ON SUMS OF POWERS OF ALMOST EQUAL PRIMES

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Abstract. Let $k \geqslant 2$ and s be positive integers, and let n be a large positive integer subject to certain local conditions. We prove that if $s \geqslant k^2+k+1$ and $\theta > 31/40$, then n can be expressed as a sum $p_1^k + \cdots + p_s^k$, where $p_1, \dots, p_s$ are primes with $|p_j - (n/s)^{1/k}| \leqslant n^{\theta/k}$. This improves on earlier work by Wei and Wooley [15] and by Huang [8] who proved similar theorems when $\theta > 19/24$.

1. Introduction

The study of additive representations of integers as sums of powers of primes goes back to the work of Hua [6, 7]. In particular, Hua proved that when k and s are positive integers with $s > 2^k$, every sufficiently large natural number n satisfying certain local solubility conditions can be represented as

$$n = p_1^k + \cdots + p_s^k, \tag{1.1}$$

where $p_1, \dots, p_s$ are prime numbers. (Henceforth, the letter p, with or without subscripts, always denotes a prime number.) To describe the local conditions, we let $\tau = \tau(k,p)$ be the largest integer with $p^\tau \mid k$, and then define

$$K(k) = \prod_{(p-1)|k} p^{\gamma(k,p)}, \qquad \gamma(k,p) = \begin{cases} \tau(k,p)+2 & \text{when } p=2, \tau>0, \\ \tau(k,p)+1 & \text{otherwise.} \end{cases}$$

One typically studies (1.1) for n restricted to the congruence class

$$\mathcal{H}_{k,s} = \big\{ n \in \mathbb{N} : n \equiv s \pmod{K(k)} \big\}.$$

In this paper, we are interested in the additive representations of the form (1.1) with "almost equal" primes. Given a large integer $n \in \mathcal{H}_{k,s}$, we ask whether it is possible to solve (1.1) in primes subject to

$$\big| p_j - (n/s)^{1/k} \big| \leqslant H \qquad (1 \leqslant j \leqslant s), \tag{1.2}$$

where $H = o(n^{1/k})$. There is a long list of results on sums of five or fewer almost equal squares ($k=2$, $3 \leqslant s \leqslant 5$), beginning with the work of Liu and Zhan [11] and culminating with the results of Kumchev and Li [10] (see [10] for a detailed history of that problem). In particular, Kumchev and Li showed that when $k=2$ and $s=5$ the problem has solutions with $H = n^{\theta/2}$ for any fixed $\theta > 8/9$. They were also the first to obtain results on sums of more than five almost equal squares, where the extra variables are used to reduce the admissible size of H. Let $\theta_{k,s}$ denote the least exponent θ such that (1.1) and (1.2) with $H = n^{\theta/k}$ can be solved for sufficiently large $n \in \mathcal{H}_{k,s}$ whenever $\theta > \theta_{k,s}$. Kumchev and Li [10] proved that $\theta_{2,s} \leqslant 19/24$ when $s \geqslant 17$. The lower bound on s in this theorem was reduced to $s \geqslant 7$ in a recent paper by Wei and Wooley [15], in which those authors also established surprisingly strong results for higher values of k: they proved that if $s > 2k(k-1)$, one has

$$\theta_{k,s} \leqslant \begin{cases} 4/5 & \text{if } k=3, \\ 5/6 & \text{if } k \geqslant 4. \end{cases} \tag{1.3}$$

Huang [8] further reduced the bound (1.3) to $\theta_{k,s} \leqslant 19/24$ for all $k \geqslant 3$ and $s > 2k(k-1)$.

The main goal of the present work is to establish the bound $\theta_{k,s} \leqslant 31/40$ for all $k \geqslant 2$. We also make use of a recent breakthrough by Bourgain, Demeter and Guth [2] to reduce the lower bound on s when $k \geqslant 4$. Our main result is as follows.

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Theorem 1. *Let $k \geqslant 2$, $s \geqslant k^2 + k + 1$, and $\theta > 31/40$. When $n \in \mathcal{H}_{k,s}$ is sufficiently large, equation* (1.1) *has solutions in primes $p_1, \dots, p_s$ satisfying* (1.2) *with $H = n^{\theta/k}$.*

Circle method experts will not be surprised that our methods lead also to improvements on the results established by Wei and Wooley [15] and by Huang [8] on solubility for "almost all" n and on the number of exceptions for representations by six almost equal squares. Indeed, by adapting the ideas in [15, §9], we obtain the following theorems.

Theorem 2. *Let $k \geqslant 2$, $s > k(k+1)/2$, $\theta > 31/40$, and $N \to \infty$. There is a fixed $\delta > 0$ such that equation* (1.1) *has solutions in primes $p_1, \dots, p_s$ satisfying* (1.2) *with $H = n^{\theta/k}$ for all but $O(N^{1-\delta})$ integers $n \leqslant N$ subject to $n \in \mathcal{H}_{k,s}$ (and, when $k = 3$ and $s = 7$, also $9 \nmid n$).*

Theorem 3. *Let $\theta > 31/40$, and $N \to \infty$. Let $E_6(N; H)$ denote the number of integers $n \equiv 6 \pmod{24}$, with $|n - N| \leqslant HN^{1/2}$, such that equation* (1.1) *with $k = 2$ and $s = 6$ has solutions in primes $p_1, \dots, p_6$ satisfying* (1.2). *There is a fixed $\delta > 0$ such that*

$$E_6(N; N^{\theta/2}) \ll N^{(1-\theta)/2-\delta}.$$

Notation. Throughout the paper, the letter ϵ denotes a sufficiently small positive real number. Any statement in which ϵ occurs holds for each positive ϵ, and any implied constant in such a statement is allowed to depend on ϵ. The letter c denotes a constant that depends at most on k and s, not necessarily the same in all occurrences. As usual in number theory, $\mu(n)$, $\Lambda(n)$, $\phi(n)$, and $\tau(n)$ denote, respectively, the Möbius function, von Mangoldt's function, Euler's totient function, and the number of divisors function. We write $e(x) = \exp(2\pi i x)$ and $(a, b) = \gcd(a, b)$, and we use $m \sim M$ as an abbreviation for the condition $M \leqslant m < 2M$. If χ denotes a Dirichlet character, we set $\delta_\chi = 1$ or 0 according as χ is principal or not. The sums $\sum_{\chi \bmod q}$ and $\sum^*_{\chi \bmod q}$ denote summations over all the characters modulo q and over the primitive characters modulo q, respectively.

2. Outline of the proof

Let $x = (n/s)^{1/k}$, $y = x^\theta$, $\mathcal{I} = (x - y, x + y]$, and write

$$R_{k,s}(n) = \sum_{\substack{n = p_1^k + \cdots + p_s^k \\ p_i \in \mathcal{I}}} 1.$$

Let $\mathbf{1}_{\mathbb{P}}$ denote the indicator function of the primes, and suppose that we have arithmetic functions $\lambda^\pm$ such that, for $m \in \mathcal{I}$,

$$\lambda^-(m) \leqslant \mathbf{1}_{\mathbb{P}}(m) \leqslant \lambda^+(m). \tag{2.1}$$

Then the vector sieve of Brüdern and Fouvry [3, Lemma 13] yields

$$\mathbf{1}_{\mathbb{P}}(m_1) \cdots \mathbf{1}_{\mathbb{P}}(m_5) \geqslant \sum_{i=1}^{5} \lambda^-(m_i) \prod_{j \neq i} \lambda^+(m_j) - 4\lambda^+(m_1) \cdots \lambda^+(m_5). \tag{2.2}$$

Thus, by the symmetry of the problem, we have

$$R_{k,s}(n) \geqslant 5R_{k,s}(n, \lambda^-) - 4R_{k,s}(n, \lambda^+), \tag{2.3}$$

where

$$R_{k,s}(n, \lambda) = \sum_{\substack{n = p_1^k + \cdots + p_{s-5}^k + m_1^k + \cdots + m_5^k \\ p_i, m_j \in \mathcal{I}}} \lambda(m_1)\lambda^+(m_2) \cdots \lambda^+(m_5).$$

To prove the theorem, we show that one can choose sieve functions $\lambda^\pm$ satisfying (2.1) so that the right side of (2.3) is positive. Our choice of $\lambda^\pm$ is borrowed from Baker, Harman and Pintz [1]—namely, λ^- and λ^+ are, respectively, the functions a_0 and a_1 constructed in §4 of that paper. In many ways, the functions $\lambda^\pm$ imitate the indicator function $\mathbf{1}_{\mathbb{P}}$ of the primes $p \in \mathcal{I}$. We will discuss the similarities in detail later (see §3 below) and will focus here on their most crucial property:

(A0) Let $A, B > 0$ be fixed (possibly large) numbers and let $x \to \infty$. If χ is a Dirichlet character modulo $q \leqslant (\log x)^B$ and $x^{11/20+\epsilon} \leqslant y \leqslant x \exp\big(-(\log x)^{1/3}\big)$, then one has

$$\sum_{|m-x| \leqslant y} \lambda^{\pm}(m)\chi(m) = \frac{2y}{\phi(q)\log x}\big(\delta_\chi \kappa_{\pm} + O\big((\log x)^{-A}\big)\big), \tag{2.4}$$

where $\kappa_{\pm}$ are absolute constants satisfying

$$\kappa_- > 0.99, \qquad \kappa_+ < 1.01. \tag{2.5}$$

We now sketch the application of the circle method to $R_{k,s}(n, \lambda)$. Let $\delta > 0$ be a fixed number, to be chosen later sufficiently small in terms of k, s and θ, and set

$$P = y^{\delta}, \quad Q = x^{k-2}y^2P^{-1}, \quad L = \log x. \tag{2.6}$$

We write

$$\mathfrak{M}(q, a) = \big\{\alpha \in \mathbb{R} : |q\alpha - a| \leqslant Q^{-1}\big\},$$

and define the sets of major and minor arcs by

$$\mathfrak{M} = \bigcup_{\substack{1 \leqslant a \leqslant q \leqslant P \\ (a,q)=1}} \mathfrak{M}(q, a) \quad \text{and} \quad \mathfrak{m} = \big[Q^{-1}, 1 + Q^{-1}\big] \backslash \mathfrak{M}, \tag{2.7}$$

respectively. Further, for any Lebesgue measurable set $\mathfrak{B}$, we write

$$R_{k,s}(n, \lambda; \mathfrak{B}) = \int_{\mathfrak{B}} f(\alpha, \mathbf{1}_{\mathbb{P}})^{s-5} f(\alpha, \lambda) f(\alpha, \lambda^+)^4 e(-n\alpha)\, d\alpha,$$

where

$$f(\alpha, \lambda) = \sum_{m \in \mathcal{I}} \lambda(m) e(m^k \alpha). \tag{2.8}$$

By orthogonality and (2.7), we have

$$R_{k,s}(n, \lambda) = R_{k,s}(n, \lambda; \mathfrak{M}) + R_{k,s}(n, \lambda; \mathfrak{m}). \tag{2.9}$$

In §4, we show that when $s \geqslant k^2 + k + 1$, $\delta < 1/(16k)$, and $\theta \geqslant 31/40$, one has

$$R_{k,s}(n, \lambda; \mathfrak{m}) \ll y^{s-1-\delta/(3k)} x^{1-k}. \tag{2.10}$$

Then, in §5, we show that when $\delta \leqslant 2(\theta - 31/40)$, one has

$$R_{k,s}(n, \lambda^{\pm}; \mathfrak{M}) = \mathfrak{C}(n) y^{s-1} x^{1-k} L^{-s}\big(\kappa_{\pm}\kappa_+^4 + O(L^{-1})\big), \tag{2.11}$$

where $1 \ll \mathfrak{C}(n) \ll 1$ for sufficiently large $n \in \mathcal{H}_{k,s}$, and $\kappa_{\pm}$ are the constants from (2.4). Theorem 1 follows from (2.3), (2.5), and (2.9)–(2.11). □

3. The sieve weights

As we said before, we use sieve weights $\lambda^{\pm}$ constructed by Baker, Harman and Pintz [1] to have properties (2.1) and (A0) above. We remark that (A0) is a short-interval version of the Siegel–Walfisz theorem: when the functions $\lambda^{\pm}$ are replaced by $\mathbf{1}_{\mathbb{P}}$, the asymptotic formula (2.4) with $\kappa = 1$ and $y \geqslant x^{7/12+\epsilon}$ is a well-known extension of a celebrated result of Huxley [9]. In this section, we record some additional properties of the weights $\lambda^{\pm}$ that we will need later in the paper:

(A1) The functions $\lambda^{\pm}(m)$ vanish if m has a prime divisor $p < x^{1/10}$.

(A2) Let $\mathbb{S} = \{p^j : p \in \mathbb{P}, j \geqslant 2\}$. When $m \sim 2x/3$, one can express $\lambda^{\pm}(m)$ as a linear combination of a bounded function supported on $\mathbb{S}$ and of $O(L^c)$ triple convolutions of the form

$$\sum_{\substack{m=uvw \\ u \sim U,\ v \sim V}} \xi_u \eta_v \zeta_w,$$

where $|\xi_u| \leqslant \tau(u)^c$, $|\eta_v| \leqslant \tau(v)^c$, $\max(U, V) \ll x^{11/20}$, and either $\zeta_w = 1$ for all w, or $|\zeta_w| \leqslant \tau(w)^c$ and $UV \gg x^{27/35}$.

(A3) Let $A, B, \epsilon > 0$ be fixed, let χ be a Dirichlet character modulo $q \leqslant L^B$, and put $T_0 = \exp(L^{1/3})$ and $T_1 = x^{9/20-\epsilon}$. Then

$$\int_{T_0}^{T_1} \Big| \sum_{m \sim 2x/3} \lambda^{\pm}(m)\chi(m)m^{-1/2-it} \Big| \, dt \ll x^{1/2}L^{-A}.$$

Of the three properties above, (A3) is the easiest to justify, since it is a part of the proof of (A0) in [1]. Indeed, the method of Baker, Harman and Pintz reduces (2.4) to the classical Siegel–Walfisz theorem by decomposing $\lambda^{\pm}$ into a linear combination of $O(L^c)$ arithmetic functions for which (A3) holds and then applying [1, Lemma 11] to each of them. In order to justify that the functions $\lambda^{\pm}$ have also properties (A1) and (A2), we need to provide some details on their construction.

The core idea behind the construction of $\lambda^{\pm}$ is explained in [1, pages 32–33, 41–42]. It amounts to setting

$$\lambda^{\pm}(m) = \mathbf{1}_{\mathbb{P}}(m) \pm \sum_{j=1}^{J^{\pm}} \lambda_j^{\pm}(m) \tag{3.1}$$

where $J^{\pm} = O(1)$ and the arithmetic functions $\lambda_j^{\pm}$ have the form

$$\lambda_j^{\pm}(m) = \sum_{m=u_1 \cdots u_{d+1}} \xi(u_1, \dots, u_{d+1}) \qquad (4 \leqslant d \leqslant 7),$$

with $\xi(u_1, \dots, u_{d+1}) = 1$ or 0. The latter functions impose various restrictions on the sizes and arithmetic properties of $u_1, \dots, u_{d+1}$ that amount to restricting the support of $\lambda_j^{\pm}$ to integers m with very specific (undesirable) factorizations. Moreover:

(i) Only the cases $d = 4$ and $d = 6$ occur in the construction of λ^-, whereas only $d = 5$ and $d = 7$ occur in the construction of λ^+.

(ii) $\xi(u_1, \dots, u_{d+1}) = 0$ if any of $u_1, \dots, u_{d+1}$ has a prime divisor $< x^{1/10}$. Note that property (A1) is an immediate consequence of this observation.

(iii) When $d = 5$, λ_j^+ is supported on integers m that have a divisor u in the range $x^{0.46} \leqslant u \leqslant x^{1/2}$: see [1, p. 42].

(iv) When $d = 4$, λ_j^- is supported on integers $m = n_1n_2n_3$, where $n_i = x^{\alpha_i}$ with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ lying in one of regions Γ, Δ_2, Δ_3, or Δ_4 in [1, Diagram 1 on p. 33].

We now turn to property (A2). We note that when $\lambda_j^{\pm}$ is supported on integers $m = uv$, with $x^{9/20} \leqslant u \leqslant x^{11/20}$, it has property (A2). Thus, by (iii) above, property (A2) holds for all terms λ_j^+ with $d = 5$. Moreover, the same is true for λ_j^- with $d = 4$ and $\boldsymbol{\alpha}$ in one of the regions Δ_3 or Δ_4: we have $0.46 \leqslant \alpha_1 \leqslant 0.5$ when $\boldsymbol{\alpha} \in \Delta_4$, and $0.46 \leqslant \alpha_1 + \alpha_2 \leqslant 0.54$ when $\boldsymbol{\alpha} \in \Delta_3$.

We next consider the case $d \geqslant 6$ and suppose that the variables u_i have been labelled so that $u_1 \geqslant u_2 \geqslant \cdots \geqslant u_{d+1}$. When $\lambda_j^{\pm}$ is supported on integers $m = u_1 \cdots u_{d+1}$ with $u_4 \cdots u_{d+1} \geqslant x^{11/20}$, we have

$$u_1u_2u_3 \ll x^{9/20} \quad \text{and} \quad u_4 \leqslant \sqrt[3]{u_1u_2u_3} \ll x^{3/20}.$$

Since $u_5 \cdots u_{d+1} \ll x^{1/2}$, we can then verify that $\lambda_j^{\pm}$ has property (A2) by grouping the variables $u_1, \dots, u_{d+1}$ into $u = u_1u_2u_3$, $v = u_5 \cdots u_{d+1}$, and $w = u_4$. On the other hand, when $\lambda_j^{\pm}$ is supported on integers $m = u_1 \cdots u_{d+1}$ with $u_4 \cdots u_{d+1} \leqslant x^{11/20}$, we note that

$$u_1u_2 \ll x^{1/2} \quad \text{and} \quad u_3 \leqslant \sqrt[3]{u_1u_2u_3} \ll x^{1/5}.$$

Thus, we can verify that $\lambda_j^{\pm}$ has property (A2) by grouping the variables $u_1, \dots, u_{d+1}$ into $u = u_1u_2$, $v = u_4 \cdots u_{d+1}$, and $w = u_3$.

The functions λ_j^- with $d = 4$ and $\boldsymbol{\alpha} \in \Delta_2$ are supported on integers $m = u_1 \cdots u_5$, where

$$x^{1/10} \leqslant u_4 \leqslant u_3 \leqslant u_2 \leqslant u_1, \quad \text{and} \quad x^{0.32} \leqslant u_1u_2 \leqslant x^{0.36}. \tag{3.2}$$

(These functions arise by "decomposing twice the variable n_3" in [1, (4.24)], so we have $u_1u_2 = x^{\alpha_1+\alpha_2}$.) Since the inequalities (3.2) imply that

$$x^{1/10} \leqslant u_4 \leqslant u_3 \leqslant x^{0.18}, \quad u_1u_2u_3 \leqslant x^{0.54}, \quad u_5 \ll x^{0.48},$$

we can verify that λ_j^- has property (A2) by grouping the variables $u_1, \dots, u_5$ into $u = u_1u_2u_3$, $v = u_5$, and $w = u_4$. Similarly, the functions λ_j^- with $d = 4$ and $\boldsymbol{\alpha} \in \Gamma$ are supported on integers $m = u_1 \cdots u_5$, where

$$x^{0.32} \ll u_1u_2, u_3u_4 \ll x^{0.36}, \quad \text{and} \quad u_5 \leqslant x^{1/3}.$$

(In this case, we have $u_1u_2 = x^{\alpha_1}$ and $u_5 = x^{\alpha_2}$.) If we assume that the variables are labelled so that $u_1 \leqslant u_2$ and $u_3 \leqslant u_4$, we have

$$u_2u_4 \leqslant x^{0.72}/(u_1u_3) \leqslant x^{0.52}, \quad u_1u_5 \leqslant x^{0.18}x^{1/3} < x^{0.52}, \quad u_3 \leqslant x^{0.18}.$$

Hence, we can once again verify that λ_j^- has property (A2) by grouping the variables $u_1, \dots, u_5$ into $u = u_2u_4$, $v = u_1u_5$, and $w = u_3$.

We have shown that each term $\lambda_j^\pm$ on the right side of (3.1) satisfies (A2). It remains to show that so does the indicator function $\mathbf{1}_\mathbb{P}$. The proof of [4, Theorem 1.1] uses Heath-Brown's identity to establish (A2) for von Mangoldt's function. In the case of $\mathbf{1}_\mathbb{P}$, we can use a variant of that argument based on Linnik's identity instead of Heath-Brown's.

4. The minor arcs

In this section, we establish inequality (2.10). Our main tools are Propositions 1 and 2 below.

Proposition 1. *Suppose that $k \geqslant 2$, $s \geqslant k^2 + k$, and $y \geqslant x^{1/2}$. Then for any bounded arithmetic function λ, one has*

$$I_s(\lambda) := \int_0^1 |f(\alpha, \lambda)|^s \, d\alpha \ll y^{s-1}x^{1-k+\epsilon}. \tag{4.1}$$

Proposition 2. *Let $k \geqslant 2$, $0 < \delta < 1/(16k)$, and $y \geqslant x^{31/40}$, and suppose that $\alpha \in \mathfrak{m}$. Then*

$$f(\alpha, \mathbf{1}_\mathbb{P}) \ll y^{1-\delta/(2k)+\epsilon}.$$

It is straightforward to deduce (2.10) from these propositions. First, we remark that the functions $\lambda^\pm$ are bounded by construction—they are linear combinations of a bounded number of indicator functions. Thus, we may apply Proposition 1 to $\lambda = \lambda^\pm$. By Hölder's inequality,

$$|R_{k,s}(n, \lambda; \mathfrak{m})| \leqslant \Big(\sup_{\alpha \in \mathfrak{m}} |f(\alpha, \mathbf{1}_\mathbb{P})|\Big) I_{s-1}(\lambda)^u I_{s-1}(\lambda^+)^{4u} I_{s-1}(\mathbf{1}_\mathbb{P})^{1-5u},$$

where $u = (s-1)^{-1}$. Thus, when $s \geqslant k^2 + k + 1$, we may use Propositions 1 and 2 to get

$$R_{k,s}(n, \lambda; \mathfrak{m}) \ll y^{1-\delta/(2k)+\epsilon} y^{s-2} x^{1-k+\epsilon} \ll y^{s-1-\delta/(3k)} x^{1-k},$$

provided that δ and y satisfy the hypotheses of Proposition 2 and ϵ is chosen sufficiently small; this verifies (2.10). In the remainder of this section, we prove the propositions.

4.1. **Proof of Proposition 1.** This is a variant of [15, Proposition 2.2], which we have extended in two ways. First, we have included the arbitrary coefficients λ. This is straightforward, due to the "maximal inequality"

$$\int_0^1 |f(\alpha, \lambda)|^s \, d\alpha \ll y^{s-k^2-k} \int_0^1 |f(\alpha, \mathbf{1})|^{k^2+k} \, d\alpha, \tag{4.2}$$

where $\mathbf{1}$ is the constant function $\mathbf{1}(n) = 1$ (compare this to [15, p. 1136]). Like Wei and Wooley, we estimate the right side of (4.2) by means of [5, Theorem 3] and standard bounds for Vinogradov's mean-value integral. In particular, the recent work of Bourgain, Demeter and Guth [2] allows us to reduce the lower bound on s to the one stated above. □

4.2. **Proof of Proposition 2.** Although it looks somewhat different, Proposition 2 is merely a slight variation of the main theorem of Huang [8], and our proof follows closely Huang's. We first obtain variants of some technical estimates from [8] by making some slight changes to Huang's arguments.

Lemma 1. *Let $k \geqslant 2$ be an integer and ρ be real, with $0 < \rho \leqslant t_k^{-1}$, where*

$$t_k = \begin{cases} 2 & \text{if } k = 2, \\ k^2 - k + 1 & \text{if } k \geqslant 3. \end{cases}$$

Suppose also that $y = x^\theta$, where

$$\frac{1}{2 - t_k\rho} \leqslant \theta \leqslant 1.$$

Then either

$$\sum_{x<m\leqslant x+y} e(m^k\alpha) \ll y^{1-\rho+\epsilon},$$

or there exist integers a, q such that

$$1 \leqslant q \leqslant y^{k\rho}, \quad (a,q) = 1, \quad |q\alpha - a| \leqslant x^{1-k}y^{k\rho-1},$$

and

$$\sum_{x<m\leqslant x+y} e(m^k\alpha) \ll y^{1-\rho+\epsilon} + \frac{y}{(q + yx^{k-1}|q\alpha - a|)^{1/k}}.$$

Proof. When $k \geqslant 3$, we follow the argument of Huang [8, Lemma 1] with $\gamma = \rho^{-1}(t_k - 1)^{-1}$. Within that argument, we apply the latest version of Vinogradov's mean-value theorem due to Bourgain, Demeter and Guth [2] in place of the earlier version by Wooley [16] used by Huang. When $k = 2$, we follow the same argument with $\gamma = (2\rho)^{-1}$ but observe that in this case the bound at the top of [8, p. 512] can be improved to

$$\Delta \ll q^{1/2+\epsilon}(1 + x^2(qQ_0)^{-1})^{1/2} \ll P_0^{1/2+\epsilon}xy^{-1}.$$

This slight improvement is possible, because in the quadratic case, Daemen's proof of [5, (3.5)] does not require the iterative process in [5, p. 78]. Thus, we need not incur a loss of a factor of $q^{-1/2}$ in the above bound which the iterative method causes when $k \geqslant 3$. □

Lemma 2 (Type II sum). *Let $k \geqslant 2$ be an integer, let ρ be real, with $0 < \rho \leqslant \min\left((4t_k)^{-1}, \frac{1}{20}\right)$, and suppose that $y = x^\theta$, where*

$$\frac{3}{4 - 4t_k\rho} \leqslant \theta \leqslant 1. \tag{4.3}$$

Suppose also that $\alpha \in \mathfrak{m}$ and that the coefficients ξ_u, η_v satisfy $\xi_u \ll \tau(u)^c$ and $\eta_v \ll \tau(v)^c$. Then

$$\sum_{u\sim U}\sum_{uv\in\mathcal{I}} \xi_u\eta_v e(u^kv^k\alpha) \ll y^{1-\rho+\epsilon} + y^{1+\epsilon}P^{-1/(2k)},$$

provided that

$$xy^{-1+2\rho} \ll U \ll y^{1-2\rho}. \tag{4.4}$$

Proof. This is a version of [8, Proposition 2] that applies Lemma 1 above in place of [8, Lemma 1]. We have also altered slightly the choice of ν in Huang's argument by choosing it so that $Y^\nu = y^{2\rho}L^{-1}$ as opposed to $Y^\nu = x^{2\rho}L^{-1}$ (see [8, p. 515]). □

Lemma 3 (Type I sum). *Let $k \geqslant 2$ be an integer, let ρ be real, with $0 < \rho \leqslant \min\left((4t_k)^{-1}, \frac{1}{20}\right)$, and suppose that $y = x^\theta$, with θ in the range (4.3). Suppose also that $\alpha \in \mathfrak{m}$ and that the coefficients ξ_u satisfy $\xi_u \ll \tau(u)^c$. Then*

$$\sum_{u\sim U}\sum_{uv\in\mathcal{I}} \xi_u e(u^kv^k\alpha) \ll y^{1-\rho+\epsilon} + y^{1+\epsilon}P^{-1/(2k)},$$

provided that

$$U \ll y^{1-2\rho}. \tag{4.5}$$

Proof. This is a version of [8, Proposition 1]. Following the proof of that result, with our Lemma 1 in place of [8, Lemma 1] and with ν chosen so that $Y^\nu = y^\rho L^{-1}$, one obtains the above bound when

$$U \ll x^{-1}y^{2-t_k\rho}, \qquad U^{2k} \ll x^{k-1}y^{1-2k\rho}.$$

On the other hand, when either of these inequalities fails, one has $U \gg xy^{-1+2\rho}$ and the result follows from Lemma 2. □

Proof of Proposition 2. It suffices to bound $f(\alpha, \Lambda)$, where Λ is von Mangoldt's function. Let $\rho = (31t_k)^{-1}$ and $X = xy^{-1+2\rho}$. We note that this choice of ρ ensures that (4.3) holds for all $\theta \geqslant 31/40$ and that $X \leqslant x^{9/40+(31\rho)/20} \leqslant x^{1/4}$. We may thus apply Vaughan's identity for Λ (see [14, p. 28]) to decompose $f(\alpha, \Lambda)$ into $O(L)$ type I sums with $U \leqslant X^2$ and $O(L)$ type II sums with $X \leqslant U \leqslant xX^{-1}$. By the choice of X and ρ, Lemma 2 can be applied to the arising type II sums. Moreover, since $X^2 \leqslant xX^{-1} = y^{1-2\rho}$, Lemma 3 can be applied to the type I sums. We conclude that when $\alpha \in \mathfrak{m}$, one has

$$f(\alpha, \Lambda) \ll y^{1-\rho+\epsilon} + y^{1-\delta/(2k)+\epsilon}.$$

Since the hypothesis $\delta < 1/(16k)$ ensures that $\delta/(2k) < \rho$, this completes the proof. □

5. The major arcs

In this section, we establish (2.11). First, we need to introduce some notation. We write

$$S(q, a) = \sum_{\substack{1 \leqslant h \leqslant q \\ (h,q)=1}} e(ah^k/q), \quad v(\beta; s) = \int_{\mathcal{I}} u^{s-1} e(u^k\beta)\, du,$$

and define the singular series $\mathfrak{S}(n)$ and the singular integral $\mathfrak{J}(n)$ by

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \phi(q)^{-s} \sum_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} S(q, a)^s e(-an/q), \quad \mathfrak{J}(n) = \int_{\mathbb{R}} v(\beta; 1)^s e(-n\beta)\, d\beta.$$

If λ denotes one of the functions $\lambda^\pm$ and κ the respective constant $\kappa_\pm$, we define a function $f^*(\alpha, \lambda)$ on the major arcs $\mathfrak{M}$ by setting

$$f^*(\alpha, \lambda) = \kappa\phi(q)^{-1}S(q, a)v(\beta; 1)L^{-1} \quad \text{if } \alpha \in \mathfrak{M}(q, a).$$

This is the "major arc approximation" to $f(\alpha, \lambda)$. We also define a major arc approximation to $f(\alpha, \mathbf{1}_{\mathbb{P}})$ by

$$f^*(\alpha) = \phi(q)^{-1}S(q, a)v(\beta; 1)L^{-1} \quad \text{if } \alpha \in \mathfrak{M}(q, a).$$

Finally, we adopt the convention that for any arithmetic function λ, there is an associated Dirichlet polynomial $F(s, \lambda)$, given by

$$F(s, \lambda) = \sum_{m \sim 2x/3} \lambda(m) m^{-s}.$$

5.1. Some technical estimates.

Lemma 4. *Let $x^{11/20} \leqslant y \leqslant x$ and suppose that P, Q satisfy*

$$PQ \leqslant yx^{k-1}, \quad Q \geqslant x^{k-9/20}.$$

Suppose also that g is a positive integer, $\nu > 1$, and λ is a bounded arithmetic function satisfying hypothesis (A2) *above. Then*

$$\sum_{r \leqslant P} [g, r]^{-\nu} \sideset{}{^*}\sum_{\chi \bmod r} \left(\int_{-1/(rQ)}^{1/(rQ)} |f(\beta, \lambda\chi)|^2 d\beta \right)^{1/2} \ll g^{-\nu+\epsilon} y^{1/2} x^{(1-k)/2} L^c. \tag{5.1}$$

Proof. When $k = 2$ and $\nu = 1 - \epsilon$, this is [10, Lemma 4.5]. The proof for general $k \geqslant 2$ and $\nu \geqslant 1$ uses the same argument with some obvious changes: e.g., $T_1 = \Delta x^k$ and $H \ll \Delta^{-1}x^{1-k}$ in place of the respective statements in [10, p. 618]. □

Lemma 5. *Let x be a large integer, and suppose that y, b, T are reals with: $y = o(x)$, $\|y\| = 1/2$, $0 < b \leqslant 1$, and $1 \leqslant T \leqslant x^{1/2}$. Suppose also that λ is a bounded arithmetic function. Then*

$$f(\beta, \lambda) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s, \lambda) v(\beta; s)\, ds + O\big((1 + yx^{k-1}|\beta|)xLT^{-1}\big).$$

Proof. For any $u \in \mathcal{I}$ with $\|u\| = 1/2$, Perron's formula (see [12, Corollary 5.3]) gives

$$\sum_{x-y<m\leqslant u} \lambda(m) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s, \lambda) \frac{u^s - (x-y)^s}{s}\, ds + O(xLT^{-1}). \tag{5.2}$$

If we change u in (5.2) to u_1, where $|u_1 - u| \leqslant 1/2$, the left side will change by $O(1)$ and the integral on the right side will change by $O(T)$. Hence, the integral representation (5.2) can be extended to all $u \in \mathcal{I}$. The conclusion of the lemma then follows by partial summation. □

Lemma 6. *Under the assumptions of Lemma 4, we have*

$$\sum_{r\leqslant P} [g, r]^{-\nu} \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\beta|\leqslant 1/(rQ)} |f(\beta, \lambda\chi)| \ll g^{-\nu+\epsilon} y L^c. \tag{5.3}$$

Furthermore, for any given $A > 0$, there is a $B = B(A, \nu) > 0$ such that

$$\sum_{L^B\leqslant r\leqslant P} r^{-\nu} \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\beta|\leqslant 1/(rQ)} |f(\beta, \lambda\chi)| \ll yL^{-A}. \tag{5.4}$$

Proof. Let $1 \leqslant R_0 \leqslant P$. By a simple splitting argument,

$$\sum_{R_0\leqslant r\leqslant P} [g, r]^{-\nu} \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\beta|\leqslant 1/(rQ)} |f(\beta, \lambda\chi)| \ll (gR)^{-\nu} L \sum_{\substack{d|g \\ d\leqslant 2R}} d^\nu S(R, d), \tag{5.5}$$

where $R_0 \leqslant R \leqslant P$ and

$$S(R, d) = \sum_{\substack{r\sim R \\ d|r}} \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\beta|\leqslant 1/(RQ)} |f(\beta, \lambda\chi)|.$$

We now estimate $S(R, d)$. The contribution to $S(R, d)$ from any powers of primes in the support of λ can be bounded trivially as $O(yx^{-1/2}(R^2/d))$. Under the assumptions of the lemma, we have $P \leqslant yx^{-11/20}$, so this contribution can be absorbed into the term $y(R/d)L$ on the right side of (5.8) below. Thus, we may assume that λ is merely the linear combination of triple convolutions of the kind described in (A2). We may also assume that $x \in \mathbb{Z}$ and $\|y\| = 1/2$.

Let $0 < b \leqslant 1$, $|\beta| \leqslant (RQ)^{-1}$, $T_1 = 3k\pi x^k Q^{-1}$, and $T_0 = T_1/R$. Then, by Lemma 5 with $T = T_1$,

$$f(\beta, \lambda\chi) = \frac{1}{2\pi i} \int_{b-iT_1}^{b+iT_1} F(s, \lambda\chi) v(\beta; s)\, ds + O(yR^{-1}L). \tag{5.6}$$

Letting $b \downarrow 0$ in (5.6), we obtain

$$f(\beta, \lambda\chi) = \frac{1}{2\pi} \int_{-T_1}^{T_1} F(it, \lambda\chi) v(\beta; it)\, dt + O(yR^{-1}L). \tag{5.7}$$

When $|\beta| \leqslant (RQ)^{-1}$ and $|t| \geqslant T_0$, we have

$$v(\beta; it) \ll |t|^{-1},$$

by the first-derivative test for exponential integrals (see [13, Lemma 4.5]). Combining this bound with (5.7) and the trivial estimate $|v(\beta; it)| \ll yx^{-1}$, we find that

$$f(\beta, \lambda\chi) \ll yx^{-1} \int_{-T_0}^{T_0} |F(it, \lambda\chi)|\, dt + \int_{T_0\leqslant|t|\leqslant T_1} |F(it, \lambda\chi)| \frac{dt}{|t|} + yR^{-1}L.$$

Summing this inequality over r and χ and then splitting the range of t in the second integral into dyadic intervals, we deduce that

$$S(R, d) \ll yx^{-1} S_1(R, d; T_0) + \sum_{2^j\leqslant R} (2^j T_0)^{-1} S_1(R, d; 2^j T_0) + y(R/d)L, \tag{5.8}$$

where

$$S_1(R,d;T) = \sum_{\substack{r\sim R\\ d|r}} \sum_{\chi \bmod r}^{*} \int_{-T}^{T} |F(it,\lambda\chi)|\,dt.$$

Since λ is assumed to be a linear combination of convolutions of the type in (A2), we may apply [4, Theorem 2.1] to obtain the bound

$$S_1(R,d;T) \ll \big(x + (R^2T/d)x^{11/20}\big)L^c.$$

Combining this bound, (5.5) and (5.8), we conclude that the left side of (5.3) is

$$\ll g^{-\nu+\epsilon} y\big(1 + x^{k-9/20}Q^{-1} + x^{1-k}y^{-1}PQ + Px^{11/20}y^{-1}\big)L^c.$$

This establishes the first claim of the lemma.

When $g = 1$, the above argument yields the bound

$$\ll yR_0^{1-\nu}\big(1 + x^{k-9/20}Q^{-1} + x^{1-k}y^{-1}PQ + Px^{11/20}y^{-1}\big)L^c$$

for the left side of (5.4). When $R_0 = L^B$ for a sufficiently large $B > 0$, this establishes the second claim of the lemma. □

Lemma 7. *Let $x^{11/20+2\epsilon} \leqslant y \leqslant x^{1-\epsilon}$ and suppose that P, Q satisfy*

$$PQ \leqslant yx^{k-1}, \quad Q \geqslant x^{k-9/20+2\epsilon}. \tag{5.9}$$

Suppose also that $\nu > 1$ and λ is a bounded arithmetic function that satisfies hypotheses (A0)*,* (A2) *and* (A3) *above. Then, for any given $A > 0$,*

$$\sum_{r\leqslant P} r^{-\nu} \sum_{\chi \bmod r}^{*} \max_{|\beta|\leqslant 1/(rQ)} |f(\beta,\lambda\chi) - \rho_\chi v(\beta;1)| \ll yL^{-A}, \tag{5.10}$$

where $\rho_\chi = \delta_\chi \kappa L^{-1}$, κ being the constant in hypothesis (A0) *for λ.*

Proof. By the second part of Lemma 6, it suffices to show that

$$\max_{|\beta|\leqslant 1/Q} |f(\beta,\lambda\chi) - \rho_\chi v(\beta;1)| \ll yL^{-B-A} \tag{5.11}$$

for all primitive characters χ with moduli $r \leqslant L^B$, where $B = B(A,\nu)$ is the number that appears in (5.4). Let χ be such a character and suppose that $|\beta| \leqslant Q^{-1}$. By Lemma 5 with $b = 1/2$ and $T = T_1 = x^{9/20-\epsilon}$,

$$f(\beta,\lambda\chi) = \frac{1}{2\pi i}\int_{1/2-iT_1}^{1/2+iT_1} F(s,\lambda\chi)v(\beta;s)\,ds + O\big(yx^{-\epsilon/2} + yx^{k-9/20+\epsilon}Q^{-1}L\big). \tag{5.12}$$

Since $v(\beta;1/2+it) \ll yx^{-1/2}$, we deduce from (5.12) and hypothesis (A3) that

$$f(\beta,\lambda\chi) = \frac{1}{2\pi i}\int_{1/2-iT_0}^{1/2+iT_0} F(s,\lambda\chi)v(\beta;s)\,ds + O(yL^{-B-A}),$$

where $T_0 = \exp(L^{1/3})$. Note that when $\mathrm{Re}(s) = 1/2$,

$$v(\beta;s) - x^{s-1}v(\beta;1) \ll (|s|+1)y^2x^{-3/2}.$$

Hence,

$$f(\beta,\lambda\chi) = \frac{v(\beta;1)}{2\pi i}\int_{1/2-iT_0}^{1/2+iT_0} F(s,\lambda\chi)x^{s-1}\,ds + O(yL^{-B-A}). \tag{5.13}$$

When $\beta = 0$, we can evaluate the left side of (5.13) directly by means of hypothesis (A0). Thus,

$$\frac{1}{2\pi i}\int_{1/2-iT_0}^{1/2+iT_0} F(s,\lambda\chi)x^{s-1}\,ds = \rho_\chi + O(L^{-B-A}). \tag{5.14}$$

The desired inequality (5.11) follows from (5.13) and (5.14). □

Lemma 8. *Let $x^{7/12+2\epsilon} \leqslant y \leqslant x^{1-\epsilon}$ and suppose that P, Q satisfy*

$$PQ \leqslant yx^{k-1}, \quad Q \geqslant x^{k-5/12+\epsilon}.$$

Suppose also that $\nu > 1$. Then, for any given $A > 0$,

$$\sum_{r \leqslant P} r^{-\nu} \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\beta| \leqslant 1/(rQ)} |f(\beta, \mathbf{1}_{\mathbb{P}}\chi) - \delta_\chi L^{-1} v(\beta; 1)| \ll yL^{-A}. \tag{5.15}$$

Proof. This is a slight variation of [10, Lemma 4.7]. We use the same argument, but we alter slightly the choice of T in [10, p. 620]: instead of $T = (x/y)^2 x^{3\epsilon}$, we choose

$$T = x^\epsilon \max\left(xy^{-1}, x^k Q^{-1}\right),$$

which suffices to complete the proof. $\square$

5.2. **The asymptotic formula for $R_{k,s}(n, \lambda; \mathfrak{M})$.** We have

$$R_{k,s}(n, \lambda; \mathfrak{M}) = \sum_{p_1, \dots, p_t \in \mathcal{I}} \int_{\mathfrak{M}} f(\alpha, \lambda) f(\alpha, \lambda^+)^4 e(-n_{\mathbf{p}}\alpha)\, d\alpha, \tag{5.16}$$

where $t = s - 5$ and $n_{\mathbf{p}} = n - p_1^k - \cdots - p_t^k$. We now proceed to show that, for any fixed $A > 0$, one has

$$\int_{\mathfrak{M}} \left(f(\alpha, \lambda) f(\alpha, \lambda^+)^4 - f^*(\alpha, \lambda) f^*(\alpha, \lambda^+)^4\right) e(-n_{\mathbf{p}}\alpha)\, d\alpha \ll y^4 x^{1-k} L^{-A}. \tag{5.17}$$

Let $\alpha \in \mathfrak{M}(q, a)$ and write $\beta = \alpha - a/q$. Since $q \leqslant P$, property (A1) ensures that the function λ is supported on integers m with $(m, q) = 1$. Hence, by the orthogonality of the characters modulo q, we have

$$\begin{aligned} f(\alpha, \lambda) &= \sum_{\substack{1 \leqslant h \leqslant q \\ (h,q)=1}} e(ah^k/q) \sum_{\substack{m \in \mathcal{I} \\ m \equiv h \ (\mathrm{mod}\ q)}} \lambda(m) e(m^k \beta) \\ &= \phi(q)^{-1} \sum_{\chi \bmod q} S(\chi, a) f(\beta, \lambda\chi), \end{aligned}$$

where

$$S(\chi, a) = \sum_{h=1}^{q} \bar{\chi}(h) e(ah^k/q).$$

Hence,

$$f(\alpha, \lambda) = f^*(\alpha, \lambda) + \Delta(\alpha, \lambda), \tag{5.18}$$

where

$$\begin{aligned} \Delta(\alpha, \lambda) &= \phi(q)^{-1} \sum_{\chi \bmod q} S(\chi, a) W(\beta, \lambda\chi), \\ W(\beta, \lambda\chi) &= f(\beta, \lambda\chi - \rho_\chi), \quad \rho_\chi = \delta_\chi \kappa L^{-1}. \end{aligned}$$

Using (5.18), we can express the integral in (5.17) as the linear combination of integrals of the form

$$\int_{\mathfrak{M}} f^*(\alpha, \lambda)^a \Delta(\alpha, \lambda)^{1-a} f^*(\alpha, \lambda^+)^b \Delta(\alpha, \lambda^+)^{4-b} e(-n_{\mathbf{p}}\alpha)\, d\alpha, \tag{5.19}$$

where $a \in \{0, 1\}$, $b \in \{0, 1, \cdots, 4\}$ and $a + b < 5$. The estimation of all those integrals follows the same pattern, so we shall focus on the most troublesome among them, namely,

$$\int_{\mathfrak{M}} \Delta(\alpha, \lambda) \Delta(\alpha, \lambda^+)^4 e(-n_{\mathbf{p}}\alpha)\, d\alpha. \tag{5.20}$$

We can rewrite (5.20) as the multiple sum

$$\sum_{q \leqslant P} \sum_{\chi_1 \bmod q} \cdots \sum_{\chi_5 \bmod q} B(q; \chi_1, \dots, \chi_5) J(q; \chi_1, \dots, \chi_5), \tag{5.21}$$

where

$$B(q;\chi_1,\dots,\chi_5)=\phi(q)^{-5}\sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}}S(\chi_1,a)\cdots S(\chi_5,a)e(-an_{\mathbf{p}}/q),$$

$$J(q;\chi_1,\dots,\chi_s)=\int_{-1/qQ}^{1/qQ}W(\beta,\lambda\chi_1)W(\beta,\lambda^+\chi_2)\cdots W(\beta,\lambda^+\chi_5)e(-n_{\mathbf{p}}\beta)\,d\beta.$$

First, we reduce (5.21) to a sum over primitive characters. If χ is a Dirichlet character modulo q that is induced by a primitive character χ^* modulo r, $r\mid q$, then by property (A1), $\lambda^\pm\chi=\lambda^\pm\chi^*$. Thus,

$$W(\beta,\lambda^\pm\chi)=W(\beta,\lambda^\pm\chi^*).\tag{5.22}$$

Let χ_i^* modulo r_i, $r_i|q$, be the primitive character inducing χ_i and set $q_0=[r_1,\dots,r_5]$. By (5.22), we have

$$J(q;\chi_1,\dots,\chi_5)=J(q;\chi_1^*,\dots,\chi_5^*).$$

Therefore, the sum (5.21) does not exceed

$$\sum_{r_1\leqslant P}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\cdots\sum_{r_5\leqslant P}\sideset{}{^*}\sum_{\chi_5 \bmod r_5}J_0(\chi_1,\dots,\chi_5)B_0(\chi_1,\dots,\chi_5),$$

where

$$B_0(\chi_1,\dots,\chi_5)=\sum_{\substack{q\leqslant P\\q_0|q}}|B(q;\chi_1,\dots,\chi_5)|,$$

$$J_0(\chi_1,\dots,\chi_5)=\int_{-1/(q_0Q)}^{1/(q_0Q)}|W(\beta,\lambda\chi_1)W(\beta,\lambda^+\chi_2)\cdots W(\beta,\lambda^+\chi_5)|\,d\beta.$$

Recalling the bound (see [15, Lemma 6.1])

$$B_0(\chi_1,\dots,\chi_5)\ll q_0^{-3/2+\epsilon}L^c,$$

we conclude that the sum (5.21) is

$$\ll L^c\sum_{r_1\leqslant P}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\cdots\sum_{r_5\leqslant P}\sideset{}{^*}\sum_{\chi_5 \bmod r_5}q_0^{-3/2+\epsilon}V(\lambda\chi_1)V(\lambda^+\chi_2)V(\lambda^+\chi_3)W(\lambda^+\chi_4)W(\lambda^+\chi_5),\tag{5.23}$$

where for a character χ modulo r, we write

$$V(\lambda\chi)=\max_{|\beta|\leqslant 1/(rQ)}|W(\beta,\lambda\chi)|,$$

$$W(\lambda\chi)=\left(\int_{-1/(rQ)}^{1/(rQ)}|W(\beta,\lambda\chi)|^2\,d\beta\right)^{1/2}.$$

Next, we proceed to estimate the sum in (5.23) by Lemmas 4, 6 and 7, which we will denote by Σ. When $y=x^\theta$ with $\theta>31/40$ and $\delta\leqslant 2(\theta-31/40)$, the definitions of P and Q (recall (2.6)) ensure that they satisfy inequalities (5.9). Since the sieve functions $\lambda^\pm$ have properties (A0)–(A3), this means that all the hypotheses of the lemmas are in place.

To begin the estimation of Σ, we note that Lemma 4 yields

$$\sum_{r\leqslant P}\sideset{}{^*}\sum_{\chi \bmod r}[g,r]^{-\nu}W(\lambda^+\chi)\ll g^{-\nu+\epsilon}y^{1/2}x^{(1-k)/2}L^c+g^{-\nu}I_0^{1/2},\tag{5.24}$$

where

$$\begin{aligned}I_0=\int_{-1/Q}^{1/Q}|v(\beta;1)|^2\,d\beta&\ll\iint_{\mathcal{I}^2}\frac{du_1du_2}{Q+|u_1^k-u_2^k|}\\&\ll yx^{1-k}+yLQ^{-1}\ll yx^{1-k}.\end{aligned}\tag{5.25}$$

(We remark that the second term on the right side of (5.24) accounts for the contribution of ρ_χ to $W(\beta, \lambda\chi)$—which is present only when $r = 1$.) Similarly, the first part of Lemma 6 yields

$$\sum_{r \leqslant P} \sideset{}{^*}\sum_{\chi \bmod r} [g, r]^{-\nu} V(\lambda^+\chi) \ll g^{-\nu+\epsilon} y L^c. \tag{5.26}$$

Applying (5.24) to the summations over r_5 and r_4 in Σ and then (5.26) to the summations over r_3 and r_2, we obtain

$$\Sigma \ll y^3 x^{1-k} L^c \sum_{r \leqslant P} \sideset{}{^*}\sum_{\chi \bmod r} r^{-3/2+5\epsilon} V(\lambda\chi).$$

Finally, we apply Lemma 7 to the last sum and conclude that

$$\Sigma \ll y^4 x^{1-k} L^{-A}$$

for any fixed $A > 0$. This inequality and its variants for other integrals of the form (5.19) establish (5.17).

Having established (5.17), we can combine it with (5.16) to get

$$R_{k,s}(n, \lambda; \mathfrak{M}) = \int_{\mathfrak{M}} f(\alpha, \mathbf{1}_{\mathbb{P}})^t f^*(\alpha, \lambda) f^*(\alpha, \lambda^+)^4 e(-n\alpha)\, d\alpha + O\big(y^{s-1} x^{1-k} L^{-A}\big).$$

We now define a new, slimmer set of major arcs $\mathfrak{M}_0$, given by (2.7) with $Q_0 = x^{k-1} y P^{-1}$ in place of Q. From the bound

$$f^*(\alpha, \lambda^{\pm}) \ll y q^{-1/2+\epsilon} \big(1 + y x^{k-1} |\alpha - a/q|\big)^{-1/2} \quad \text{if } \alpha \in \mathfrak{M}(q, a),$$

we find that

$$\begin{aligned}\int_{\mathfrak{M} \setminus \mathfrak{M}_0} \big|f(\alpha, \mathbf{1}_{\mathbb{P}})^t f^*(\alpha, \lambda) f^*(\alpha, \lambda^+)^4\big|\, d\alpha &\ll \sum_{\substack{1 \leqslant a \leqslant q \leqslant P \\ (a,q)=1}} \int_{|\beta| \geqslant 1/(qQ_0)} \frac{y^s q^{-5/2+\epsilon}}{(1 + y x^{k-1} |\beta|)^{5/2}}\, d\beta \\ &\ll y^{s-1} x^{1-k} P^{-1/2+\epsilon}.\end{aligned}$$

Hence, for any fixed $A > 0$, we have

$$R_{k,s}(n, \lambda; \mathfrak{M}) = \int_{\mathfrak{M}_0} f(\alpha, \mathbf{1}_{\mathbb{P}})^t f^*(\alpha, \lambda) f^*(\alpha, \lambda^+)^4 e(-n\alpha)\, d\alpha + O\big(y^{s-1} x^{1-k} L^{-A}\big). \tag{5.27}$$

Finally, we have

$$\int_{\mathfrak{M}_0} \big(f(\alpha, \mathbf{1}_{\mathbb{P}})^t - f^*(\alpha)^t\big) f^*(\alpha, \lambda) f^*(\alpha, \lambda^+)^4 e(-n\alpha)\, d\alpha \ll y^{s-1} x^{1-k} L^{-A}. \tag{5.28}$$

The proof of this inequality is simlar to the proof of (5.17), except that we do not need to use Lemma 4 (the bound (5.25) can be used instead) and we use Lemma 8 instead of Lemma 7. We remark that during the process, we need to verify the hypotheses $Q \geqslant x^{k-9/20}$ and $Q \geqslant x^{k-5/12+\epsilon}$ of those lemmas for $Q = Q_0$; with our choice of Q_0, those hypotheses are satisfied when $y \geqslant x^{7/12+\delta}$.

By (5.27) and (5.28), we have

$$R_{k,s}(n, \lambda; \mathfrak{M}) = \kappa\kappa_+^4 \int_{\mathfrak{M}_0} f^*(\alpha)^s e(-n\alpha)\, d\alpha + O\big(y^{s-1} x^{1-k} L^{-A}\big).$$

The evaluation of the last integral uses standard major arc techniques (e.g., see Wei and Wooley [15, pp. 1150–1151]), so we can omit it and report that

$$\int_{\mathfrak{M}_0} f^*(\alpha)^s e(-n\alpha)\, d\alpha = \mathfrak{S}(n) \mathfrak{I}(n) L^{-s} + O\big(y^{s-1} x^{1-k} P^{-1}\big).$$

We note that $\mathfrak{S}(n)$ is the standard singular series in the Waring–Goldbach problem for s kth powers. In particular, it is known that $1 \ll \mathfrak{S}(n) \ll 1$ when $n \in \mathcal{H}_{k,s}$. Since the inequality

$$y^{s-1} x^{1-k} \ll \mathfrak{I}(n) \ll y^{s-1} x^{1-k}$$

is also standard (compare to [15, (6.5)]), we conclude that (2.11) holds with

$$\mathfrak{C}(n) = \mathfrak{S}(n) \mathfrak{I}(n) y^{1-s} x^{k-1}.$$

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