ON WEYL SUMS OVER PRIMES IN SHORT INTERVALS

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1. Introduction

In this note we pursue bounds for exponential sums of the form

$$f_k(\alpha; x, y) = \sum_{x < n \le x+y} \Lambda(n) e\left(\alpha n^k\right), \tag{1.1}$$

where $k \ge 2$ is an integer, $2 \le y \le x$, $\Lambda(n)$ is von Mangoldt's function, and $e(z) = e^{2\pi i z}$. When $y = x^\theta$ with $\theta < 1$, such exponential sums play a central role in applications of the Hardy–Littlewood circle method to additive problems with almost equal prime unknowns (see [6, 7, 9]). When α is closely approximated by a rational number with a small denominator (i.e., when α is on a "major arc"), Liu, Lü and Zhan [5] bounded $f_k(\alpha; x, x^\theta)$ using methods from multiplicative number theory. Their result, which generalizes earlier work by Ren [8], can be stated as follows.

Theorem 1. *Let $k \ge 1$, $7/10 < \theta \le 1$ and $0 < \rho \le \min\{(8\theta - 5)/(6k+6), (10\theta - 7)/15\}$. Suppose that α is real and that there exist integers a and q satisfying*

$$1 \le q \le P, \quad (a, q) = 1, \quad |q\alpha - a| \le x^{-k+2(1-\theta)} P, \tag{1.2}$$

with $P = x^{2k\rho}$. Then, for any fixed $\varepsilon > 0$,

$$f_k\left(\alpha; x, x^\theta\right) \ll x^{\theta - \rho + \varepsilon} + x^{\theta + \varepsilon} \Xi(\alpha)^{-1/2},$$

where $\Xi(\alpha) = q + x^{k - 2(1-\theta)} |q\alpha - a|$.

For a given P, let $\mathfrak{M}(P)$ denote the set of real α that have rational approximations of the form (1.2), and let $\mathfrak{m}(P)$ denote the complement of $\mathfrak{M}(P)$. In the terminology of the circle method, $\mathfrak{M}(P)$ is a set of *major arcs* and $\mathfrak{m}(P)$ is the respective set of *minor arcs*. The main goal of this note is to bound $f_k(\alpha; x, x^\theta)$, $k \ge 3$, on sets of minor arcs by extending a theorem of the author [4, Theorem 1], which gives the best known bound for $f_k(\alpha; x, x)$. We establish the following theorem.

Theorem 2. *Let $k \ge 3$ and θ be a real number with $(2k+2)/(2k+3) < \theta \le 1$. Suppose that $0 < \rho \le \rho_k(\theta)$, where*

$$\rho_k(\theta) = \begin{cases} \min\left(\frac{1}{14}(2\theta - 1), \frac{1}{6}(9\theta - 8)\right) & \text{if } k = 3, \\ \min\left(\frac{\sigma_k}{6}(3\theta - 1), \frac{1}{6}((2k+3)\theta - 2k - 2)\right) & \text{if } k \ge 4, \end{cases}$$

with σ_k defined by $\sigma_k^{-1} = \min(2^{k-1}, 2k(k-2))$. Then, for any fixed $\varepsilon > 0$,

$$\sup_{\alpha \in \mathfrak{m}(P)} \left| f_k\left(\alpha; x, x^\theta\right)\right| \ll x^{\theta - \rho + \varepsilon} + x^{\theta + \varepsilon} P^{-1/2}. \tag{1.3}$$

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When $\theta = 1$ and $k \leq 7$, this theorem recovers the respective cases of [4, Theorem 3]. On the other hand, when $k \geq 8$, the bound (1.3) is technically new even in the case $\theta = 1$, as we use the occasion to put on the record an almost automatic improvement of the theorems in [4] that results from a recent breakthrough by Wooley [11, 12].

Notation. Throughout the paper, the letter ε denotes a sufficiently small positive real number. Any statement in which ε occurs holds for each positive ε, and any implied constant in such a statement is allowed to depend on ε. The letter p, with or without subscripts, is reserved for prime numbers. As usual in number theory, $\mu(n)$, $\tau(n)$ and $\|x\|$ denote, respectively, the Möbius function, the number of divisors function and the distance from x to the nearest integer. We write $(a, b) = \gcd(a, b)$, and we use $m \sim M$ as an abbreviation for the condition $M < m \leq 2M$.

2. Auxiliary results

When $k \geq 3$, we define the multiplicative function $w_k(q)$ by

$$w_k\left(p^{ku+v}\right) = \begin{cases} kp^{-u-1/2}, & \text{if } u \geq 0, v = 1, \\ p^{-u-1}, & \text{if } u \geq 0, v = 2, \dots, k. \end{cases}$$

By the argument of [10, Theorem 4.2], we have

$$\sum_{1 \leq x \leq q} e\left(ax^k/q\right) \ll qw_k(q) \ll q^{1-1/k} \tag{2.1}$$

whenever $k \geq 3$ and $(a, q) = 1$. We also need several estimates for sums involving the function $w_k(q)$. We list those in the following lemma.

Lemma 2.1. *Let $w_k(q)$ be the multiplicative function defined above. Then the following inequalities hold for any fixed $\varepsilon > 0$:*

$$\sum_{q \sim Q} w_k(q)^j \ll \begin{cases} Q^{-1+\varepsilon} & \text{if } k = 3, j = 4, \\ Q^{-1+1/k} & \text{if } k \geq 4, j = k; \end{cases} \tag{2.2}$$

$$\sum_{n \sim N} w_k\left(\frac{q}{(q, n^j)}\right) \ll q^\varepsilon w_k(q) N \qquad (1 \leq j \leq k); \tag{2.3}$$

$$\sum_{\substack{n \sim N \\ (n,h)=1}} w_k\left(\frac{q}{(q, R(n, h))}\right) \ll q^\varepsilon w_k(q) N + q^\varepsilon, \tag{2.4}$$

where $R(n, h) = \left((n + h)^k - n^k\right)/h$.

Proof. See Lemmas 2.3 and 2.4 and inequality (3.11) in Kawada and Wooley [3]. □

Lemma 2.2. *Let $k \geq 3$ be an integer and let $0 < \rho \leq \sigma_k$, where $\sigma_k^{-1} = \min(2^{k-1}, 2k(k-2))$. Suppose that $y \leq x$, $x^k \leq y^{k+1-2\rho}$, and $\mathcal{I}$ is a subinterval of $(x, x + y]$. Then either*

$$\sum_{n \in \mathcal{I}} e\left(\alpha n^k\right) \ll y^{1-\rho+\varepsilon}, \tag{2.5}$$

or there exist integers a and q such that

$$1 \leq q \leq y^{k\rho}, \quad (a, q) = 1, \quad |q\alpha - a| \leq x^{1-k} y^{k\rho-1}, \tag{2.6}$$

and

$$\sum_{n\in\mathcal{I}} e\left(\alpha n^k\right) \ll \frac{w_k(q)y}{1+yx^{k-1}|\alpha-a/q|} + x^{k/2+\varepsilon}y^{(1-k)/2}. \tag{2.7}$$

Proof. By Dirichlet's theorem on Diophantine approximation, there exist integers a and q with

$$1 \le q \le y^{k-1}, \quad (a,q)=1, \quad |q\alpha - a| \le y^{1-k}. \tag{2.8}$$

When $q > y$, we rewrite the sum on the left of (2.5) as

$$\sum_{1\le n\le z} e\left(\alpha n^k + \alpha_{k-1}n^{k-1} + \cdots + \alpha_0\right),$$

where $z \le y$ and $\alpha_j = \binom{k}{j}\alpha u^{k-j}$, with u a fixed integer. Hence, (2.5) follows from Weyl's bound

$$\sum_{1\le n\le z} e\left(\alpha n^k + \alpha_{k-1}n^{k-1} + \cdots + \alpha_0\right) \ll y^{1-\sigma_k+\varepsilon}.$$

Under (2.8), this follows from [10, Lemma 2.4] when $\sigma_k = 2^{1-k}$ and from a recent bound of Wooley [12, Theorem 11.1] otherwise. When $q \le y$, we deduce (2.7) from [10, Theorem 4.1], (2.1) and a variant of [10, Lemma 6.2]. Thus, at least one of (2.5) and (2.7) always holds. The lemma follows on noting that when conditions (2.6) fail, inequality (2.5) follows from (2.7) and the hypothesis $x^k \le y^{k+1-2\rho}$. □

The following lemma is a slight variation of [1, Lemma 6]. The proof is the same.

Lemma 2.3. *Let q and N be positive integers exceeding 1 and let $0 < \delta < \frac{1}{2}$. Suppose that $q \nmid a$ and denote by $\mathcal{S}$ the number of integers n such that*

$$N < n \le 2N, \quad (n,q)=1, \quad \left\|an^k/q\right\| < \delta.$$

Then

$$\mathcal{S} \ll \delta q^{\varepsilon}(q+N).$$

3. Multilinear Weyl sums

We write

$$\delta = x^{\theta-1}, \quad L = \log x, \quad \mathcal{I} = \left(x, x+x^{\theta}\right].$$

We also set

$$Q = \left(\delta x^{k-2\rho}\right)^{k/(2k-1)}. \tag{3.1}$$

Recall that, by Dirichlet's theorem on Diophantine approximations, every real number α has a rational approximation a/q, where a and q are integers subject to

$$1 \le q \le Q, \quad (a,q)=1, \quad |\alpha - a/q| < (qQ)^{-1}. \tag{3.2}$$

Lemma 3.1. *Let $k \ge 3$ and $0 < \rho < \sigma_k/(2+2\sigma_k)$. Suppose that α is real and that there exist integers a and q such that (3.2) holds with Q given by (3.1). Let $|\xi_m| \le 1$, $|\eta_n| \le 1$, and define*

$$S(\alpha) = \sum_{m\sim M}\sum_{mn\in\mathcal{I}} \xi_m\eta_n e\left(\alpha(mn)^k\right).$$

Then

$$S(\alpha) \ll x^{\theta-\rho+\varepsilon} + \frac{w_k(q)^{1/2} x^{\theta+\varepsilon}}{(1+\delta^2 x^k|\alpha - a/q|)^{1/2}},$$

provided that

$$\delta^{-1} \max\left(x^{2\rho/\sigma_k}, \delta^{-k}x^{4\rho}, \left(\delta^{2k-2}x^{k-1+4k\rho}\right)^{1/(2k-1)}\right) \ll M \ll x^{\theta-2\rho}. \tag{3.3}$$

Proof. Set $H = \delta M$ and $N = xM^{-1}$ and define ν by $H^\nu = x^{2\rho}L^{-1}$. By (3.3), we have $\nu < \sigma_k$. For $n_1, n_2 \le 2N$, let

$$\mathcal{M}(n_1, n_2) = \left\{m \in (M, 2M] : mn_1, mn_2 \in \mathcal{I}\right\}.$$

By Cauchy's inequality and an interchange of the order of summation,

$$|S(\alpha)|^2 \ll x^\theta M + MT_1(\alpha), \tag{3.4}$$

where

$$T_1(\alpha) = \sum_{n_1<n_2} \left| \sum_{m\in\mathcal{M}(n_1,n_2)} e\left(\alpha\left(n_2^k - n_1^k\right)m^k\right)\right|.$$

Let $\mathcal{N}$ denote the set of pairs (n_1, n_2) with $n_1 < n_2$ and $\mathcal{M}(n_1, n_2) \neq \varnothing$ for which there exist integers b and r such that

$$1 \le r \le H^{k\nu}, \quad (b, r) = 1, \quad \left|r\left(n_2^k - n_1^k\right)\alpha - b\right| \le H^{k\nu}(\delta M^k)^{-1}. \tag{3.5}$$

We remark that there are $O(\delta N^2)$ pairs (n_1, n_2) with $\mathcal{M}(n_1, n_2) \neq \varnothing$. Since $\nu < \sigma_k$ and $M^k \le H^{k+1-2\nu}$, we can apply Lemma 2.2 with $\rho = \nu$, $x = x/n_1 \asymp M$ and $y = 2H$ to the inner summation in $T_1(\alpha)$. We get

$$T_1(\alpha) \ll x^{2\theta-2\rho+\varepsilon}M^{-1} + T_2(\alpha), \tag{3.6}$$

where

$$T_2(\alpha) = \sum_{(n_1,n_2)\in\mathcal{N}} \frac{w_k(r)H}{1 + \delta M^k\left|\left(n_2^k - n_1^k\right)\alpha - b/r\right|}.$$

We now change the summation variables in $T_2(\alpha)$ to

$$d = (n_1, n_2), \quad n = n_1/d, \quad h = (n_2 - n_1)/d.$$

We obtain

$$T_2(\alpha) \ll \sum_{dh\le\delta N}\sum_n \frac{w_k(r)H}{1 + \delta M^k\left|hd^kR(n,h)\alpha - b/r\right|}, \tag{3.7}$$

where $R(n, h) = \left((n + h)^k - n^k\right)/h$ and the inner summation is over n with $(n, h) = 1$ and $(nd, (n + h)d) \in \mathcal{N}$. For each pair (d, h) appearing in the summation on the right side of (3.7), Dirichlet's theorem on Diophantine approximation yields integers b_1 and r_1 with

$$1 \le r_1 \le x^{-2k\rho}(\delta M^k), \quad (b_1, r_1) = 1, \quad \left|r_1hd^k\alpha - b_1\right| \le x^{2k\rho}(\delta M^k)^{-1}. \tag{3.8}$$

As $R(n, h) \le 3^kN^{k-1}$, combining (3.3), (3.5) and (3.8), we get

$$\begin{aligned}|b_1rR(n, h) - br_1| &\le r_1H^{k\nu}(\delta M^k)^{-1} + rR(n, h)x^{2k\rho}(\delta M^k)^{-1}\\ &\le L^{-k} + 3^k\delta^{-1}x^{k-1+4k\rho}M^{1-2k}L^{-k} < 1.\end{aligned}$$

Hence,

$$\frac{b}{r} = \frac{b_1 R(n,h)}{r_1}, \quad r = \frac{r_1}{(r_1, R(n,h))}. \tag{3.9}$$

Combining (3.7) and (3.9), we obtain

$$T_2(\alpha) \ll \sum_{dh \le \delta N} \frac{H}{1 + \delta M^k N_d^{k-1} \left| hd^k\alpha - b_1/r_1 \right|} \sum_{\substack{n \sim N_d \\ (n,h)=1}} w_k\left(\frac{r_1}{(r_1, R(n,h))}\right),$$

where $N_d = Nd^{-1}$. Using (2.4), we deduce that

$$T_2(\alpha) \ll \delta x^{\theta+\varepsilon} + T_3(\alpha), \tag{3.10}$$

where

$$T_3(\alpha) = \sum_{dh \le \delta N} \frac{r_1^{\varepsilon} w_k(r_1) H N_d}{1 + \delta M^k N_d^{k-1} \left| hd^k\alpha - b_1/r_1 \right|}.$$

We now write $\mathcal{H}$ for the set of pairs (d,h) with $dh \le \delta N$ for which there exist integers b_1 and r_1 subject to

$$1 \le r_1 \le x^{2k\rho}, \quad (b_1, r_1) = 1, \quad \left| r_1 hd^k\alpha - b_1 \right| \le x^{-k+1+2k\rho} H^{-1}. \tag{3.11}$$

We have

$$T_3(\alpha) \ll x^{2\theta - 2\rho + \varepsilon} M^{-1} + T_4(\alpha), \tag{3.12}$$

where

$$T_4(\alpha) = \sum_{(d,h) \in \mathcal{H}} \frac{r_1^{\varepsilon} w_k(r_1) H N_d}{1 + \delta M^k N_d^{k-1} \left| hd^k\alpha - b_1/r_1 \right|}.$$

For each $d \le \delta N$, Dirichlet's theorem on Diophantine approximation yields integers b_2 and r_2 with

$$1 \le r_2 \le \tfrac{1}{2} x^{k-1-2k\rho} H, \quad (b_2, r_2) = 1, \quad \left| r_2 d^k\alpha - b_2 \right| \le 2x^{-k+1+2k\rho} H^{-1}. \tag{3.13}$$

Combining (3.11) and (3.13), we obtain

$$\begin{aligned} |b_2 r_1 h - b_1 r_2| &\le (r_2 + 2r_1 h) x^{-k+1+2k\rho} H^{-1} \\ &\le \tfrac{1}{2} + 2x^{-k+2+4k\rho} M^{-2} < 1, \end{aligned}$$

whence

$$\frac{b_1}{r_1} = \frac{hb_2}{r_2}, \quad r_1 = \frac{r_2}{(r_2, h)}.$$

We write $Z_d = \delta M^k N_d^{k-1} \left| d^k\alpha - b_2/r_2 \right|$ and we use (2.3) to get

$$T_4(\alpha) \le \sum_{dh \le \delta N} \frac{r_2^{\varepsilon} H N_d}{1 + Z_d h} w_k\left(\frac{r_2}{(r_2, h)}\right) \ll \sum_{d \le \delta N} \frac{w_k(r_2) x^{2\theta+\varepsilon} M^{-1}}{d^2 (1 + \delta Z_d N_d)}.$$

Hence,

$$T_4(\alpha) \ll x^{2\theta - 2\rho + \varepsilon} M^{-1} + T_5(\alpha), \tag{3.14}$$

where

$$T_5(\alpha) = \sum_{d \in \mathcal{D}} \frac{w_k(r_2) x^{2\theta+\varepsilon} M^{-1}}{d^2 \left(1 + \delta^2 (x/d)^k \left| d^k\alpha - b_2/r_2 \right|\right)}$$

and $\mathcal{D}$ is the set of integers $d \le x^{2\rho}$ for which there exist integers b_2 and r_2 with

$$1 \le r_2 \le x^{2k\rho}, \quad (b_2, r_2) = 1, \quad \left| r_2 d^k \alpha - b_2 \right| \le \delta^{-2} x^{-k+2k\rho}. \tag{3.15}$$

Combining (3.1), (3.2) and (3.15), we deduce that

$$\begin{aligned} \left| r_2 d^k a - b_2 q \right| &\le r_2 d^k Q^{-1} + q\delta^{-2} x^{-k+2k\rho} \\ &\le x^{4k\rho} Q^{-1} + \delta^{-2} x^{-k+2k\rho} Q < 1, \end{aligned}$$

whence

$$\frac{b_2}{r_2} = \frac{d^k a}{q}, \quad r_2 = \frac{q}{(q, d^k)}.$$

Thus, recalling (2.3), we get

$$T_5(\alpha) \ll \frac{x^{2\theta+\varepsilon} M^{-1}}{1 + \delta^2 x^k |\alpha - a/q|} \sum_{d \le x^{2\rho}} w_k \left(q/(q, d^k) \right) d^{-2} \ll \frac{w_k(q) x^{2\theta+\varepsilon} M^{-1}}{1 + \delta^2 x^k |\alpha - a/q|}. \tag{3.16}$$

The lemma follows from (3.3), (3.4), (3.6), (3.10), (3.12), (3.14) and (3.16). □

Lemma 3.2. *Let $k \ge 3$ and $0 < \rho < \sigma_k$. Suppose that α is real and that there exist integers a and q such that (3.2) holds with Q given by (3.1). Let $|\xi_{m_1,m_2}| \le 1$, and define*

$$S(\alpha) = \sum_{m_1 \sim M_1} \sum_{m_2 \sim M_2} \sum_{m_1 m_2 n \in \mathcal{I}} \xi_{m_1,m_2} e\left(\alpha (m_1 m_2 n)^k \right).$$

Then

$$S(\alpha) \ll x^{\theta-\rho+\varepsilon} + \frac{w_k(q) x^{\theta+\varepsilon}}{1 + \delta x^k |\alpha - a/q|},$$

provided that

$$M_1^{2k-1} \ll \delta x^{k-(2k+1)\rho}, \quad M_1 M_2 \ll \min(\delta x^{1-\rho/\sigma_k}, \delta^{k+1} x^{1-2\rho}), \quad M_1 M_2^2 \ll \delta^{1/k} x^{1-2\rho}. \tag{3.17}$$

Proof. Set $N = x(M_1 M_2)^{-1}$ and $H = \delta N$ and define ν by $H^\nu = x^\rho L^{-1}$. Note that, by (3.17), we have $\nu < \sigma_k$. We denote by $\mathcal{M}$ the set of pairs (m_1, m_2), with $m_1 \sim M_1$ and $m_2 \sim M_2$, for which there exist integers b_1 and r_1 with

$$1 \le r_1 \le H^{k\nu}, \quad (b_1, r_1) = 1, \quad \left| r_1 (m_1 m_2)^k \alpha - b_1 \right| \le H^{k\nu} (\delta N^k)^{-1}. \tag{3.18}$$

We apply Lemma 2.2 to the summation over n and get

$$S(\alpha) \ll x^{\theta-\rho+\varepsilon} + T_1(\alpha), \tag{3.19}$$

where

$$T_1(\alpha) = \sum_{(m_1, m_2) \in \mathcal{M}} \frac{w_k(r_1) H}{1 + \delta N^k \left| (m_1 m_2)^k \alpha - b_1/r_1 \right|}.$$

For each $m_1 \sim M_1$, we apply Dirichlet's theorem on Diophantine approximation to find integers b and r with

$$1 \le r \le x^{-k\rho} (\delta N^k), \quad (b, r) = 1, \quad \left| r m_1^k \alpha - b \right| \le x^{k\rho} (\delta N^k)^{-1}. \tag{3.20}$$

By (3.17), (3.18) and (3.20),

$$\begin{aligned} \left| b_1 r - b m_2^k r_1 \right| &\le r H^{k\nu} (\delta N^k)^{-1} + r_1 m_2^k x^{k\rho} (\delta N^k)^{-1} \\ &\le L^{-k} + 2^k \delta^{-1} x^{-k+2k\rho} (M_1 M_2^2)^k L^{-k} < 1, \end{aligned}$$

whence

$$\frac{b_1}{r_1} = \frac{m_2^k b}{r}, \quad r_1 = \frac{r}{(r, m_2^k)}.$$

Thus, by (2.3),

$$T_1(\alpha) \ll \sum_{m_1 \sim M_1} \frac{H}{1 + \delta(M_2N)^k \left| m_1^k \alpha - b/r \right|} \sum_{m_2 \sim M_2} w_k\left(\frac{r}{(r, m_2^k)} \right) \tag{3.21}$$

$$\ll \sum_{m_1 \sim M_1} \frac{r^\varepsilon w_k(r) H M_2}{1 + \delta(M_2N)^k \left| m_1^k \alpha - b/r \right|}.$$

Let $\mathcal{M}_1$ be the set of integers $m \sim M_1$ for which there exist integers b and r with

$$1 \le r \le x^{k\rho} L^{-1}, \quad (b, r) = 1, \quad \left| r m^k \alpha - b \right| \le \delta^{-1} x^{-k+k\rho} M_1^k L^{-1}. \tag{3.22}$$

From (3.21),

$$T_1(\alpha) \ll x^{\theta - \rho + \varepsilon} + T_2(\alpha), \tag{3.23}$$

where

$$T_2(\alpha) = \sum_{m \in \mathcal{M}_1} \frac{r^\varepsilon w_k(r) H M_2}{1 + \delta(M_2N)^k \left| m^k \alpha - b/r \right|}.$$

We now consider two cases depending on the size of q in (3.2).

Case 1: $q \le \delta x^{k-k\rho} M_1^{-k}$. In this case, we estimate $T_2(\alpha)$ as in the proof of Lemma 3.1. Combining (3.1), (3.2), (3.17) and (3.22), we obtain

$$\begin{aligned} \left| r m^k a - bq \right| &\le q \delta^{-1} x^{-k+k\rho} M_1^k L^{-1} + r m^k Q^{-1} \\ &\le L^{-1} + 2^k x^{k\rho} M_1^k Q^{-1} L^{-1} < 1. \end{aligned}$$

Therefore,

$$\frac{b}{r} = \frac{m^k a}{q}, \quad r = \frac{q}{(q, m^k)},$$

and by (2.3),

$$T_2(\alpha) \ll \frac{q^\varepsilon H M_2}{1 + \delta x^k |\alpha - a/q|} \sum_{m \sim M_1} w_k\left(\frac{q}{(q, m^k)} \right) \ll \frac{w_k(q) x^{\theta + \varepsilon}}{1 + \delta x^k |\alpha - a/q|}. \tag{3.24}$$

Case 2: $q > \delta x^{k-k\rho} M_1^{-k}$. We remark that in this case, the choice (3.1) and the second hypothesis in (3.17) imply that $M_1 \ge x^\rho$. By a standard splitting argument,

$$T_2(\alpha) \ll \sum_{d|q} \sum_{m \in \mathcal{M}_d(R,Z)} \frac{w_k(r) H M_2 x^\varepsilon}{1 + \delta(M_2N)^k (RZ)^{-1}}, \tag{3.25}$$

where

$$1 \le R \le x^{k\rho} L^{-1}, \quad \delta x^{k-k\rho} M_1^{-k} L \le Z \le \delta (x/M_1)^k R^{-1}, \tag{3.26}$$

and $\mathcal{M}_d(R, Z)$ is the subset of $\mathcal{M}_1$ containing integers m subject to

$$(m, q) = d, \quad r \sim R, \quad \left| r m^k \alpha - b \right| < Z^{-1}.$$

We now estimate the inner sum on the right side of (3.25). We have

$$\sum_{m \in \mathcal{M}_d(R,Z)} w_k(r) \ll \sum_{r \sim R} w_k(r) \mathcal{S}_0(r), \tag{3.27}$$

where $\mathcal{S}_0(r)$ is the number of integers $m \sim M_1$ with $(m,q) = d$ for which there exists an integer b such that

$$(b,r) = 1 \quad \text{and} \quad \left| rm^k \alpha - b \right| < Z^{-1}. \tag{3.28}$$

Since for each $m \sim M_1$ there is at most one pair (b,r) satisfying (3.28) and $r \sim R$, we have

$$\sum_{r \sim R} \mathcal{S}_0(r) \le \sum_{\substack{m \sim M_1 \\ (m,q)=d}} 1 \ll M_1 d^{-1} + 1. \tag{3.29}$$

Hence,

$$\sum_{\substack{r \sim R \\ (q, rd^k) = q}} w_k(r) \mathcal{S}_0(r) \ll R^{-1/k} \left(M_1 d^{-1} + 1 \right) \ll M_1 q^{-1/k} + 1, \tag{3.30}$$

on noting that the sum on the left side is empty unless $Rd^k \gg q$.

When $(q, rd^k) < q$, we make use of Lemma 2.3. By (3.2), (3.26) and (3.28),

$$\mathcal{S}_0(r) \le \mathcal{S}(r), \tag{3.31}$$

where $\mathcal{S}(r)$ is the number of integers m subject to

$$m \sim M_1 d^{-1}, \quad (m, q_1) = 1, \quad \left\| ard^{k-1} m^k / q_1 \right\| < \Delta,$$

with $q_1 = qd^{-1}$ and $\Delta = Z^{-1} + 2^{k+1} R M_1^k (qQ)^{-1}$. Since (3.17) implies $M_1 \le \delta x^{k - k\rho} M_1^{-k} < q$, we obtain

$$\mathcal{S}(r) \ll \Delta q^{\varepsilon} d^{-1} (M_1 + q) \ll \Delta q^{1+\varepsilon}. \tag{3.32}$$

Combining (3.31) and (3.32), we get

$$\mathcal{S}_0(r) \ll \Delta q^{1+\varepsilon}. \tag{3.33}$$

We now apply Hölder's inequality, (2.2), (3.29), and (3.33) and obtain

$$\begin{aligned} \sum_{\substack{r \sim R \\ (q, rd^3) < q}} w_3(r) \mathcal{S}_0(r) &\ll \left(\Delta q^{1+\varepsilon} \right)^{1/4} \left(\sum_{r \sim R} w_3(r)^4 \right)^{1/4} \left(\sum_{r \sim R} \mathcal{S}_0(r) \right)^{3/4} \\ &\ll \Delta^{1/4} q^{1/4+\varepsilon} R^{-1/4} M_1^{3/4}. \end{aligned} \tag{3.34}$$

Similarly, when $k \ge 4$, we have

$$\begin{aligned} \sum_{\substack{r \sim R \\ (q, rd^k) < q}} w_k(r) \mathcal{S}_0(r) &\ll \left(\Delta q^{1+\varepsilon} \right)^{1/k} \left(\sum_{r \sim R} w_k(r)^k \right)^{1/k} \left(\sum_{r \sim R} \mathcal{S}_0(r) \right)^{1-1/k} \\ &\ll \Delta^{1/k} q^{1/k+\varepsilon} R^{(1-k)/k^2} M_1^{(k-1)/k}. \end{aligned} \tag{3.35}$$

Combining (3.27), (3.30), (3.34) and (3.35), we deduce

$$\sum_{m\in\mathcal{M}_d(R,Z)} w_3(r) \ll \Delta^{1/4}q^{1/4+\varepsilon}R^{-1/4}M_1^{3/4} + M_1q^{-1/3} + 1 \tag{3.36}$$

and

$$\sum_{m\in\mathcal{M}_d(R,Z)} w_k(r) \ll \Delta^{1/k}q^{1/k+\varepsilon}R^{(1-k)/k^2}M_1^{(k-1)/k} + M_1q^{-1/k} + 1 \tag{3.37}$$

for $k \geq 4$.

Substituting (3.36) into (3.25), we get

$$\begin{aligned} T_2(\alpha) &\ll \frac{x^{\theta+\varepsilon}M_1^{-1/4}}{1+\delta(M_2N)^3(RZ)^{-1}}\left(\frac{Q}{RZ}+\frac{M_1^3}{Q}\right)^{1/4} + x^{\theta+\varepsilon}q^{-1/3} + x^{\theta+\varepsilon}M_1^{-1} \\ &\ll (\delta^3xM_1^2Q)^{1/4+\varepsilon} + x^{\theta+\varepsilon}\left(M_1^2Q^{-1}\right)^{1/4} + x^{\rho+\varepsilon}M_1 + x^{\theta-\rho+\varepsilon}. \end{aligned}$$

The hypotheses of the lemma ensure that

$$M_1 \leq \min\left(\delta^{1/2}x^{3/2-2\rho}Q^{-1/2}, Q^{1/2}x^{-2\rho}, x^{\theta-2\rho}\right),$$

and so when $k=3$,

$$T_2(\alpha) \ll x^{\theta-\rho+\varepsilon}. \tag{3.38}$$

When $k \geq 4$, by (3.25) and (3.37),

$$\begin{aligned} T_2(\alpha) &\ll \frac{x^{\theta+\varepsilon}M_1^{-1/k}R^{1/k^2}}{1+\delta(M_2N)^k(RZ)^{-1}}\left(\frac{Q}{RZ}+\frac{M_1^k}{Q}\right)^{1/k} + x^{\theta+\varepsilon}q^{-1/k} + x^{\theta+\varepsilon}M_1^{-1} \\ &\ll \left(x^\rho Q(\delta M_1)^{k-1}\right)^{1/k+\varepsilon} + x^{\theta+\varepsilon}\left(x^\rho M_1^{k-1}Q^{-1}\right)^{1/k} + x^{\rho+\varepsilon}M_1 + x^{\theta-\rho+\varepsilon}, \end{aligned}$$

and using (3.1) and (3.17), we find that (3.38) holds in this case as well.

The desired estimate follows from (3.19), (3.23), (3.24) and (3.38). □

4. Proof of Theorem 2

In this section we deduce Theorem 2 from Lemmas 3.1 and 3.2 and Heath-Brown's identity for $\Lambda(n)$. We apply Heath-Brown's identity in the following form [2, Lemma 1]: if $n \leq X$ and J is a positive integer, then

$$\Lambda(n) = \sum_{j=1}^{J}\binom{J}{j}(-1)^j \sum_{\substack{n=n_1\cdots n_{2j}\\ n_1,\dots,n_j\leq X^{1/J}}} \mu(n_1)\cdots\mu(n_j)(\log n_{2j}). \tag{4.1}$$

Let $\alpha \in \mathfrak{m}(P)$. By Dirichlet's theorem on Diophantine approximation, there exist integers a and q such that (3.2) holds with Q given by (3.1). Let β be defined by

$$x^\beta = \min\left(\delta^2x^{1-2\rho(\sigma_k^{-1}+1)}, \delta^{k+2}x^{1-6\rho}, \left(\delta^{2k}x^{k-(8k-2)\rho}\right)^{1/(2k-1)}\right),$$

and suppose that ρ and δ are chosen so that

$$\delta^{-1}x^{\beta+2\rho} \geq 2x^{1/3}, \quad x^\beta \geq \delta^{-1}x^{2\rho}. \tag{4.2}$$

We apply (4.1) with $X = x + x^\theta$ and $J \geq 3$ chosen so that $x^{1/J} \leq x^\beta$. After a standard splitting argument, we have

$$\sum_{n \in \mathcal{I}} \Lambda(n) e\left(\alpha n^k\right) \ll \sum_{\mathbf{N}} \left| \sum_{n \in \mathcal{I}} c(n;\mathbf{N}) e\left(\alpha n^k\right) \right|, \tag{4.3}$$

where $\mathbf{N}$ runs over $O(L^{2J-1})$ vectors $\mathbf{N} = (N_1, \dots, N_{2j})$, $j \leq J$, subject to

$$N_1, \dots, N_j \ll x^{1/J}, \qquad x \ll N_1 \cdots N_{2j} \ll x,$$

and

$$c(n;\mathbf{N}) = \sum_{\substack{n = n_1 \cdots n_{2j} \\ n_i \sim N_i}} \mu(n_1) \cdots \mu(n_j)(\log n_{2j}).$$

In fact, since the coefficient $\log n_{2j}$ can be removed by partial summation, we may assume that

$$c(n;\mathbf{N}) = L \sum_{\substack{n = n_1 \cdots n_{2j} \\ N_i < n_i \leq N_i'}} \mu(n_1) \cdots \mu(n_j),$$

where $N_i < N_i' \leq 2N_i$ (in reality, $N_i' = 2N_i$ except for $i = 2j$). We also assume (as we may) that the summation variables $n_{j+1}, \dots, n_{2j}$ are labeled so that $N_{j+1} \leq \cdots \leq N_{2j}$. Next, we show that each of the sums occurring on the right side of (4.3) satisfies the bound

$$\sum_{n \in \mathcal{I}} c(n;\mathbf{N}) e\left(\alpha n^k\right) \ll x^{\theta - \rho + \varepsilon} + \frac{w_k(q)^{1/2} x^{\theta + \varepsilon}}{\left(1 + \delta^2 x^k |\alpha - a/q|\right)^{1/2}}. \tag{4.4}$$

The analysis involves several cases depending on the sizes of $N_1, \dots, N_{2j}$.

Case 1: $N_1 \cdots N_j \gg \delta^{-1} x^{2\rho}$. Since none of the N_i's exceeds x^β, there must be a set of indices $S \subset \{1, \dots, j\}$ such that

$$\delta^{-1} x^{2\rho} \leq \prod_{i \in S} N_i \leq \delta^{-1} x^{\beta + 2\rho}. \tag{4.5}$$

Hence, we can rewrite $c(n;\mathbf{N})$ in the form

$$c(n;\mathbf{N}) = \sum_{\substack{mr = n \\ m \asymp M}} \xi_m \eta_r, \tag{4.6}$$

where $|\xi_m| \ll m^\varepsilon$, $|\eta_r| \ll r^\varepsilon$ and $M = \prod_{i \notin S} N_i$. By (4.5), M satisfies (3.3), so (4.4) follows from Lemma 3.1.

Case 2: $N_1 \cdots N_j < \delta^{-1} x^{2\rho}$, $j \leq 2$. When $j = 1$, (4.4) follows from Lemma 3.2 with $M_1 = N_1$, $M_2 = 1$ and $N = N_2$. When $j = 2$, we have

$$N_3 \leq (x/N_1N_2)^{1/2} \leq x^{1/2}, \quad N_1N_2N_3 \leq (xN_1N_2)^{1/2} \leq \delta^{-1/2} x^{1/2+\rho},$$
$$(N_1N_2)^2 N_3 \leq x^{1/2} (N_1N_2)^{3/2} \leq \delta^{-3/2} x^{1/2+3\rho}.$$

Hence, we can deduce (4.4) from Lemma 3.2 with $M_1 = N_3$, $M_2 = N_1N_2$ and $N = N_4$, provided that

$$x^{k-1/2} \leq \delta x^{k-(2k+1)\rho}, \quad \delta^{-3/2} x^{1/2+3\rho} \leq \delta^{1/k} x^{1-2\rho}, \tag{4.7}$$

$$\delta^{-1/2} x^{1/2+\rho} \leq \delta \min\left(x^{1-\rho/\sigma_k}, \delta^k x^{1-2\rho}\right). \tag{4.8}$$

Case 3: $N_1 \cdots N_{2j-2} < \delta^{-1} x^{2\rho}$, $j \geq 3$. Then we are in a similar situation to Case 2 with $j = 2$, with the product $N_1 \cdots N_{2j-2}$ playing the role of $N_1 N_2$ in Case 2. Thus, we can again use Lemma 3.2 to obtain (4.4).

Case 4: $N_1 \cdots N_j < \delta^{-1} x^{2\rho} \leq N_1 \cdots N_{2j-2}$, $j \geq 3$. In this case, we have

$$N_{j+1}, \ldots, N_{2j-2} \leq 2x^{1/3} \leq \delta^{-1} x^{\beta + 2\rho}.$$

If $N_{2j-2} \geq \delta^{-1} x^{2\rho}$, we can write $c(n; \mathbf{N})$ in the form (4.6) where $M = \prod_{i \neq 2j-2} N_i$. We can then appeal to Lemma 3.1 to show that (4.4) holds. On the other hand, if $N_{2j-2} < \delta^{-1} x^{2\rho}$, then $N_{j+1}, \ldots, N_{2j-2} \leq x^{\beta}$ (by (4.2)). Thus, we can use the product $N_1 \cdots N_{2j-2}$ in a similar fashion to the product $N_1 \cdots N_j$ in Case 1 to obtain a set of indices $S \subset \{1, 2, \ldots, 2j-2\}$ such that (4.5) holds. Hence, we can again represent $c(n; \mathbf{N})$ in the form (4.6) and then appeal to Lemma 3.1 to show that (4.4) holds one last time.

By the above analysis,

$$\sum_{n \in \mathcal{I}} \Lambda(n) e\left(\alpha n^k\right) \ll x^{\theta - \rho + \varepsilon} + \frac{w_k(q)^{1/2} x^{\theta + \varepsilon}}{\left(1 + \delta^2 x^k |\alpha - a/q|\right)^{1/2}}, \tag{4.9}$$

provided that conditions (4.2), (4.7) and (4.8) hold. Altogether, those conditions are equivalent to the inequality

$$\begin{aligned} x^{\rho} \ll \min\Big(&(\delta^3 x^2)^{\sigma_k/6}, (\delta^2 x)^{1/(4k+2)}, (\delta^3 x)^{\sigma_k/(2+2\sigma_k)}, \delta^{(2k+3)/6} x^{1/6}, \\ &(\delta^{4k-1} x^k)^{1/(12k-4)}, (\delta^3 x)^{\sigma_k/(2+4\sigma_k)}, \delta^{(k+3)/8} x^{1/8}, \\ &\delta^{(k+1)/4} x^{1/6}, \delta^{(3k+2)/(10k)} x^{1/10}, \delta^{1/(4k)} x^{(k+1)/(12k)} \Big). \end{aligned} \tag{4.10}$$

When $\delta \geq x^{-3/11}$ and $k \geq 3$, we have

$$\delta^{(2k+3)/6} x^{1/6} \leq \delta^{(k+1)/4} x^{1/6}, \quad (\delta^2 x)^{1/(4k+2)} \leq \delta^{1/(4k)} x^{(k+1)/(12k)},$$
$$(\delta^{4k-1} x^k)^{1/(12k-4)} \leq (\delta^{3k+2} x^k)^{1/(12k-4)} \leq (\delta^{3k+2} x^k)^{1/(10k)}.$$

so the last three terms in the above minimum are superfluous. Clearly, so is the third. Furthermore,

$$\min\left((\delta^2 x)^{1/(4k+2)}, \delta^{(2k+3)/6} x^{1/6}\right) \leq \left((\delta^2 x)^{1/(4k+2)}\right)^a \left(\delta^{(2k+3)/6} x^{1/6}\right)^{1-a} = \delta^{(k+3)/8} x^{\mu},$$
$$\min\left((\delta^2 x)^{1/(4k+2)}, \delta^{(2k+3)/6} x^{1/6}\right) \leq \left((\delta^2 x)^{1/(4k+2)}\right)^b \left(\delta^{(2k+3)/6} x^{1/6}\right)^{1-b} = (\delta^{4k-1} x^{\nu})^{1/(12k-4)},$$

where

$$a = \frac{6k^2 + 21k - 15}{16k^2 + 32k - 12}, \quad b = \frac{24k^2 - 30k + 9}{(3k-1)(8k^2 + 16k - 6)}, \quad \mu \leq \frac{13}{114}, \quad \nu \leq \frac{52k}{57}.$$

When $k = 3$ and $\delta \geq x^{-1/9}$, we have

$$\delta^{1/7} x^{1/14} \leq \delta^{1/4} x^{1/12} \leq \delta^{1/8} x^{1/12}.$$

Hence, in this case, (4.10) is equivalent to

$$\rho \leq \min\left(\frac{2\theta - 1}{14}, \frac{9\theta - 8}{6}\right).$$

On the other hand, when $k \geq 4$, we have

$$(\delta^3 x^2)^{\sigma_k/6} \leq (\delta^2 x)^{1/(4k+2)} \qquad \text{when } \delta \geq x^{-2/7};$$
$$(\delta^3 x^2)^{\sigma_k/6} \leq (\delta^3 x)^{\sigma_k/(2+4\sigma_k)} \qquad \text{when } \delta \geq x^{-2/21}.$$

Hence, in this case, (4.10) is equivalent to

$$\rho \leq \min\left(\frac{\sigma_k(3\theta - 1)}{6}, \frac{(2k+3)\theta - 2k - 2}{6} \right).$$

Therefore, (4.10) is a direct consequence of the hypotheses of the theorem and the proof of (4.9) is complete.

If either $q \geq x^{2k\rho}$ or $|q\alpha - a| \geq \delta^{-2} x^{k-2k\rho}$, we can use (2.1) to show that the second term on the right side of (4.9) is smaller than the first. Thus,

$$\sup_{\alpha \in \mathfrak{m}(x^{2k\rho})} \left| f_k\left(\alpha; x, x^\theta\right) \right| \ll x^{\theta - \rho + \varepsilon}. \tag{4.11}$$

This establishes the theorem when $P \geq x^{2k\rho}$. When $P < x^{2k\rho}$, we combine (4.11) with the inequality

$$\sup_{\alpha \in \mathfrak{m}(P) \cap \mathfrak{M}(x^{2k\rho})} \left| f_k\left(\alpha; x, x^\theta\right) \right| \ll x^{\theta - \rho + \varepsilon} + x^{\theta + \varepsilon} P^{-1/2},$$

which follows from Theorem 1, provided that $\rho \leq (8\theta - 5)/(6k + 6)$. To complete the proof, we note that the last condition on ρ is implied by the hypotheses of the theorem. $\square$

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