ON THE CONVERGENCE OF SOME ALTERNATING SERIES

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1. Introduction

This note is motivated by a question that a colleague of the author's often challenges calculus students with: Does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n |\sin n|}{n} \tag{1}$$

converge? This series combines features of several series commonly studied in calculus:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \quad \sum_{n=1}^{\infty} \frac{|\sin n|}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$$

come to mind. However, unlike these familiar examples, the series (1) seems to live on the fringes, just beyond the reach of standard convergence tests like the alternating series test or the tests of Abel and Dirichlet. It is therefore tempting for an infinite series aficionado to study (1) in hope to find some clever resolution of the question of its convergence. Yet, the author's colleague reports that although he has posed the above question to many calculus students, he has never received an answer. It turns out that there is a good reason for that: the question is quite delicate and is intimately connected to deep facts about Diophantine approximation—facts which the typical second-semester calculus student is unlikely to know.

The series (1) is obtained by perturbation of the moduli of the alternating harmonic series, which is the simplest conditionally convergent alternating series one can imagine. In this note, we study the convergence sets of similar perturbations of a wide class of alternating series. In particular, the convergence of (1) follows from our results and classical work by Mahler [5] on the rational approximations to π.

Let $\mathfrak{F}$ denote the class of continuous, decreasing functions $f : [1, \infty) \to \mathbb{R}$ such that

$$\lim_{x \to \infty} f(x) = 0, \qquad \int_1^{\infty} f(x)\, dx = \infty.$$

When $f \in \mathfrak{F}$, the alternating series $\sum_n (-1)^n f(n)$ is conditionally convergent. Our goal is to describe the convergence set of the related series

$$\sum_{n=1}^{\infty} (-1)^n f(n) |\sin(n\pi\alpha)|, \tag{2}$$

where α is a real number.

It is natural to start one's investigation of (2) with the case when α is rational, since in that case the sequence $\{(-1)^n |\sin(n\pi\alpha)|\}_{n=1}^{\infty}$ is periodic, and one may hope to see a pattern. Indeed, this turns out to be the case, and one quickly discovers the following result.

Theorem 1. *Suppose that $f \in \mathfrak{F}$ and that $\alpha = p/q$, with $p \in \mathbb{Z}$, $q \in \mathbb{Z}^+$, and $\gcd(p, q) = 1$. Then the series (2) converges if and only if q is odd.*

Given Theorem 1, one may expect that when α is irrational, the convergence of (2) should depend on the nature and quality of the rational approximations to α. The next result sheds some light on the dependence.

Theorem 2. *Suppose that $f \in \mathfrak{F}$ and $\alpha \notin \mathbb{Q}$, and let $\{q_n\}_{n=1}^{\infty}$ be the sequence of denominators of convergents to the continued fraction of α. If the series*

$$\sum_{\substack{n=1 \\ q_n \text{ even}}}^{\infty} \frac{1}{q_n^2} \int_1^{q_{n+1}} f(x)\,dx \tag{3}$$

converges, then so does the series (2)*.*

By combining Theorem 2 with various facts about Diophantine approximation, we obtain the following corollaries.

Corollary 3. *There is a set $D \subset \mathbb{R}$, with Lebesgue measure zero, such that the series* (2) *converges for all real $\alpha \notin D$ and all $f \in \mathfrak{F}$.*

Corollary 4. *Suppose that $f \in \mathfrak{F}$ and α is an algebraic irrationality. Then the series* (2) *converges.*

Corollary 5. *The series* (1) *converges.*

Corollary 3 follows from Theorem 2 and Khinchin's theorem [2] on metric Diophantine approximation. In particular, the set D can be chosen to be the set of real α such that the inequality $q_{n+1} \leq q_n^{3/2}$ fails for infinitely many n. Similarly, Corollary 4 follows from Theorem 2 and Roth's celebrated result [8] on Diophantine approximations to algebraic irrationalities. Finally, to derive Corollary 5 directly from Theorem 2, we need to establish the convergence of (3) in the special case when $\alpha = 1/\pi$ and $f(x) = x^{-1}$. To that end, it suffices to know that the sequence $\{q_n\}_{n=1}^{\infty}$ of denominators of convergents to the continued fraction of π satisfies an inequality of the form $q_{n+1} \ll q_n^C$ for some absolute constant C. The first such results were obtained by Mahler in the early 1950s. In particular, he showed in [5] that $C = 41$ is acceptable. More recent improvements on Mahler's work by Hata [1] and Salikhov [9] allow us to take C even smaller ($C = 7.02$ and $C = 6.61$, respectively), but that has no effect on Corollary 5. The interested reader can easily obtain further corollaries along the lines of Corollaries 4 and 5 by appealing to other results in transcendental number theory. For example, one can use irrationality measures of numbers of the form $\ln a$ or $(\ln a)/(\ln b)$, with a and b real algebraic, to obtain other explicit examples of convergent series similar to (1).

Theorem 2 provides a sufficient condition for convergence of alternating series of the form (2). It is natural to ask how far is this condition from being also necessary. A closer look at the special case $f(x) = x^{-\theta}$, $0 < \theta \leq 1$, reveals that sometimes the convergence of (3) is, in fact, equivalent to the convergence of (2). We have the following result.

Theorem 6. *Suppose that $\alpha \notin \mathbb{Q}$, and let $\{q_n\}_{n=1}^{\infty}$ be the sequence of denominators of convergents to the continued fraction of α. When $0 < \theta \leq 1$, the series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n |\sin(n\pi\alpha)|}{n^{\theta}} \tag{4}$$

converges if and only if the series

$$\sum_{\substack{n=1 \\ q_n \text{ even}}}^{\infty} \frac{1}{q_n^2} \int_1^{q_{n+1}} x^{-\theta}\,dx \tag{5}$$

does.

In particular, it follows from Theorem 6 that the divergence set of (2) can be uncountable. Let a, b be integers, with $\gcd(a, b) = 1$ and $b \neq 0$, and $\{d_k\}_{k=1}^{\infty}$ be an infinite sequence of 1's and 3's, and consider the series

$$\lambda = \frac{a}{b} + \sum_{k=1}^{\infty} d_k 10^{-k!}.$$

When $a = 0$ and $d_k = 1$ for all k, this is the Liouville constant, the first example of a transcendental number constructed by Liouville [4]. It is not difficult to show that when λ is of the above form, the respective series (5) diverges for $0 < \theta < 1$. Furthermore, this classical example can be easily modified to construct numbers for which the series (5) converges even when $\theta = 1$. Since the set L of such λ's is uncountable and dense in $\mathbb{R}$, we obtain the following corollary.

Corollary 7. *There is an uncountable set $L \subset \mathbb{R}$, dense in $\mathbb{R}$, such that the series* (4) *diverges for all $\alpha \in L$ and all $\theta \in (0, 1]$.*

As is stated in the opening paragraph, the present work is motivated by a calculus question. Thus, the author decided to keep the mathematical prerequisites of the paper relatively low. The statements of the theorems should be accessible to a good undergraduate student familiar with calculus and elementary number theory, and the statements of the corollaries require only slightly higher mathematical sophistication. Some high-powered mathematics is involved in the proofs of the results on Diophantine approximation used to derive the corollaries from the main theorems, but the proofs of the theorems themselves do not require advanced techniques. It is, perhaps, possible to go further by using more sophisticated harmonic analysis, but the author deliberately avoided going down that road. For example, one may be tempted to generalize the proof of Theorem 6 as to turn Theorem 2 into a necessary and sufficient condition similar to Theorem 6. However, such an attempt leads to an analysis of the asymptotic behavior as $\xi \downarrow 0$ of the Fourier transform $\hat{f}(\xi)$ of a function $f \in \mathfrak{F}$. While the author is quite interested to see what the proper such version of Theorem 2 looks like, he decided that this paper is not the place for it.

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2. Preliminaries

We start by introducing some notation. Throughout the remainder of the paper, we write $e(z) = e^{2\pi iz}$ and $\|x\|$ for the distance from the real number x to the nearest integer. We also use Landau's big-O notation and Vinogradov's $\ll$-notation: if $B > 0$, we write $A = O(B)$ or $A \ll B$, if there exists a constant $c > 0$ such that $|A| \leq cB$. We remark that in our proofs, the real numbers α, θ and the function $f \in \mathfrak{F}$ are considered fixed, and so implied constants are allowed to depend on those; dependence on other parameters will be stated explicitly when it occurs.

We need to recall some basic facts about continued fractions. For definitions, proofs and further details, see Khinchin's book [3]. Let α be an irrational real number. Then α has a unique representation as an (infinite) continued fraction

$$\alpha = [a_0; a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}.$$

Let p_n/q_n, where $\gcd(p_n, q_n) = 1$, denote the nth convergent to the continued fraction of α,

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_{n-1}}}}.$$

Then, by a variant of Khinchin [3, Theorem 1], the sequences $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ can be defined recursively by the conditions

$$p_0 = 1, \quad q_0 = 0, \quad p_1 = a_0, \quad q_1 = 1,$$
$$p_{n+1} = a_n p_n + p_{n-1}, \quad q_{n+1} = a_n q_n + q_{n-1} \qquad (n \geq 1).$$

It then follows (see Khinchin [3, Theorems 9 and 13]) that

$$\frac{1}{2q_n q_{n+1}} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}. \tag{6}$$

Furthermore, the denominators q_n grow exponentially and satisfy

$$q_n \geq 2^{n/2-1} \qquad (n \geq 1). \tag{7}$$

Next, we state some estimates that will be needed in the sequel. Our first lemma is well-known to number theorists; see Montgomery [6, pp. 39–40], for example.

Lemma 1. *Suppose that N is a positive integer and $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. Then*

$$\left| \sum_{n=0}^{N-1} e(\alpha n) \right| \leq \min\left(N, \frac{1}{2\|\alpha\|} \right). \tag{8}$$

Lemma 2. *Suppose that $1 \leq \nu < \mu \leq \infty$ and $\theta > 0$. Then*

$$\int_{\nu}^{\mu} t^{-\theta} \cos t \, dt \ll \nu^{-\theta}. \tag{9}$$

Proof. Partial integration gives

$$\int_{\nu}^{\mu} t^{-\theta} \cos t \, dt = t^{-\theta} \sin t \Big|_{\nu}^{\mu} + \theta \int_{\nu}^{\mu} \frac{\sin t}{t^{\theta+1}} \, dt,$$

and (9) follows. □

Lemma 3. *Suppose that $0 < \theta < 1$ and $0 < \nu < 1 < \mu$. Then*

$$\int_{\nu}^{\mu} t^{-\theta} \cos t \, dt = A_\theta + O\big(\mu^{-\theta} + \nu^{1-\theta}\big), \tag{10}$$

where

$$A_\theta = \int_0^{\infty} t^{-\theta} \cos t \, dt = \Gamma(1-\theta) \sin(\pi\theta/2) > \frac{\theta}{1-\theta}. \tag{11}$$

Proof. Inequality (10) follows from Lemma 2 and the bound

$$\left| \int_0^{\nu} t^{-\theta} \cos t \, dt \right| \leq \int_0^{\nu} t^{-\theta} \, dt \ll \nu^{1-\theta}.$$

The closed-form expression for A_θ is a standard Fourier cosine-transform formula. Its most natural proof (which uses the theory of contour integration) can be found in the solution of Problem III.151 in Pólya and Szegö [7, p. 331]. To justify the final inequality in (11), we use that $\sin(\pi\theta/2) > \theta$ (by the concavity of the sine function) and

$$(1-\theta)\Gamma(1-\theta) = \Gamma(2-\theta) \geq \Gamma(1) = 1.$$ □

3. Proof of Theorem 1

We derive the theorem from Cauchy's criterion. Consider the sum

$$S(\alpha; M, N) = \sum_{n=N+1}^{N+M} (-1)^n f(n) |\sin(n\pi\alpha)|, \tag{12}$$

where M, N are positive integers. Splitting $S(p/q; M, N)$ according to the residue class of n modulo q, we have

$$\begin{aligned} S(p/q; M, N) &= \sum_{h=1}^{q} \sum_{\substack{n=N+1 \\ n\equiv h \ (\mathrm{mod}\ q)}}^{N+M} (-1)^n f(n) |\sin(\pi p n/q)| \\ &= \sum_{h=1}^{q} |\sin(\pi p h/q)| \sum_{\substack{n=N+1 \\ n\equiv h \ (\mathrm{mod}\ q)}}^{N+M} (-1)^n f(n). \end{aligned} \tag{13}$$

We now consider separately the cases of odd and even q.

Case 1: q odd. Then the sum over n in (13) is alternating, and hence, bounded by its first term, which is at most $f(N)$. Thus,

$$|S(p/q; M, N)| \leq q f(N).$$

This establishes the convergence case of the theorem.

Case 2: q even. Then p is odd and we have $(-1)^n = (-1)^h = (-1)^{ph}$ in the sum over n in (13). Hence,

$$S(p/q; M, N) = \sum_{h=1}^{q} (-1)^{ph} |\sin(\pi p h/q)| \sum_{\substack{n=N+1 \\ n\equiv h \ (\mathrm{mod}\ q)}}^{N+M} f(n).$$

We apply Euler's summation formula to the sum over n and deduce that

$$S(p/q; M, N) = \frac{I_f}{q} \sum_{h=1}^{q} (-1)^{ph} |\sin(\pi p h/q)| + O\left(q f(N)\right), \tag{14}$$

where

$$I_f = I_f(M, N) = \int_N^{N+M} f(x)\, dx.$$

Since $\gcd(p, q) = 1$, the sets $\{p, 2p, \ldots, qp\}$ and $\{1, 2, \ldots, q\}$ are equal modulo q. Thus,

$$\sum_{h=1}^{q} (-1)^{ph} |\sin(\pi p h/q)| = \sum_{h=1}^{q} (-1)^h \sin(\pi h/q) = \sum_{h=1}^{q} \sin(\pi h(1 + 1/q)).$$

We evaluate the last sum using the well-known formula

$$\sum_{h=1}^{q} \sin(hx) = \frac{\sin((q+1)x/2) \sin(qx/2)}{\sin(x/2)},$$

and (after some simplification) we obtain

$$\sum_{h=1}^{q} (-1)^{ph} |\sin(\pi p h/q)| = -\tan\left(\pi/2q\right). \tag{15}$$

Substituting (15) into the right side of (14), we conclude that

$$S(p/q; M, N) = C_q I_f(M, N) + O\left(qf(N)\right),$$

where $C_q = -q^{-1}\tan(\pi/2q) < 0$. Since $\int_N^\infty f(x)\,dx$ diverges, this completes the proof of the divergence case.

4. Proof of Theorem 2

To prove Theorem 2, we again estimate the sum $S(\alpha; M, N)$ defined by (12). It is convenient to assume that N is even—as we may, since

$$S(\alpha; M, N) = S(\alpha; M, N+1) + O\left(f(N)\right).$$

We start by expanding the function $|\sin(n\pi\alpha)|$ in a Fourier series. It is known and easily verified that the Fourier expansion

$$|\sin(x/2)| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(kx)}{4k^2-1}$$

is valid for $|x| \leq \pi$, and consequently by periodicity for all real x. It follows that

$$S(\alpha; M, N) = \frac{2}{\pi}\sum_{k=-\infty}^{\infty}\frac{-1}{4k^2-1}\sum_{n=N+1}^{N+M}(-1)^n f(n)e(\alpha kn).$$

The contribution from $k = 0$ is

$$\frac{2}{\pi}\sum_{n=N+1}^{N+M}(-1)^n f(n) \ll f(N).$$

Thus, combining the terms with $k = \pm m$, $m \geq 1$, we obtain

$$S(\alpha; M, N) = \frac{4}{\pi}\,\mathrm{Re}\left\{\sum_{k=1}^{\infty}\frac{-1}{4k^2-1}\sum_{n=N+1}^{N+M}(-1)^n f(n)e(\alpha kn)\right\} + O\left(f(N)\right). \tag{16}$$

We estimate trivially the contribution to the right side of (16) from terms with $k > M$, and we get

$$S(\alpha; M, N) = \frac{4}{\pi}\,\mathrm{Re}\left\{\sum_{k=1}^{M}\frac{-e(\alpha kN)}{4k^2-1}\sum_{n=1}^{M}(-1)^n g(n)e(\alpha kn)\right\} + O\left(f(N)\right), \tag{17}$$

where $g(x) = f(N+x)$. By partial summation,

$$\sum_{n=1}^{M}(-1)^n g(n)e(\beta n) = g(M)U(\beta; M) - \sum_{m=1}^{M-1}\Delta g(m)U(\beta; m), \tag{18}$$

where $\Delta g(m) = g(m+1) - g(m)$ and

$$U(\beta; m) = \sum_{n=1}^{m}(-1)^n e(\beta n) = \sum_{n=1}^{m} e((\beta + 1/2)n).$$

Substituting (18) into the right side of (17), we obtain

$$S(\alpha; M, N) = \frac{4}{\pi}\,\mathrm{Re}\left\{g(M)V(\alpha; M) - \sum_{m=1}^{M-1}\Delta g(m)V(\alpha; m)\right\} + O(f(N)), \tag{19}$$

where

$$V(\alpha;m)=\sum_{k=1}^{M}\frac{-e(\alpha kN)}{4k^2-1}\,U(k\alpha;m).$$

In order to estimate the right side of (19), we break the sum $V(\alpha;m)$ into blocks depending on the denominators of the rational approximations to α. Let $\{p_n/q_n\}_{n=0}^{\infty}$ be the sequence of convergents to the continued fraction of α. We decompose $V(\alpha;m)$ into blocks $V_j(\alpha;m)$ defined by

$$V_j(\alpha;m)=\sum_{k\in\mathcal{K}_j(M)}\frac{-e(\alpha kN)}{4k^2-1}\,U(k\alpha;m), \tag{20}$$

where $\mathcal{K}_j(M)$ is the set of positive integers $k\le M$ such that $q_j<4k\le q_{j+1}$. We obtain different estimates for $V_j(\alpha;m)$, depending on the size and parity of q_j.

4.1. **Estimation of $V_j(\alpha;m)$ for small j.** When j is bounded above by an absolute constant, we appeal to (8) and get

$$|V_j(\alpha;m)|\le\sum_{q_j<4k\le q_{j+1}}\frac{\|k\alpha+1/2\|^{-1}}{8k^2-2}=K_j(\alpha),\quad\text{say.} \tag{21}$$

4.2. **Estimation of $V_j(\alpha;m)$ for odd q_j.** Suppose that q_j is odd and sufficiently large, and write $p=p_j$, $q=q_j$, $r=q_{j+1}$, and $\beta=\alpha-p/q$. From (6), when $4k\le r$,

$$k|\beta|<\frac{k}{qr}\le\frac{1}{4q}.$$

Since q is odd, we have $2q\nmid(2pk+q)$ and

$$\delta_{p,q}(k)=\left\|\frac{2pk+q}{2q}\right\|\ge\frac{1}{2q}.$$

Hence,

$$\left\|k\alpha+\frac12\right\|\ge\delta_{p,q}(k)-k|\beta|\ge\delta_{p,q}(k)-\frac{1}{4q}\ge\frac{\delta_{p,q}(k)}{2}.$$

Using (21), we obtain

$$\begin{aligned}|V_j(\alpha;m)|&\le\sum_{q<4k\le r}\frac{\delta_{p,q}(k)^{-1}}{4k^2-1}=\sum_{\substack{1\le h\le 2q\\ 2\nmid h}}\ \sum_{\substack{q<4k\le r\\ 2pk+q\equiv h \ (\mathrm{mod}\ 2q)}}\frac{\delta_{p,q}(k)^{-1}}{4k^2-1}\\ &=\sum_{\substack{1\le h\le 2q\\ 2\nmid h}}\left\|\frac{h}{2q}\right\|^{-1}\sum_{\substack{q<4k\le r\\ 2pk\equiv h \ (\mathrm{mod}\ q)}}\frac{1}{4k^2-1}.\end{aligned} \tag{22}$$

Upon noting that the sum over k on the right side of (22) is $O(q^{-2})$, we deduce from (22) that

$$\begin{aligned}|V_j(\alpha;m)|&\ll q^{-2}\sum_{h=1}^{q}\left\|\frac{2h-1}{2q}\right\|^{-1}\\ &\ll q^{-1}\sum_{h=1}^{(q+1)/2}\frac{1}{2h-1}\ll q_j^{-1}\ln q_j.\end{aligned} \tag{23}$$

4.3. **Estimation of $V_j(\alpha;m)$ for even q_j.** Suppose that q_j is even, and let p,q,r, and β be as in §4.2. Except when $2k \equiv q \pmod{2q}$, we can argue similarly to §4.2. Hence, when q is even, we get

$$V_j(\alpha;m) = V_j'(\alpha;m) + O\left(q^{-1}\ln q\right), \tag{24}$$

where

$$V_j'(\alpha;m) = \sum_{\substack{k\in\mathcal{K}_j(M)\\ 2k\equiv q \pmod{2q}}} \frac{-e(\alpha kN)}{4k^2-1}\,U(k\alpha;m). \tag{25}$$

When $2k \equiv q \pmod{2q}$, we get

$$U(k\alpha;m) = \sum_{n=1}^{m} e(\beta kn),$$

so (8) gives

$$|U(k\alpha;m)| \le \min\left(m,(2k|\beta|)^{-1}\right) \ll \min(m,r).$$

Therefore, by (25),

$$\left|V_j'(\alpha;m)\right| \ll \sum_{\substack{q<2l\le r\\ l\equiv q \pmod{2q}}} \frac{\min(m,r)}{l^2-1} \ll q^{-2}\min(m,r). \tag{26}$$

Combining (24) and (26), we conclude that

$$|V_j(\alpha;m)| \ll q_j^{-2}\min(m,q_{j+1}) + q_j^{-1}\ln q_j. \tag{27}$$

Moreover, we note that the first term on the right side of (27) is superfluous when $2q_j > q_{j+1}$, since in that case the sum $V_j'(\alpha;m)$ is empty.

4.4. **Completion of the proof.** Let $j_0 \ge 2$ be an integer to be chosen later, and set

$$K = \sum_{j=0}^{j_0-1} K_j(\alpha) = \sum_{4k\le q_{j_0}} \frac{\|k\alpha+1/2\|^{-1}}{8k^2-2}.$$

We use (21) to estimate the contribution to $V(\alpha;m)$ from subsums $V_j(\alpha;m)$ with $j<j_0$, and we use (23) and (27) to estimate the contribution from sums $V_j(\alpha;m)$ with $j\ge j_0$. Let $\mathcal{I}_\alpha(M)$ denote the set of indices $j\ge j_0$ such that q_j is even and satisfies the inequalities $q_j\le M$ and $2q_j\le q_{j+1}$. We obtain

$$|V(\alpha;m)| \le K + c_1\sum_{j\ge j_0}\frac{\ln q_j}{q_j} + c_1\sum_{j\in\mathcal{I}_\alpha(M)} q_j^{-2}\min(m,q_{j+1}),$$

where $c_1>0$ is an absolute constant. Recalling (7), we deduce

$$|V(\alpha;m)| \le K + c_2 + 2c_1\sum_{j\in\mathcal{I}_\alpha(M)} q_j^{-2}\min(m,r_j), \tag{28}$$

where $c_2>0$ is another absolute constant and $r_j = \lceil q_{j+1}/2\rceil$. Using (28) to bound the right side of (19), we get

$$|S(\alpha;M,N)| \le 3c_1\sum_{j\in\mathcal{I}_\alpha(M)} q_j^{-2}\Sigma_j + O_{j_0}(f(N)), \tag{29}$$

where the O-implied constant depends on j_0 and

$$\Sigma_j = g(M)\min(M,r_j) - \sum_{m=1}^{M-1}\Delta g(m)\min(m,r_j).$$

Using partial summation and the monotonicity of f, we find that

$$\Sigma_j \le \sum_{1\le n\le r_j} g(n) \le \int_0^{r_j} g(x)\,dx \le \int_1^{q_{j+1}} f(x)\,dx.$$

Hence, (29) yields

$$|S(\alpha;M,N)| \le \sum_{j\in\mathcal{I}_\alpha(M)} \frac{3c_1}{q_j^2}\int_1^{q_{j+1}} f(x)\,dx + O_{j_0}(f(N)). \tag{30}$$

Finally, let us fix an $\varepsilon > 0$. Since the series (3) converges, we can find an index $j_0 = j_0(\varepsilon)$ such that

$$\sum_{\substack{j=j_0\\ q_j \text{ even}}}^{\infty} \frac{1}{q_j^2}\int_1^{q_{j+1}} f(x)\,dx < \frac{\varepsilon}{6c_1}.$$

We choose j_0 above to be such an integer and fix it. Then

$$\sum_{j\in\mathcal{I}_\alpha(M)} \frac{1}{q_j^2}\int_1^{q_{j+1}} f(x)\,dx \le \sum_{\substack{j=j_0\\ q_j \text{ even}}}^{\infty} \frac{1}{q_j^2}\int_1^{q_{j+1}} f(x)\,dx < \frac{\varepsilon}{6c_1},$$

and (30) yields

$$|S(\alpha;M,N)| < \frac{\varepsilon}{2} + O_\varepsilon(f(N)),$$

where the O-implied constant depends on ε. Therefore, we can find an integer $N_0 = N_0(\varepsilon,\alpha,f)$ such that when $N \ge N_0$, one has

$$|S(\alpha;M,N)| < \varepsilon.$$

This establishes the convergence of the series (2).

5. Proof of Theorem 6

We assume that the series (5) diverges and consider the sum $S(\alpha;M,N)$ one last time. We will use (19) to show that $S(\alpha;M,N)$ can approach ∞ as $M,N\to\infty$. We retain the notation introduced in the proof of Theorem 2 and proceed with the estimation of $S(\alpha;M,N)$.

Let $j_0 \ge 2$ be a fixed integer chosen so that q_{j_0} is sufficiently large, and let N be a large even integer. We restrict the choice of N to integers of the form $q_j - b$, with $j > j_0$ and $b\in\{1,2\}$. Using (19), (21), (23), (24), and (27), we obtain the following version of (29):

$$S(\alpha;M,N) = \sum_{j\in\mathcal{I}_\alpha(M)} S_j(\alpha;M,N) + O\big(N^{-\theta}\big), \tag{31}$$

where $\mathcal{I}_\alpha(M)$ is the set of indices defined in §4.4,

$$S_j(\alpha;M,N) = \frac{4}{\pi}\,\mathrm{Re}\left\{ g(M)V_j'(\alpha;M) - \sum_{m=1}^{M-1} \Delta g(m) V_j'(\alpha;m)\right\},$$

and $V_j'(\alpha;m)$ is the sum defined by (25). Furthermore, by (26) and the choice of N, for indices j with $q_j \le N$, we have

$$\big|V_j''(\alpha;M,N)\big| \ll q_j^{-2}q_{j+1} \ll q_j^{-2}N,$$

whence

$$|S_j(\alpha;M,N)| \ll q_j^{-2}N.$$

Thus, from (31),

$$S(\alpha;M,N) = \sum_{j\in\mathcal{I}'_\alpha(M,N)} S_j(\alpha;M,N) + O(N), \tag{32}$$

where $\mathcal{I}'_\alpha(M,N)$ is the set of indices $j \in \mathcal{I}_\alpha(M)$ such that $q_j > N$.

We now proceed to obtain an approximation for $V'_j(\alpha;m)$, which we will then use to estimate the right side of (32). Let p, q, r and β be as in §4.3. When $2k \equiv q \pmod{2q}$, we have

$$U(k\alpha;m) = \sum_{n=1}^{m} e(\beta kn) = \frac{e(km\beta)-1}{1-e(k\beta)} + O(1).$$

Using the Taylor expansion $e(z) = 1 + 2\pi iz + O(|z|^2)$, we deduce that when $k \le r$,

$$U(k\alpha;m) = \frac{e(km\beta)-1}{-2\pi ik\beta} + O(1).$$

We substitute this approximation in (25) and obtain

$$\begin{aligned} V'_j(\alpha;m) &= \sum_{k\in\mathcal{L}_j(M)} \frac{e(kN\beta/2)}{k^2-1}\,\frac{e(km\beta/2)-1}{\pi ik\beta} + O\left(q^{-2}\right) \\ &= \sum_{k\in\mathcal{L}_j(M)} \frac{1}{k^2-1}\int_N^{N+m} e(kt\beta/2)\,dt + O\left(q^{-2}\right), \end{aligned}$$

where $\mathcal{L}_j(M)$ denotes the set of even integers k such that $\frac{1}{2}k \in \mathcal{K}_j(M)$ and $k \equiv q \pmod{2q}$. Hence,

$$S_j(\alpha;M,N) = \frac{4}{\pi}\sum_{k\in\mathcal{L}_j(M)} \frac{\operatorname{Re}\Xi_k(\alpha;M,N)}{k^2-1},$$

where

$$\Xi_k(\alpha;M,N) = g(M)\int_N^{N+M} e(kt\beta/2)\,dt - \sum_{m=1}^{M-1}\Delta g(m)\int_N^{N+m} e(kt\beta/2)\,dt.$$

Recall that here $g(x) = (N+x)^{-\theta}$. Using partial summation, we find that

$$\begin{aligned} \Xi_k(\alpha;M,N) &= \int_N^{N+M} g(\lceil t\rceil - N)e(kt\beta/2)\,dt \\ &= \int_N^{N+M} t^{-\theta}e(kt\beta/2)\,dt + O\big(N^{-\theta}\big). \end{aligned}$$

Hence,

$$\begin{aligned} S_j(\alpha;M,N) &= \frac{4}{\pi}\sum_{k\in\mathcal{L}_j(M)} \frac{1}{k^2-1}\int_N^{N+M} t^{-\theta}\cos(\pi kt\beta)\,dt + O\big(q^{-2}N^{-\theta}\big) \\ &= \frac{4}{\pi}\sum_{k\in\mathcal{L}_j(M)} \frac{(\pi k|\beta|)^{\theta-1}}{k^2-1}\int_{\nu_k}^{\nu_k+\mu_k} t^{-\theta}\cos t\,dt + O\big(q^{-2}N^{-\theta}\big), \end{aligned} \tag{33}$$

where $\nu_k = \pi k|\beta|N$ and $\mu_k = \pi k|\beta|M$. Summing over j, we deduce from (32) and (33) that

$$S(\alpha;M,N) = \frac{4}{\pi}\sum_{j\in\mathcal{I}'_\alpha(M,N)}\sum_{k\in\mathcal{L}_j(M)} \frac{(\pi k|\beta_j|)^{\theta-1}}{k^2-1}\int_{\nu_k}^{\nu_k+\mu_k} t^{-\theta}\cos t\,dt + O\left(N\right), \tag{34}$$

where $|\beta_j| = q_j^{-1}\|q_j\alpha\|$.

In order to estimate the right side of (34), we will impose some restrictions on the choice of M. Let $\mathcal{Q}_\alpha$ be the set of even members of $\{q_n\}_{n=1}^\infty$ such that

$$\int_1^{q_{n+1}} x^{-\theta}\,dx > q_n^{1-\theta/2}.$$

We remark that the contribution to the series (5) from terms with $q_n \notin \mathcal{Q}_\alpha$ is dominated by the convergent series $\sum_q q^{-1-\theta/2}$. Thus, the divergence of (5) implies the divergence of the series

$$\sum_{q_n \in \mathcal{Q}_\alpha} \frac{1}{q_n^2} \int_1^{q_{n+1}} x^{-\theta}\, dx. \tag{35}$$

In particular, the set $\mathcal{Q}_\alpha$ is infinite. We restrict M to the sequence of numbers of the form $\lceil q^{1+\theta/3} \rceil$, with $q \in \mathcal{Q}_\alpha$.

Let $J = J(M,N)$ denote the largest index in the set $\mathcal{I}'_\alpha(M,N)$. Using our restriction on the choice of M and the bound

$$\left| \int_{\nu_k}^{\nu_k+\mu_k} t^{-\theta} \cos t\, dt \right| \ll \begin{cases} \mu_k^{1-\theta} & \text{if } \theta < 1, \\ \ln M & \text{if } \theta = 1, \end{cases}$$

we find that the term with $j = J$ in (34) is

$$\ll \sum_{\substack{q \le 2k \le M/2 \\ k \equiv q \pmod{2q}}} \frac{q^{1-\theta/2}}{k^2 - 1} \ll q^{-1},$$

where $q = q_J$ and the implied constant depends on θ. Hence,

$$S(\alpha; M, N) = \frac{4}{\pi} \sum_{j \in \mathcal{I}''_\alpha(M,N)} \sum_{k \in \mathcal{L}_j(M)} \frac{(\pi k |\beta_j|)^{\theta - 1}}{k^2 - 1} \int_{\nu_k}^{\nu_k+\mu_k} t^{-\theta} \cos t\, dt + O(N), \tag{36}$$

where

$$\mathcal{I}''_\alpha(M,N) = \mathcal{I}'_\alpha(M,N) \setminus \{J\}.$$

Since the integrals on the right side of (36) behave somewhat differently when $\theta = 1$ and when $0 < \theta < 1$, we now consider these two cases separately.

5.1. **The case $\theta = 1$.** When $j \in \mathcal{I}''_\alpha(M,N)$ and $k \in \mathcal{L}_j(M)$, we have $\mu_k = \pi k |\beta_j| M > 1$. Hence, by Lemma 2,

$$\begin{aligned} \int_{\nu_k}^{\nu_k+\mu_k} t^{-1} \cos t\, dt &= \int_{\nu_k}^{1} t^{-1} \cos t\, dt + O(1) \\ &= \int_{\nu_k}^{1} t^{-1} \left(1 + O\left(t^2\right)\right) dt + O(1) \\ &= -\ln \nu_k + O(1) = \ln q_{j+1} + O(\ln(kN)). \end{aligned}$$

From this inequality and (36), we obtain

$$\begin{aligned} S(\alpha; M, N) &= \frac{4}{\pi} \sum_{j \in \mathcal{I}''_\alpha(M,N)} \sum_{k \in \mathcal{L}_j(M)} \frac{\ln q_{j+1}}{k^2 - 1} + O(N) \\ &\ge \frac{4}{\pi} \sum_{j \in \mathcal{I}''_\alpha(M,N)} \frac{\ln q_{j+1}}{q_j^2 - 1} + O(N) \\ &\ge \sum_{q_n \in \mathcal{Q}'_\alpha(M,N)} \frac{1}{q_n^2} \int_1^{q_{n+1}} \frac{dt}{t} + O(N), \end{aligned} \tag{37}$$

where $\mathcal{Q}'_\alpha(M,N)$ is the set of those $q_n \in \mathcal{Q}'_\alpha$ for which $q_n > N$ and $q_{n+1} \le M$. In view of the divergence of the series (35), this establishes that

$$\limsup_{M \to \infty} S(\alpha; M, N) = \infty.$$

5.2. **The case** $0 < \theta < 1$. When $j \in \mathcal{I}''_\alpha(M,N)$ and $k \in \mathcal{L}_j(M)$, by Lemma 3,

$$\int_{\nu_k}^{\nu_k+\mu_k} t^{-\theta} \cos t \, dt = A_\theta + O\big(\nu_k^{1-\theta} + \mu_k^{-\theta}\big),$$

where A_θ is the Fourier integral (11). From this inequality and (36), we obtain

$$\begin{aligned} S(\alpha; M, N) &= \frac{4A_\theta}{\pi} \sum_{j \in \mathcal{I}''_\alpha(M,N)} \sum_{k \in \mathcal{L}_j(M)} \frac{(\pi k|\beta_j|)^{\theta-1}}{k^2-1} + O\left(N + \Delta\right) \\ &\geq \frac{4A_\theta}{\pi^{2-\theta}} \sum_{j \in \mathcal{I}''_\alpha(M,N)} \frac{q_{j+1}^{1-\theta}}{q_j^2 - 1} + O\left(N + \Delta\right), \end{aligned}$$

where

$$\Delta = M^{-\theta} \sum_{j \in \mathcal{I}''_\alpha(M,N)} \sum_{k \in \mathcal{L}_j(M)} \frac{(k|\beta_j|)^{-1}}{k^2-1}.$$

By (6) and the restriction on M,

$$\Delta \ll M^{-\theta} \sum_{j \in \mathcal{I}''_\alpha(M,N)} \frac{q_{j+1}}{q_j^2 - 1} \ll M^{-\gamma} \sum_{j \in \mathcal{I}''_\alpha(M,N)} \frac{q_{j+1}^{1-\theta}}{q_j^2-1},$$

where $\gamma = \frac{1}{4}\theta^2 > 0$. Thus, for sufficiently large values of M, we obtain

$$\begin{aligned} S(\alpha; M, N) &\geq \frac{2A_\theta}{\pi^{2-\theta}} \sum_{j \in \mathcal{I}''_\alpha(M,N)} \frac{q_{j+1}^{1-\theta}}{q_j^2 - 1} + O\left(N\right), \\ &\geq A'_\theta \sum_{q_n \in \mathcal{Q}'_\alpha(M,N)} \frac{1}{q_n^2} \int_1^{q_{n+1}} t^{-\theta} \, dt + O\left(N\right), \end{aligned} \tag{38}$$

where $A'_\theta = 2\pi^{\theta-2}(1-\theta)A_\theta > 0$ and $\mathcal{Q}'_\alpha(M,N)$ is defined as in §5.1. Therefore, once again, using (38) and the divergence of the series (35), we conclude that

$$\limsup_{M \to \infty} S(\alpha; M, N) = \infty.$$

This completes the proof of the theorem.

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