THE STRONG SYMMETRIC GENUS SPECTRUM OF ABELIAN GROUPS

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Abstract. Let $\mathcal{S}$ denote the set of positive integers that may appear as the strong symmetric genus of a finite abelian group. We obtain a set of (simple) necessary and sufficient conditions for an integer g to belong to $\mathcal{S}$. We also prove that the set $\mathcal{S}$ has an asymptotic density and approximate its value.

1. Introduction

Let G be a finite group. Among the various genus parameters associated with G, the most classical is perhaps the *strong symmetric* genus $\sigma^0(G)$, the minimum genus of any Riemann surface on which G acts faithfully and preserving orientation. Work on this parameter dates back over a century and includes the fundamental bound $\sigma^0(G) \leq 84(g-1)$ due to Hurwitz [4].

A natural problem is to determine the positive integers that occur as the strong symmetric genus of a group (or a particular type of group), that is, to determine the strong symmetric genus spectrum for the particular type of group. This basic problem was settled for the family of all finite groups by May and Zimmerman [6]: there is a group of strong symmetric genus g, for all $g \in \mathbb{N}$. Our focus here is to describe the strong symmetric genus spectrum of abelian groups.

Let

$$\mathcal{S} = \big\{ g \in \mathbb{N} : g = \sigma^0(A) \text{ for some abelian group } A \big\}$$

denote the strong symmetric genus spectrum of abelian groups. We will refer to $\mathcal{S}$ simply as the "spectrum." The abelian groups of strong symmetric genus zero are exactly the cyclic groups and the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$, and those of strong symmetric genus one are exactly the abelian groups of rank 2 and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. These facts are a direct consequence

Date: November 16, 2016.

2010 *Mathematics Subject Classification.* Primary 57M60; Secondary 11B05, 11N25, 11N37, 20F38, 30F99.

Key words and phrases. Strong symmetric genus, Riemann surface, genus spectrum, abelian groups, asymptotic density, unions of arithmetic progressions.

of the classification of the groups of strong symmetric genus zero or one (see Gross and Tucker [2, §6.3]). In the remainder of this note we shall focus on abelian groups of rank $r \geq 3$.

One can find the strong symmetric genus of any abelian group of rank $r \geq 3$ by applying a classical result due to Maclachlan [5, Theorem 4]. Recall that every finite abelian group A has a canonical representation $A \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_r}$, with standard invariants $m_1, m_2, \dots, m_r$ subject to $m_1 > 1$ and $m_i | m_{i+1}$ for $1 \leq i < r$. We extend the list of standard invariants by adding $m_0 = 1$ to it. Maclachlan [5] proved that if A is an abelian group of rank $r \geq 3$, with $|A| \geq 10$, then

$$\sigma^0(A) = 1 + \frac{|A|}{2} \min_{0 \leq \gamma \leq r/2} \left\{ 2\gamma - 2 + \sum_{i=1}^{r-2\gamma} \left(1 - \frac{1}{m_i}\right) + \left(1 - \frac{1}{m_{r-2\gamma}}\right) \right\}. \tag{1.1}$$

For example, when $a > 1$ and $a^3 n \geq 10$, Maclachlan's formula yields

$$\sigma^0(\mathbb{Z}_a \times \mathbb{Z}_a \times \mathbb{Z}_{an}) = 1 - a^2 + a^2(a-1)n. \tag{1.2}$$

In particular, when $a = 2$, this reveals that $\mathcal{S}$ contains the entire residue class $g \equiv 1 \pmod 4$. The first goal of the present paper is to take such observations a step further and to provide a relatively simple test that can be used to check whether a given positive integer g belongs to the spectrum $\mathcal{S}$. Theorem 1 below is exactly such a test, as each of conditions (ii)–(iv) can be checked easily given the prime factorization of the integer $g - 1$.

Theorem 1. *Let $g \geq 2$. Then $g \in \mathcal{S}$ if and only if g satisfies one of the following conditions:*

(i) $g \equiv 1 \pmod 4$ *or* $g \equiv 55 \pmod{81}$;
(ii) $g - 1$ *is divisible by* p^4 *for some odd prime* p;
(iii) $g - 1$ *is divisible by* a^2 *for some odd integer* a *with* $(a-1) \mid g$;
(iv) $g - 1$ *is divisible by* $b^2 a^2 (a-1)$ *for some odd integers* $a, b > 1$, *with* $a \equiv 3 \pmod 4$.

We note two more results on the structure of $\mathcal{S}$. While the next two results are direct consequences of Theorem 1 and its proof, they are of independent interest.

Theorem 2. *If $g \geq 2$ and $g - 1$ is squarefree, then $g \notin \mathcal{S}$.*

Theorem 3. *Suppose that A is an abelian group of rank 5 or higher. Then there exists an abelian group B of rank 3 or 4 such that $\sigma^0(A) = \sigma^0(B)$.*

If $\mathcal{A}$ is a set of integers, its *lower* and *upper asymptotic densities*, denoted $\underline{\delta}(\mathcal{A})$ and $\overline{\delta}(A)$, are given by

$$\underline{\delta}(\mathcal{A}) = \liminf_{X\to\infty} X^{-1}A(X) \qquad \text{and} \qquad \overline{\delta}(\mathcal{A}) = \limsup_{X\to\infty} X^{-1}A(X),$$

where $A(X) = |\mathcal{A} \cap [1, X]|$. A set $\mathcal{A}$ is said to have an *asymptotic density*, if $\underline{\delta}(\mathcal{A}) = \overline{\delta}(A)$; when $\mathcal{A}$ does have an asymptotic density, it is denoted $\delta(\mathcal{A})$. Since the set of squarefree integers is known to have an asymptotic density of $6\pi^{-2} \approx 0.6079$ (see Montgomery and Vaughan [7, Theorem 2.2]), we find as a direct corollary of Theorem 2 that $\overline{\delta}(\mathcal{S}) \leq 0.3921$. On the other hand, since all the integers in the congruence class $g \equiv 1 \pmod 4$ are in the spectrum, we have $\underline{\delta}(\mathcal{S}) \geq 0.25$. It is therefore natural to ask whether the spectrum $\mathcal{S}$ has an asymptotic density—which is not obvious—and what its potential value is. The second main result of the paper establishes that the asymptotic density does indeed exist.

Theorem 4. *The spectrum $\mathcal{S}$ has an asymptotic density $\delta(\mathcal{S}) \approx 0.3284$.*

We remark that this shows that the lower bound $\underline{\delta}(\mathcal{S}) \geq 0.3175...$ given by Borror, Morris and Tarr [1] is quite tight.

2. The structure of $\mathcal{S}$

In this section, we prove Theorem 1.

2.1. The spectrum of groups of rank 3.

We first study genera of abelian groups of rank 3 and establish the following result.

Proposition 5. *The spectrum of abelian groups of rank 3 consists of the congruence class $g \equiv 1 \pmod 4$ and the integers g satisfying conditions (iii) or (iv) of Theorem 1.*

Proof. Recall that by (1.2) with $a = 2$, all integers $g \equiv 1 \pmod 4$ are part of the spectrum of abelian groups of rank 3. The spectrum of the groups of type $\mathbb{Z}_a \times \mathbb{Z}_a \times \mathbb{Z}_{an}$, with a odd, are the integers g satisfying a congruence of the form

$$g \equiv 1 - a^2 \pmod{a^2(a-1)}$$

for some odd $a > 1$. These are exactly the integers described by condition (iii) of Theorem 1, since we may apply the Chinese Remainder Theorem to rewrite the above congruence as the pair of congruences

$$g \equiv 1 \pmod{a^2}, \qquad g \equiv 0 \pmod{a-1}.$$

When $b \geq 2$ and $bn > 2$, Maclachlan's formula (1.1) gives

$$\sigma^0(\mathbb{Z}_a \times \mathbb{Z}_{ab} \times \mathbb{Z}_{abn}) = 1 + b^2a^2(a-1)n. \tag{2.1}$$

When a and b are odd, with $a \equiv 3 \pmod 4$, these are the integers described in condition (iv) of Theorem 1. In the remaining cases, the expression $b^2a^2(a-1)$ is divisible by 4, so in those cases the groups $\mathbb{Z}_a \times \mathbb{Z}_{ab} \times \mathbb{Z}_{abn}$ have genera $g \equiv 1 \pmod 4$. By (1.2), the rank-3 groups of types $\mathbb{Z}_a \times \mathbb{Z}_a \times \mathbb{Z}_{an}$, with a even, also contribute also genera $g \equiv 1 \pmod 4$. Finally, the same conclusion holds for groups of type $\mathbb{Z}_a \times \mathbb{Z}_{2a} \times \mathbb{Z}_{2a}$, since

$$\sigma^0(\mathbb{Z}_a \times \mathbb{Z}_{2a} \times \mathbb{Z}_{2a}) = 1 - 5a^2 + 4a^3. \qquad \square$$

2.2. **The spectrum of groups of rank 4.** The main result of this section is the following proposition.

Proposition 6. *The spectrum of abelian groups of rank* 4 *is a subset of the integers satisfying conditions (i) or (ii) of Theorem 1. Moreover, the spectrum of abelian groups of ranks* 3 *or* 4 *contains all the integers satisfying condition (ii) of Theorem 1.*

Proof. Let $A \cong \mathbb{Z}_a \times \mathbb{Z}_{ab} \times \mathbb{Z}_{abc} \times \mathbb{Z}_{abcn}$, with $a > 1$. By (1.1),

$$\sigma^0(A) = 1 + \frac{|A|}{2} \min\left\{2, 3 - \frac{1}{a} - \frac{2}{ab}, 3 - \frac{1}{a} - \frac{1}{ab} - \frac{1}{abc} - \frac{2}{abcn}\right\}.$$

When a is even, the last expression is 1 modulo 4, so we may assume that a is odd. When $a \geq 5$ or $a = 3$ and $b \geq 2$, we deduce that $\sigma^0(A) = 1 + |A|$. In particular, $\sigma^0(A) \equiv 1 \pmod 4$ when a is odd and b is even.

Assume now that ab is odd. Then we have $\sigma^0(A) = 1 + |A|$ unless A isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3n}$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}_6$. Since the order $|A|$ is divisible by p^4 for any prime divisor p of a, we conclude that $\sigma^0(A)$ satisfies condition (ii) of the theorem unless A is one of the exceptional groups. By (1.1),

$$\sigma^0(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3n}) = 81n - 26,$$

so the congruence class $g \equiv 55 \pmod{81}$ is part of the spectrum of abelian groups of rank 4. We remark that this congruence class includes also the genus of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}_6$, since

$$\sigma^0(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}_6) = 298 \equiv 55 \pmod{81}.$$

It remains to show that all integers g satisfying condition (ii) belong to $\mathcal{S}$. For a prime $p \geq 5$, we have

$$\sigma^0(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{pn}) = 1 + p^4 n.$$

Finally, by applying (2.1) to the groups of types $\mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_{9n}$ and and (1.2) to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3n}$, we see that the congruence classes $g \equiv 1$

(mod 162) and $g \equiv 10 \pmod{18}$ are part of the spectrum of groups of rank 3. Thus, so is the class $g \equiv 1 \pmod{81}$. □

2.3. **Groups of higher ranks.** To complete the proof of Theorem 1, we now establish Theorem 3. When $\operatorname{rank}(A) \geq 5$ and the smallest invariant m_1 of A is even, we have that $\frac{1}{2}m_1 \cdots m_{r-1}$ must be a factor of $\sigma^0(A) - 1$. In particular, in this case, $\sigma^0(A) \equiv 1 \pmod 4$ and we may choose B of the form $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2n}$. On the other hand, when $\operatorname{rank}(A) \geq 5$ and m_1 odd, we have that p^{r-1} must be a factor of $\sigma^0(A) - 1$ for any prime divisor p of m_1. Thus, $\sigma^0(A) \equiv 1 \pmod{p^4}$ and the existence of B follows from Proposition 6.

3. The asymptotic density of $\mathcal{S}$

In this section, we establish Theorem 4. Let $\mathcal{S}_j$, $1 \leq j \leq 4$, denote the set of integers that satisfy the jth condition of Theorem 1 but none of the previous conditions (if any). We deal with each of these four sets separately and show that each $\mathcal{S}_j$ has an asymptotic density.

Densities of residue classes play a major role in our proofs, so we begin by recalling that the residue class $x \equiv a \pmod q$ has asymptotic density $1/q$. Also, by the Chinese Remainder Theorem, the density of the intersection of two residue classes $x \equiv a_i \pmod{q_i}$, $i = 1, 2$, has density

$$\begin{cases} [q_1, q_2]^{-1} & \text{if } (q_1, q_2) \mid (a_1 - a_2), \\ 0 & \text{otherwise.} \end{cases}$$

Here and in the sequel, for integers $a, b, \dots$, we use $(a, b, \dots)$ and $[a, b, \dots]$ as abbreviations for $\operatorname{lcm}[a, b, \dots]$ and $\gcd(a, b, \dots)$, respectively. In particular, using the inclusion-exclusion principle, we see that the density of $\mathcal{S}_1$, the set of integers g satisfying condition (i) of Theorem 1, is

$$\delta_1 = \frac{1}{4} + \frac{1}{81} - \frac{1}{324} = \frac{7}{27}.$$

3.1. **The density of $\mathcal{S}_2$.** We split $\mathcal{S}_2$ into subsets $\mathcal{S}_{2,j}$, $2 \leq j \leq 4$, subject to $g \equiv j \pmod 4$. We will prove that each of these sets has asymptotic density

$$\delta(\mathcal{S}_{2,j}) = \frac{1}{4}\left(\frac{80}{81} - \frac{79}{75\zeta(4)}\right), \tag{3.1}$$

where $\zeta(s)$ denotes the Riemann zeta-function. Thus,

$$\delta_2 = \delta(\mathcal{S}_2) = \sum_{j=2}^{4} \delta(\mathcal{S}_{2,j}) = \frac{20}{27} - \frac{79}{100\zeta(4)} \approx .0108. \tag{3.2}$$

The calculation of the density (3.1) uses some basic facts about the distribution of biquadrate-free integers. When $k \geq 2$, let $\alpha_k(n)$ denote the characteristic function of the integers n that are not divisible by p^k for any prime p. It is well-known that

$$\alpha_k(n) = \sum_{d^k | n} \mu(d), \tag{3.3}$$

where $\mu(d)$ is the Möbius function and the summation is over all kth powers that divide n. One needs little more than (3.3) to establish the next lemma (see Prachar [8]).

Lemma 7. *Let $(a, q) = 1$. Then for any fixed $\varepsilon > 0$, one has*

$$\sum_{\substack{n \leq X \\ n \equiv a \pmod{q}}} \alpha_k(n) = \frac{X}{q\zeta(k)} \prod_{p|q} \left(1 - p^{-k}\right)^{-1} + O\left(X^{1/k+\varepsilon}\right),$$

the implied constant in the O-term depending on q and ε.

Let $T_j(X)$ denote the number of integers $g \equiv j \pmod{4}$, with $g \leq X$, that satisfy condition (ii) of Theorem 1. When $j = 2$ or 4, we have

$$T_j(X) = \frac{X}{4} - \sum_{\substack{h \leq X \\ h \equiv j-1 \pmod{4}}} \alpha_4(h) + O(1),$$

and Lemma 7 yields

$$T_j(X) = \frac{X}{4}\left(1 - \frac{16}{15\zeta(4)}\right) + O\left(X^{1/3}\right). \tag{3.4}$$

When $j = 3$, we write $g = 2h + 1$ to get

$$T_3(X) = \frac{X}{4} - \sum_{\substack{h \leq X/2 \\ h \equiv 1 \pmod{2}}} \alpha_4(h) + O(1),$$

and Lemma 7 again leads to (3.4).

Next, let $T_j'(X)$ denote the number of integers g counted by $T_j(X)$ that satisfy also the congruence $g \equiv 55 \pmod{81}$. A variant of the above argument yields

$$T_j'(X) = \frac{X}{324}\left(1 - \frac{27}{25\zeta(4)}\right) + O\left(X^{1/3}\right). \tag{3.5}$$

The desired result (3.1) follows from (3.4) and (3.5), after noting that the counting function $S_{2,j}(X)$ of $\mathcal{S}_{2,j}$ can be expressed as

$$S_{2,j}(X) = T_j(X) - T_j'(X).$$

3.2. **The density of $\mathcal{S}_3$.** Recall that condition (iii) is equivalent to the requirement that

$$g \equiv 1 - a^2 \pmod{a^2(a-1)} \tag{3.6}$$

for some odd $a > 1$. Let $\mathcal{A}$ be the set of such g, and write $\mathcal{A}(X) = \mathcal{A} \cap [1, X]$. Also, let $\mathcal{A}_a(X)$ denote the set of $g \in \mathcal{A}(X)$ such that a is the least odd integer for which (3.6) is satisfied.

Let $S_3'(X)$ denote the counting function of the integers g that satisfy (3.6) but fail condition (ii) of Theorem 1. Note together these requirements restrict a to squarefree values. By (3.3),

$$\begin{aligned} S_3'(X) &= \sum_{g \in \mathcal{A}(X)} \alpha_4(g-1) + O(1) = \sum_{g \in \mathcal{A}(X)} \sum_{d^4 | (g-1)} \mu(d) + O(1) \\ &= \sum_{d \le X^{1/4}} \mu(d) \sum_{\substack{g \in \mathcal{A}(X) \\ d^4 | (g-1)}} 1 + O(1) \\ &= \sum_{d \le D} \mu(d) \sum_{\substack{g \in \mathcal{A}(X) \\ d^4 | (g-1)}} 1 + O\big(XD^{-3}\big). \end{aligned} \tag{3.7}$$

To estimate the sum on the right side of (3.7), we first observe that the contribution from residue classes (3.6) with $a > A$ is bounded above by

$$\sum_{d \le D} \sum_{a > A} \frac{X}{a^2(a-1)} = O\big(XDA^{-2}\big).$$

Upon choosing $A = D^2$, we deduce that

$$S_3'(X) = \sum_{d \le D} \mu(d) \sum_{3 \le a \le A} \sum_{\substack{g \in \mathcal{A}_a(X) \\ d^4 | (g-1)}} 1 + O\big(XD^{-3}\big). \tag{3.8}$$

We now call a set of squarefree integers $\{a_1, a_2, \dots, a_k\}$ *d-admissible* if $a_i > 1$ for all i and

$$(a_i, a_j - 1) = (a_i - 1, d) = 1 \qquad \text{for all } i, j \in \{1, 2, \dots, k\}. \tag{3.9}$$

The d-admissibility of a set $\{a_1, a_2, \dots, a_k\}$ means that the congruences (3.6) with $a = a_i$, $1 \le i \le k$, are consistent with one another and also with the condition $d^4 \mid (g-1)$. In particular, if $\{a\}$ is d-admissible (i.e., if $(a-1, d) = 1$), an inclusion-exclusion argument shows that the set $\mathcal{A}_a$ has density

$$\delta(d, a) = f_0(d, a) - \sideset{}{'}\sum_{b < a} f_1(d, a, b) + \sideset{}{'}\sum_{b_2 < b_1 < a} f_2(d, a, b_1, b_2) - \cdots,$$

where $f_k(d,a,b_1,\dots) = [d^4, a^2(a-1), b_1^2(b_1-1), \dots]^{-1}$ and the summations are over odd integers $b, b_1, b_2, \dots$ such that the sets $\{a,b\}, \{a,b_1,b_2\}, \dots$ are d-admissible. The same argument lets us rewrite (3.8) as

$$S_3'(X) = X \sum_{d \le D} \mu(d) \sideset{}{'}\sum_{3 \le a \le A} \delta(d,a) + O\big(D2^A + XD^{-3}\big).$$

Hence, if we choose $D = \lfloor \sqrt{\ln X} \rfloor$ (and keep $A = D^2$), we obtain

$$S_3'(X) = X \sum_{d \le D} \mu(d) \sideset{}{'}\sum_{3 \le a \le A} \delta(d,a) + O\big(XD^{-3}\big).$$

Recalling that a above is restricted to odd values, we conclude that

$$\lim_{X \to \infty} X^{-1} S_3'(X) = \sum_{(d,2)=1} \mu(d) \sideset{}{'}\sum_{(a,2)=1} \delta(d,a). \tag{3.10}$$

Let $S_3''(X)$ be the part of $S_3'(X)$ that counts integers g subject to $g \equiv 55 \pmod{81}$. We can estimate $S_3''(X)$ using a variant of the above argument. It is not difficult to see that the conditions

$$d^4 \mid (g-1), \quad g \equiv 55 \pmod{81}, \quad g \equiv 1 - a_i^2 \pmod{a_i^2(a_i-1)},$$

where $1 \le i \le k$, are consistent if and only if $(d,6) = 1$, the moduli $a_1, a_2, \dots, a_k$ satisfy (3.9) with $3d$ in place of d, and none of the a_i's is divisible by 9. However, since we are only interested in squarefree a_i's, the latter condition is superfluous. Thus, the argument leading to (3.10) also gives

$$\lim_{X \to \infty} X^{-1} S_3''(X) = \sum_{(d,6)=1} \mu(d) \sideset{}{'}\sum_{(a,2)=1} \delta(3d,a). \tag{3.11}$$

Finally, we note that the difference $S_3'(X) - S_3''(X)$ is exactly the counting function of $\mathcal{S}_3$. Hence, by (3.10) and (3.11), the set $\mathcal{S}_3$ has density

$$\delta_3 = \sum_{(d,2)=1} \mu(d) \sideset{}{'}\sum_{(a,2)=1} \delta(d,a) - \sum_{(d,6)=1} \mu(d) \sideset{}{'}\sum_{(a,2)=1} \delta(3d,a).$$

A simple calculation using *Mathematica* yields $\delta_3 \approx .0564$.

3.3. **The density of $\mathcal{S}_4$.** The integers g that satisfy condition (iv) of Theorem 1 never satisfy condition (iii) for parity reasons, so we only need to exclude those g that satisfy conditions (i) or (ii). Consequently, the calculation of $\delta(\mathcal{S}_4)$ is very similar to that we just went through to calculate δ_3. Let $\mathcal{B}$ be the set of biquadrate-free values that the polynomial $b^2a^2(a-1)$ takes when $a \equiv 3 \pmod 4$ and $b > 1$ is odd. Note that this restricts a and b to be squarefree and relatively prime. We reduce the calculation of the density of $\mathcal{S}_4$ to estimates for the distribution of sets of multiples of $\mathcal{B}$ in residue classes. The next lemma is a slight generalization of a classical result on the density of a set of multiples.

Lemma 8. *Let $\mathcal{B} = \{b_1, b_2, \dots\}$ be a set of positive integers such that $\sum_k b_k^{-1}$ converges, and let a, q be positive integers. Then the set*

$$\mathcal{M}(\mathcal{B}; q, a) = \big\{n \in \mathbb{N} : n \equiv a \pmod q,\ n \textit{ divisible by some } b \in \mathcal{B}\big\}$$

has an asymptotic density given by

$$\sum_{k=1}^{\infty}\bigg(\frac{\epsilon(q,a;b_k)}{[q,b_k]} - \sum_{1\le j<k}\frac{\epsilon(q,a;b_k,b_j)}{[q,b_k,b_j]} + \sum_{1\le i<j<k}\frac{\epsilon(q,a;b_k,b_j,b_i)}{[q,b_k,b_j,b_i]} - \cdots\bigg),$$

where $\epsilon(q, a; b_k, b_j, \dots)$ is the indicator function of the condition that a is divisible by $\gcd(q, [b_k, b_j, \dots])$.

Proof. The case $q = 1$ is Halberstam and Roth [3, Theorem V.9]. The proof of that result uses the inclusion-exclusion principle to count the elements of the union of the residue classes $x \equiv 0 \pmod{b_k}$. When $q > 1$, we use the Chinese Remainder Theorem to replace the latter union with the union of residue classes modulo $[q, b_k]$, defined by the conditions

$$x \equiv 0 \pmod{b_k}, \qquad x \equiv a \pmod q,$$

when those conditions are consistent (i.e., when $\epsilon(q, a; b_k) = 1$). We then follow the argument of Halberstam and Roth. □

We remark that the set $\mathcal{B}$ defined at the beginning of the section satisfies the hypothesis of Lemma 8. We may therefore apply Lemma 8 to sets $\mathcal{M}(\mathcal{B}; q, a)$ for various choices of q and a.

We have $\mathcal{S}_4 = \mathcal{S}_4' \setminus \mathcal{S}_4''$, where

$$\mathcal{S}_4' = \big\{h + 1 : h \in \mathcal{M}(\mathcal{B}; 4, 2),\ h \text{ biquadrate-free}\big\},$$
$$\mathcal{S}_4'' = \big\{h + 1 : h \in \mathcal{M}(\mathcal{B}; 324, 54),\ h \text{ biquadrate-free}\big\}.$$

Let $S_4'(X)$ and $S_4''(X)$ denote the counting functions of these two sets. Similarly to (3.7), we have

$$S_4'(X) = \sum_{\substack{d\le D\\ (d,2)=1}} \mu(d) \sum_{\substack{h\le X\\ h\in\mathcal{M}(\mathcal{B};4d^4,2d^4)}} 1 + O\big(XD^{-3}\big),$$

where D is a large integer. When X is sufficiently large in terms of D, we may use Lemma 8 to get

$$S_4'(X) = X \sum_{\substack{d\le D\\ (d,2)=1}} \mu(d)\beta(d) + O\big(XD^{-3}\big),$$

with

$$\beta(d) = \sum_{k=1}^{\infty} \left(\frac{\epsilon(4d^4, 2d^4; b_k)}{[4d^4, b_k]} - \sum_{1 \le j < k} \frac{\epsilon(4d^4, 2d^4; b_k, b_j)}{[4d^4, b_k, b_j]} + \cdots \right),$$

where $\mathcal{B} = \{b_1 < b_2 < \cdots\}$. Since $\mathcal{B}$ is contained in the residue class $x \equiv 2 \pmod 4$ and $d^4 \equiv 1 \pmod{16}$, we have

$$\frac{\epsilon(4d^4, 2d^4; b_k, b_j, \dots)}{[4d^4, b_k, b_j, \dots]} = \frac{1}{2[d^4, b_k, b_j, \dots]}.$$

Hence,

$$\beta(d) = \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{1}{[d^4, b_k]} - \sum_{1 \le j < k} \frac{1}{[d^4, b_k, b_j]} + \cdots \right).$$

By letting $X \to \infty$ and then $D \to \infty$, we conclude that

$$\delta(\mathcal{S}_4') = \frac{1}{2} \sum_{(d,2)=1} \mu(d) \sum_{k=1}^{\infty} \left(\frac{1}{[d^4, b_k]} - \sum_{1 \le j < k} \frac{1}{[d^4, b_k, b_j]} + \cdots \right).$$

A similar argument can be applied to $\mathcal{S}_4''$ to show that

$$\delta(\mathcal{S}_4'') = \frac{1}{2} \sum_{(d,6)=1} \mu(d) \sum_{k=1}^{\infty} \left(\frac{\epsilon(81, 27; b_k)}{[81d^4, b_k]} - \sum_{1 \le j < k} \frac{\epsilon(81, 27; b_k, b_j)}{[81d^4, b_k, b_j]} + \cdots \right).$$

Note that $\epsilon(81, 27; b_k, b_j)$ is 0 or 1 according as 81 divides some or none of the integers $b_k, b_j, \dots$. Recalling that the elements of $\mathcal{B}$ are biquadrate-free, we deduce that

$$\delta(\mathcal{S}_4'') = \frac{1}{2} \sum_{(d,6)=1} \mu(d) \sum_{k=1}^{\infty} \left(\frac{1}{[81d^4, b_k]} - \sum_{1 \le j < k} \frac{1}{[81d^4, b_k, b_j]} + \cdots \right).$$

We conclude that the density of $\mathcal{S}_4$ is

$$\delta_4 = \delta(\mathcal{S}_4') - \delta(\mathcal{S}_4'') \approx 0.0019$$

(after another computer calculation). This concludes the proof of Theorem 4.

References

[1] Borror, B., A. Morris and M. Tarr. *The strong symmetric genus spectrum of abelian groups*. Rose-Hulman Undergrad. Math. J. **15** (2014), 115–130.

[2] Gross, J.L. and T.W. Tucker. *Topological Graph Theory*. John Wiley & Sons, New York, 1987.

[3] Halberstam, H. and K.F. Roth. *Sequences*. Oxford University Press, Oxford, 1966.

[4] Hurwitz, A. *Über algebraische Gebilde mit eindeutigen Transformationen in sich*. Math. Ann. **41** (1893), 403–442; in German.

[5] Maclachlan, C. *Abelian groups of automorphisms of compact Riemann surfaces.* Proc. London Math. Soc. (3) **15** (1965), 699–712.
[6] May, C.L. and J. Zimmerman. *There is a group of every strong symmetric genus.* Bull. London Math. Soc. **35** (2003), 433–439.
[7] Montgomery, H.L. and R.C. Vaughan. *Multiplicative Number Theory I: Classical Theory.* Cambridge University Press, Cambridge, 2007.
[8] Prachar, K. *Über die kleinste quadratfreie Zahl einer arithmetischen Reihe.* Monatsh. Math. **62** (1958), 173–176; in German.

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