Lu, *Improvements on exponential sums related to Piatetski–Shapiro primes*, preprint, arXiv:2504.11464.

[3] A. Kumchev, *On the Piateski-Shapiro–Vinogradov theorem*, J. Théor. Nombres Bordeaux **9** (1997), 11–23.