

# ADDITIVE BASES ARISING FROM FUNCTIONS IN A HARDY FIELD

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ABSTRACT. A classical additive basis question is Waring’s problem. It has been extended to integer polynomial and non-integer power sequences. In this paper, we will consider a wider class of functions, namely functions from a Hardy field, and show that they are asymptotic bases.

## 1. INTRODUCTION

Let  $k$  be a positive integer. Waring’s problem asks whether the sequence  $1^k, 2^k, \dots$  of  $k$ th powers is an asymptotic basis. In other words, whether there are positive integers  $s$  and  $N_0$  such that every integer  $N$  greater than  $N_0$  can be written in the form

$$n_1^k + n_2^k + \dots + n_s^k = N, \quad (1.1)$$

with  $n_1, n_2, \dots, n_s \in \mathbb{N}$ . After Hilbert’s affirmative solution to this problem, several generalizations and extensions were formulated, and important methods—such as the circle and sieve methods, were developed to tackle those problems.

One variant of Waring’s problem replaces the sequence of  $k$ th powers by the range of an integer polynomial. In other words, we want to represent an integer  $N$  in the form

$$f(n_1) + f(n_2) + \dots + f(n_s) = N,$$

where  $f(x) \in \mathbb{Z}[x]$ . As a natural generalization of the classical Waring problem, this question has been studied extensively via the circle method (see Ford [6] for the history of the problem and the best results to date). In particular, it is known that the sequence  $f(n)$ ,  $n = 1, 2, \dots$ , is an asymptotic basis whenever there is no integer  $d \geq 2$  such that  $d \mid (f(n) - f(1))$  for all  $n \in \mathbb{N}$ .

Another generalization of Waring’s problem is whether *non-integer* powers form a basis? It was Segal [10] who proved that for any fixed positive real number  $c$ , the sequence  $[1^c], [2^c], \dots$  of integer parts of  $c$ th powers is an asymptotic basis. This question has been studied further by Deshouillers [4] and by Arkhipov and Zhitkov [1], among others. Whereas those authors focused on the order of the basis (i.e., the number of unknowns in an equation analogous to (1.1) that ensure solvability), in this work we ask the general question if the sequence  $[n^c]$  can be replaced by other sequences. What other sequences do we have in mind? Well, for one, take a polynomial  $p(x)$  with an irrational coefficient of a non-constant term. Is then  $[p(n)]$ ,  $n = 1, 2, \dots$  a basis? How about more general sequences such as

$$\left[ \frac{n^2}{\log n} \right], \quad \text{or} \quad \left[ \pi n^3 + \frac{n^{\sqrt{2}}}{\log \log n} \right]. \quad (1.2)$$

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In other words, we are interested in sequences of the form  $[f(1)], [f(2)], \dots$  when  $f$  is a smooth positive function such that  $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . An obvious necessary condition is that  $f$  grows no faster than a polynomial:

$$f(x) \ll (|x| + 1)^k \quad \text{for some } k \in \mathbb{N}. \quad (1.3)$$

For the rest of this note, unless we say otherwise, all functions are assumed to satisfy this growth condition.

As we mentioned, the case of rational polynomials has been extensively studied, so in this introduction we consider functions  $f(x)$  which are far away from rational polynomials in the sense that

$$|f(x) - g(x)| \rightarrow \infty \text{ for every } g(x) \in \mathbb{Q}[x]. \quad (1.4)$$

Our aim is a general theorem which will include Segal's theorem as a special case, but it also includes the case of irrational polynomials and is easily and readily applicable to somewhat wild sequences such as those in (1.2). Besides the growth condition (1.3), we need some regularity condition on  $f$ . Instead of trying to describe various growth conditions on the derivatives of the function  $f$ , we seek conditions which are easily applicable to a wide class of functions, including those appearing in (1.2). As an illustration of the type of functions we want to consider, take the so called logarithmico exponential functions of Hardy. These are functions we obtain by writing down a "formula" using the symbols  $\log, \exp, +, \cdot, x, c$  where  $x$  is a real variable and  $c$  is a real constant. This class of functions certainly includes any power function since  $x^c = \exp(c \log x)$ . It similarly includes any polynomials. Now our theorem will imply that if  $f(x)$  is a logarithmico exponential function that satisfies (1.3) and (1.4) then the sequence  $[f(1)], [f(2)], \dots$  forms a basis. We can see immediately that the sequences in (1.2) form a basis. A small technical remark is that the function  $f$  may be defined only for large enough  $x$  (for example, consider  $x^2 \log \log x$ ). In that case, we agree that we consider, instead,  $f(x+k)$  with a suitably large, fixed  $k$ .

While the logarithmico exponential functions already represent a large class of functions, sequences involving the logarithmic integral  $\text{li } x$  or  $\Gamma$ -functions are still not covered; we want to admit functions such as

$$(\text{li } x)^2 \text{ or } (\log \Gamma(x))^{\sqrt{2}}, \quad (1.5)$$

and even products, quotients, sums or differences of these functions.

The functions we want to consider are those from a Hardy field. For a more extensive introduction to Hardy fields and for further references about the facts we claim below, see Boshernitzan [2]. To define Hardy fields, we consider first the ring of continuous functions with pointwise addition and multiplication as the ring operations. Since functions such as  $\log x$ ,  $\log \log x$ , and  $\log \log \log x$  are defined only on some neighborhood of  $\infty$ , we want to consider *germs* of functions, that is, equivalence classes of functions, where we consider two functions equivalent if they are equal in a neighborhood of  $\infty$ . Let  $B$  the ring of all these germs of continuous functions. A *Hardy field* is a subring of  $B$  which is a field, and which is also closed with respect to differentiation. It is known that  $E$ , the intersection of all maximal Hardy fields, contains all logarithmico exponential functions and is closed under antidifferentiation; hence,  $\text{li } x \in E$ . Another known fact is that there is a Hardy field containing the  $\Gamma$ -function, and that the intersection of all maximal translation invariant Hardy fields contains  $\log \Gamma(x)$ .

We remark that by virtue of their very definition, functions in a Hardy field are defined only on a neighborhood of  $\infty$ . On the other hand, given a particular function in a Hardy field, we want to think of it as being defined on  $[1, \infty)$ . To fix this, we replace  $f(x)$  by  $f(x+k)$  for some sufficiently large constant  $k$ . With this caveat, our main result is the following theorem.

**Theorem A.** *Suppose that  $f$  is a function from a Hardy field that satisfies the growth condition (1.3) and the condition*

$$\lim_{x \rightarrow \infty} |f(x) - g(x)| = \infty \quad \text{for all } g(x) \in \mathbb{Q}[x]. \quad (1.4)$$

*Then the sequence  $[f(n)]$ ,  $n = 1, 2, \dots$ , is an asymptotic basis.*

The utility of the formulation of our theorem in terms of Hardy fields is that it is *easily* applicable. Indeed, by the preceding remarks, the sequences in (1.2) all form a basis as well as sequences such as  $[(\ln n)^2]$  or  $[(\log \Gamma(n))^{\sqrt{2}}]$ .

## 2. PRELIMINARIES

**Notation.** Most of our notation is standard. We write  $e(x) = e^{2\pi ix}$ . For a real number  $\theta$ ,  $[\theta]$ ,  $\{\theta\}$  and  $\|\theta\|$  denote, respectively, the integer part of  $\theta$ , the fractional part of  $\theta$  and the distance from  $\theta$  to the nearest integer. Also, we use Vinogradov's notation  $A \ll B$  and Landau's big-oh notation  $A = O(B)$  to indicate that  $|A| \leq KB$  for some constant  $K > 0$ .

**2.1. Hardy fields.** Let  $\mathfrak{U}$  denote the union of all Hardy fields. Suppose that  $f \in \mathfrak{U}$  satisfies condition (1.3) and that  $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . We say that a function  $f \in \mathfrak{U}$  is *non-polynomial* if for every  $k \in \mathbb{N}$ , we have  $f(x) = o(x^k)$  or  $x^k = o(f(x))$  as  $x \rightarrow \infty$ . Each function  $f \in \mathfrak{U}$  that satisfies the growth condition (1.3) falls in one of the following three classes (see [3]):

- i)  $f(x) = p(x)$ , where  $p(x) \in \mathbb{R}[x]$ ;
- ii)  $f(x) = r(x)$ , where  $r$  is non-polynomial;
- iii)  $f(x) = p(x) + r(x)$ , where  $p(x) \in \mathbb{R}[x]$  and  $r$  is non-polynomial, with  $r(x) = o(p(x))$  as  $x \rightarrow \infty$ .

Our proof of Theorem A will be given in two parts each utilizing different methods. The first method handles the case when  $f$  is far away from all polynomials, and the second handles the rest of the cases. More precisely, the first part of the proof is given in Section 3, and it handles the case when  $f$  either belongs to class ii) above or it belongs to class iii) but for some positive  $\delta$  we have  $r(x) \gg x^\delta$ . The second part of the proof is given in Section 4 it handles the rest of the functions, so when either  $f$  is a polynomial (class i)), or when  $f$  belongs to class iii) and  $r(x) \ll x^\delta$  for all positive  $\delta$ .

When  $f$  belongs to the classes i) or iii), these assumptions mean that the polynomial part of  $f$  has a positive degree  $d$ . For functions of class iii), we call  $d$  the *degree of  $f$* . When  $f$  is non-polynomial (so class ii)), then there exists a real number  $c \geq 0$  such that, for any fixed  $\varepsilon > 0$ , one has

$$x^{c-j-\varepsilon} \ll f^{(j)}(x) \ll x^{c-j+\varepsilon} \quad (j = 0, 1, 2, \dots), \quad (2.1)$$

the implied constants depending at most on  $f, j$  and  $\varepsilon$ . (See [3] for a proof.) For functions of class ii), we call the number  $c$  in (2.1) the *degree of  $f$* . Thus, we have

now defined the degree of any function  $f \in \mathfrak{U}$  subject to (1.3). We denote the degree of  $f$  by  $d_f$ . By (2.1),

$$x^{d_f-j-\varepsilon} \ll f^{(j)}(x) \ll x^{d_f-j+\varepsilon} \quad (j = 0, 1, \dots, d_f) \quad (2.2)$$

for any function  $f \in \mathfrak{U}$  satisfying the above hypotheses.

When  $f$  is of classes ii) or iii), we denote by  $c_f$  the degree of its non-polynomial part. Thus,  $c_f = d_f$  or  $c_f = d_r$  according as  $f$  is of class ii) or iii). In particular, we have

$$x^{c_f-j-\varepsilon} \ll f^{(j)}(x) \ll x^{c_f-j+\varepsilon} \quad (j > d_f). \quad (2.3)$$

We also recall the following result of Boshernitzan [2, Theorem 1.4].

**Lemma 2.1.** *Suppose that  $f \in \mathfrak{U}$  satisfies condition (1.3). Then the following two conditions are equivalent.*

- (a) *The sequence  $(\{f(n)\})_{n \in \mathbb{N}}$  is dense in  $[0, 1)$ .*
- (b) *For every polynomial  $g(x) \in \mathbb{Q}[x]$ , one has*

$$\lim_{x \rightarrow \infty} |f(x) - g(x)| = \infty.$$

## 2.2. Bounds on exponential sums.

**Lemma 2.2.** *Let  $k \geq 2$  be an integer and put  $K = 2^k$ . Suppose that  $a \leq b \leq a + N$  and that  $f : [a, b] \rightarrow \mathbb{R}$  has continuous  $k$ th derivative that satisfies the inequality  $0 < \lambda \leq |f^{(k)}(x)| \leq h\lambda$  for all  $x \in [a, b]$ . Then*

$$\sum_{a \leq n \leq b} e(f(n)) \ll hN(\lambda^{1/(K-2)} + N^{-2/K} + (N^k\lambda)^{-2/K}).$$

*Proof.* This is a variant of van der Corput [11, Satz 4]. □

**2.3. The Hilbert–Kamke problem.** Let  $N_1, \dots, N_k$  be large positive integers. The Hilbert–Kamke problem is concerned with the system of Diophantine equations

$$x_1^j + x_2^j + \dots + x_s^j = N_j \quad (1 \leq j \leq k).$$

It is known that this system has solutions in positive integers  $x_1, \dots, x_s$ , provided that:

- (a)  $s$  is sufficiently large;
- (b) there exist real numbers  $\mu_1, \dots, \mu_{k-1} > 1$  and  $\nu_1, \dots, \nu_{k-1} < 1$  such that

$$\mu_j N_k^{j/k} \leq N_j \leq \nu_j s^{1-j/k} N_k^{j/k} \quad (1 \leq j < k);$$

- (c) the  $N_j$ 's satisfy the congruences

$$\Delta_j(N_1, \dots, N_k) \equiv 0 \pmod{\Delta_0} \quad (1 \leq j \leq k),$$

where

$$\Delta_0 = \begin{vmatrix} 1 & 2 & \dots & k \\ 1^2 & 2^2 & \dots & k^2 \\ \vdots & \vdots & \dots & \vdots \\ 1^k & 2^k & \dots & k^k \end{vmatrix} = 1!2! \dots k! \quad (2.4)$$

and  $\Delta_j(N_1, \dots, N_k)$  is the determinant resulting from replacing the numbers  $j, \dots, j^k$  in the  $j$ th column of  $\Delta_0$  by  $N_1, \dots, N_k$ , respectively.

The reader can find a proof of the sufficiency of conditions (a)–(c) above in [7, §2.7], for example.

**2.4. Bounded gaps imply basis.** For a sequence  $\mathcal{A} = (a_n)_{n \in \mathbb{N}}$ , we define the *sumset*  $s\mathcal{A}$  by

$$s\mathcal{A} = \{a_1 + a_2 + \cdots + a_s \mid a_i \in \mathcal{A}\}.$$

In the proof, we shall need the following elementary result.

**Lemma 2.3.** *Let  $\mathcal{A} = (a_n)_{n \in \mathbb{N}}$  be an integer sequence such that:*

- (a) *for some  $s \in \mathbb{N}$ , the sumset  $s\mathcal{A}$  has bounded gaps;*
- (b)  $\gcd\{a_n - a_1 \mid n \in \mathbb{N}\} = 1$ .

*Then  $\mathcal{A}$  is an asymptotic basis.*

This lemma can be derived from more general results by Erdős and Graham [5, Theorem 1] or by Nash and Nathanson [9, Lemma 1]. For the sake of completeness, we present a direct proof.

*Proof.* By hypothesis (b), we have

$$\gcd(a_2 - a_1, a_3 - a_1, \dots, a_k - a_1) = 1$$

for some  $k \in \mathbb{N}$ . Thus, there exist integers  $x_2, \dots, x_k$  such that

$$\sum_{j=2}^k x_j (a_j - a_1) = 1. \quad (2.5)$$

Define the integers  $a'_j$  and  $a''_j$ ,  $2 \leq j \leq k$ , by

$$a'_j = \begin{cases} a_j & \text{if } x_j \geq 0, \\ a_1 & \text{if } x_j < 0; \end{cases} \quad a''_j = \begin{cases} a_1 & \text{if } x_j \geq 0, \\ a_j & \text{if } x_j < 0. \end{cases}$$

We can rewrite (2.5) as

$$\sum_{j=2}^k |x_j| a'_j = 1 + \sum_{j=2}^k |x_j| a''_j. \quad (2.6)$$

Let  $g$  and  $M$  be, respectively, the largest gap of  $s\mathcal{A}$  and the least element of  $s\mathcal{A}$ , and suppose that  $N$  is an integer with

$$N \geq M + g \sum_{j=1}^k |x_j| a''_j.$$

Then

$$N - g \sum_{j=1}^k |x_j| a''_j = b + h$$

for some integers  $b$  and  $h$ , with  $b \in s\mathcal{A}$  and  $0 \leq h \leq g$ . Hence, by (2.6),

$$N = b + h + g \sum_{j=1}^k |x_j| a''_j = b + h \sum_{j=1}^k |x_j| a'_j + (g - h) \sum_{j=1}^k |x_j| a''_j.$$

This establishes that every sufficiently large  $N$  is the sum of  $s + g \sum_j |x_j|$  elements of  $\mathcal{A}$ . Thus,  $\mathcal{A}$  is an asymptotic basis.  $\square$

3. PROOF OF THEOREM A IN CASE  $r(x) \gg x^\delta$ 

Let  $\delta > 0$  and consider a function  $f \in \mathfrak{U}$  with its non polynomial part  $r(x)$  satisfying  $r(x) \gg x^\delta$ . For simplicity, we write  $d = d_f$  and  $c = c_f$ . In the case when  $f$  is of class ii), we assume that  $d \geq 1$ ; otherwise, the sequence  $a_n = [f(n)]$  contains all sufficiently large integers, and the result is trivial.

**3.1. A variant of the circle method.** Let  $s \in \mathbb{N}$  and suppose that  $N \geq N_0(s, d, \delta)$ , where  $\delta$  is the positive number from the hypotheses of the theorem. We set  $X = N^{1/d}$  and  $N_s = N/(s+1)$ , and we choose  $X_0$  and  $X_1$  so that

$$f(X_0) = N_s, \quad f(X_1) = 2N_s.$$

Let  $R_s(N)$  denote the number of solutions of the equation

$$[f(n_1)] + [f(n_2)] + \cdots + [f(n_s)] = N$$

in integers  $n_1, n_2, \dots, n_s$  with  $X_0 < n_i \leq X_1$ . Then

$$R_s(N) = \int_{-1/2}^{1/2} S(\alpha)^s e(-\alpha N) d\alpha, \quad (3.1)$$

where

$$S(\alpha) = \sum_{X_0 < n \leq X_1} e(\alpha[f(n)]).$$

Put  $\omega = X^{1/2-d}$ . We will show that when  $s \geq 3$  and  $0 < \varepsilon < (6s)^{-1}$ , one has

$$\int_{-\omega}^{\omega} S(\alpha)^s e(-\alpha N) d\alpha \gg X^{s-d-2\varepsilon}; \quad (3.2)$$

$$\sup_{\omega \leq |\alpha| \leq 1/2} |S(\alpha)| \ll X^{1-\sigma+\varepsilon}, \quad (3.3)$$

where  $\sigma = \sigma(c, d) > 0$ . Clearly, the theorem follows from (3.1)–(3.3).

Suppose that  $|\alpha| \leq \omega$  and define

$$T(\alpha) = \sum_{X_0 < n \leq X_1} e(\alpha f(n)), \quad I(\alpha) = \int_{X_0}^{X_1} e(\alpha f(t)) dt.$$

We have

$$S(\alpha) = T(\alpha) + O(\omega X_1). \quad (3.4)$$

Furthermore, since

$$\sup_{X_0 \leq t \leq X_1} |\alpha f'(t)| \ll \omega X^{d-1+\varepsilon} < 1/2,$$

[8, Lemma 8.8] gives

$$T(\alpha) = I(\alpha) + O(1). \quad (3.5)$$

Let  $\Delta_1 = \omega X_1 + 1$ . Combining (3.4) and (3.5), we find that

$$\int_{-\omega}^{\omega} |S(\alpha)^s - I(\alpha)^s| d\alpha \ll \Delta_1 \int_{-\omega}^{\omega} |I(\alpha)|^{s-1} d\alpha + \omega \Delta_1^s. \quad (3.6)$$

Since

$$\inf_{X_0 \leq t \leq X_1} |\alpha f'(t)| \gg |\alpha| X^{d-1-\varepsilon},$$

we deduce from [8, Lemma 8.10] that

$$I(\alpha) \ll |\alpha|^{-1} X^{1-d+\varepsilon}.$$

From the last inequality and the trivial bound for  $I(\alpha)$ , we obtain

$$I(\alpha) \ll \frac{X^{1+\varepsilon}}{1 + X^d|\alpha|}. \quad (3.7)$$

Hence, for  $s \geq 3$ ,

$$\int_{-\omega}^{\omega} |I(\alpha)|^{s-1} d\alpha \ll \int_{-\omega}^{\omega} \frac{X^{s-1+(s-1)\varepsilon} d\alpha}{(1 + X^d|\alpha|)^{s-1}} \ll X^{s-d-1+(s-1)\varepsilon}.$$

Upon noting that

$$\Delta_1 \ll X^{3/2-d+\varepsilon} \ll X^{1/2+\varepsilon},$$

we deduce from (3.6) that

$$\int_{-\omega}^{\omega} |S(\alpha)^s - I(\alpha)^s| d\alpha \ll X^{s-d-1/3}. \quad (3.8)$$

We now evaluate

$$\int_{-\omega}^{\omega} I(\alpha)^s e(-\alpha N) d\alpha.$$

By (3.7),

$$\int_{|\alpha|>\omega} |I(\alpha)|^s d\alpha \ll \int_{\omega}^{\infty} \frac{X^{s+s\varepsilon} d\alpha}{(1 + \alpha X^d)^s} \ll X^{s-d-1/3},$$

so

$$\int_{-\omega}^{\omega} I(\alpha)^s e(-\alpha N) d\alpha = \int_{\mathbb{R}} I(\alpha)^s e(-\alpha N) d\alpha + O\left(X^{s-d-1/3}\right). \quad (3.9)$$

If  $g$  is the inverse function to  $f$  on the interval  $X_0 \leq t \leq X_1$ , then

$$I(\alpha) = \int_{N_s}^{2N_s} g'(u) e(\alpha u) du = V(\alpha), \quad \text{say.}$$

By Fourier's inversion formula, the integral on the right side of (3.9) equals

$$\int_{\mathcal{D}_s} g'(u_1) \cdots g'(u_{s-1}) g'(N - u_1 - \cdots - u_{s-1}) d\mathbf{u},$$

where  $\mathcal{D}_s$  is the  $(s-1)$ -dimensional region defined by the inequalities

$$N_s \leq u_1, \dots, u_{s-1} \leq 2N_s, \quad N_s \leq N - u_1 - \cdots - u_{s-1} \leq 2N_s.$$

Note that  $\mathcal{D}_s$  contains the  $(s-1)$ -dimensional box

$$N/(s+1) \leq u_1, \dots, u_{s-1} \leq N/s$$

and

$$\inf_{N_s \leq u \leq 2N_s} g'(u) \gg X_1^{1-d-\varepsilon/s}.$$

We deduce that

$$\int_{\mathbb{R}} I(\alpha)^s e(-\alpha N) d\alpha \gg N^{s-1} X_1^{s-sd-\varepsilon} \gg X^{s-d-2\varepsilon}. \quad (3.10)$$

Combining (3.8)–(3.10), we obtain (3.2).

We now proceed to the estimation of  $S(\alpha)$  on the two minor arcs. For non-integer reals  $x, \alpha$  and  $K \geq 2$ , we have

$$e(-\alpha\{x\}) = c(\alpha) \sum_{|k| \leq K} \frac{e(kx)}{k + \alpha} + O(\Phi(x; K) \log K), \quad (3.11)$$

where  $|c(\alpha)| \leq \|\alpha\|$  and  $\Phi(x; K) = (1 + K\|x\|)^{-1}$ . Furthermore,

$$\Phi(x; K) = \sum_{k \in \mathbb{Z}} b_k e(kx), \quad |b_k| \ll \frac{K \log K}{K^2 + |k|^2}. \quad (3.12)$$

By (3.11),

$$\begin{aligned} S(\alpha) &= \sum_{X_0 < n \leq X_1} e(\alpha f(n)) e(-\alpha \{f(n)\}) \\ &= \sum_{|k| \leq K} \frac{|c(\alpha)|}{|k + \alpha|} |T(k + \alpha)| + O(\Delta(K) \log K), \end{aligned} \quad (3.13)$$

where

$$\Delta(K) = \sum_{X_0 < n \leq X_1} \Phi(f(n); K) \ll \sum_{k \in \mathbb{Z}} |b_k| |T(k)|. \quad (3.14)$$

Combining (3.12)–(3.14), we obtain

$$\sup_{\omega \leq |\alpha| \leq 1/2} |S(\alpha)| \ll \left( \sup_{\omega \leq |\beta| \leq K^2} |T(\beta)| + X_1 K^{-1} \right) \log^2 K. \quad (3.15)$$

The estimate (3.3) follows readily from (3.15) with  $K = X^\sigma$  and the bound

$$\sup_{\omega \leq |\beta| \leq X^{2\sigma}} |T(\beta)| \ll X^{1-\sigma}, \quad (3.16)$$

which we establish in the next section.

**3.2. Estimation of  $T(\beta)$ .** We now establish (3.16). By a standard dyadic argument, we can reduce (3.16) to the estimation of the exponential sum

$$W(\beta) = \sum_{P < n \leq P_1} e(\beta f(n)), \quad (3.17)$$

where  $X_0 \leq P < P_1 \leq \min(2P, X_1)$  and  $P^{1/2-d-\varepsilon} \leq |\beta| \leq P^{2\sigma+\varepsilon}$ . We also assume, as we may, that  $0 < \sigma < \frac{1}{4}$ .

Suppose first that  $f$  is of class iii) and set  $\eta = \frac{1}{4} \min(1, c)$ . We consider two cases depending on the relative sizes of  $\beta$  and  $P$ .

**Case 1:**  $P^{1/2-d-\varepsilon} \leq |\beta| \leq P^{-c+\eta}$ . Let

$$p(x) = \alpha x^d + \cdots + \alpha_k x^k$$

be the polynomial part of  $f$ . When  $d \geq 2$ , we apply Lemma 2.2 with  $k = d$ ,  $N = P$ ,  $\lambda = |\beta|$ , and  $h = P^\varepsilon$ . We obtain

$$W(\beta) \ll P^{1+\varepsilon} (|\beta|^{1/(K-2)} + P^{-2/K} + (|\beta|P^d)^{-2/K}) \ll P^{1-\sigma_1+\varepsilon},$$

where  $K = 2^d$  and  $\sigma_1 = \min(1, c/2)K^{-1}$ . When  $d = 1$ , we have

$$|\beta|P^{-\varepsilon} \ll |\beta f'(x)| \ll |\beta|P^\varepsilon,$$

so the Kuzmin–Landau inequality (see [8, Corollary 8.11]) gives

$$W(\beta) \ll P^\varepsilon |\beta|^{-1} \ll P^{1/2+2\varepsilon}.$$



**Case 2:**  $P^{-c+\eta} \leq |\beta| \leq P^{2\sigma+\varepsilon}$ . We recall (2.3) and apply Lemma 2.2 with  $k = d+1$ ,  $N = P$ ,  $\lambda = |\beta|P^{c-k-\varepsilon/4}$ , and  $h = P^{\varepsilon/2}$ . We get

$$W(\beta) \ll P^{1+\varepsilon} ( (|\beta|P^{c-k})^{1/(J-2)} + P^{-2/J} + (|\beta|P^c)^{-2/J} ),$$

where  $J = 2^{d+1}$ . Since

$$(|\beta|P^{c-k})^{1/(J-2)} \ll P^{(2\sigma-1+\varepsilon)/(J-2)} \ll P^{-1/(2J)},$$

we deduce that

$$W(\beta) \ll P^{1+\varepsilon} (P^{-1/(2J)} + P^{-2\eta/J}).$$

Combining the above estimates, we conclude that when  $f$  is of class iii), (3.16) holds with  $\sigma = 2^{-d-2} \min(c, 1)$ . When  $f$  is of class ii), we can argue similarly to Case 2 above to show that (3.16) holds with  $\sigma = 2^{-k-1}$ , where  $k = \lceil d+1 \rceil$ .  $\square$

#### 4. PROOF OF THEOREM A WHEN $r(x) \ll x^\delta$ FOR ALL $\delta$

In this section, we assume that  $f$  is either of class i) or it is of class iii) with  $r(x) \ll x^\delta$  for all  $\delta$ . In this case, Theorem A follows from Lemma 2.3. Under the hypotheses of Theorem A, we have

$$\lim_{x \rightarrow \infty} |q^{-1}f(x) - g(x)| = \infty$$

whenever  $q \in \mathbb{N}$  and  $g(x) \in \mathbb{Q}[x]$ . Hence, by Lemma 2.1, the fractional parts  $\{q^{-1}f(n)\}$  are dense in  $[0, 1)$ . In particular, the Diophantine inequality

$$\frac{a-1}{q} \leq \left\{ \frac{f(n)}{q} \right\} < \frac{a}{q}$$

has solutions for any given integers  $a$  and  $q$ , with  $1 \leq a \leq q$ . Therefore, every arithmetic progression  $a \pmod{q}$  contains an element of  $\mathcal{A}$ . This establishes that  $\mathcal{A}$  satisfies hypothesis (b) of Lemma 2.3.

We see, that it is enough to show that the set of sums  $[f(x_1)] + \cdots + [f(x_s)]$  has bounded gaps when  $s$  is sufficiently large. It suffices to show that every large real  $N$  lies within a bounded distance from a sum  $f(x_1) + \cdots + f(x_s)$ . Suppose that  $f(x) = p(x) + r(x)$ , where

$$p(x) = \alpha_k x^k + \cdots + \alpha_1 x, \quad r(x) \ll (|x| + 1)^\delta,$$

with  $0 < \delta < 1/2$ . We define  $U$  and  $V$  in terms of  $N$  by the equations

$$N = sf(U+V) + s\alpha_k V^k, \quad V = U^{1-\delta}.$$

These are well-defined, since the function  $f(U+V) + \alpha_k V^k$  is strictly increasing for  $U$  sufficiently large. We then set  $X = [U]$  and  $Y = V + \{U\}$ , so that  $X+Y = U+V$ . Using the Taylor expansion

$$f(X+Y) = f(X) + \sum_{j=1}^k \frac{f^{(j)}(X)}{j!} Y^j + O(X^{-\delta}),$$

we obtain

$$N = sf(X) + \sum_{j=1}^{k-1} \frac{f^{(j)}(X)}{j!} (sY^j) + s\alpha_k (Y^k + V^k) + O(X^{-\delta}). \quad (4.1)$$

We are going to use the result on the Hilbert–Kamke problem to replace the terms  $sY^j$  in (4.1) by  $s$ -fold sums of  $j$ th powers. Let  $\Delta_0$  denote the determinant in (2.4). The idea is to approximate each  $sY^j$  by a suitable multiple of  $\Delta_0$  and carry

the residual error to the next step. We first find a positive integer  $M_1$  such that  $0 < sY^1 - \Delta_0 M_1 \leq \Delta_0$ . Let  $E_1 = sY - \Delta_0 M_1$  be the residual error. Then (4.1) becomes

$$\begin{aligned} N &= sf(X) + \frac{f'(X)}{1!} \Delta_0 M_1 + \frac{f'(X)}{1!} E_1 + \frac{f''(X)}{2!} (sY^2) + \dots \\ &\quad + s\alpha_k(Y^k + V^k) + O(X^{-\delta}) \\ &= sf(X) + \frac{f'(X)}{1!} \Delta_0 M_1 + \frac{f''(X)}{2!} \left( sY^2 + \frac{2f'(X)}{f''(X)} E_1 \right) + \dots \\ &\quad + s\alpha_k(Y^k + V^k) + O(X^{-\delta}). \end{aligned}$$

Next, we find a positive integer  $M_2$  such that

$$0 < sY^2 + \frac{2f'(X)}{f''(X)} E_1 - \Delta_0 M_2 \leq \Delta_0.$$

We then carry the residual error

$$E_2 = sY^2 + \frac{2f'(X)}{f''(X)} E_1 - \Delta_0 M_2$$

to next step and repeat the process. Upon setting  $E_0 = 0$ , this process yields a recursive integer sequence  $M_1, M_2, \dots, M_k$ , defined by the conditions

$$\begin{aligned} 0 < E_j &= sY^j + \frac{jf^{(j-1)}(X)}{f^{(j)}(X)} E_{j-1} - \Delta_0 M_j \leq \Delta_0 \quad (1 \leq j < k); \\ 0 < E_k &= s(Y^k + V^k) + \frac{kf^{(k-1)}(X)}{f^{(k)}(X)} E_{k-1} - \Delta_0 M_k \leq \Delta_0. \end{aligned}$$

Substituting into (4.1), we get

$$N = sf(X) + \sum_{j=1}^k \frac{f^{(j)}(X)}{j!} \Delta_0 M_j + O(1), \quad (4.2)$$

By the choices of the  $M_j$ 's, we have

$$\begin{aligned} \Delta_0 M_j &= sY^j(1 + O(XY^{-2})) \quad (1 \leq j < k), \\ \Delta_0 M_k &= 2sY^k(1 + O(XY^{-2})), \end{aligned}$$

whence

$$\Delta_0 M_j = 2^{-j/k} s^{1-j/k} (\Delta_0 M_k)^{j/k} (1 + O(XY^{-2})) \quad (1 \leq j < k).$$

Recalling that  $Y \gg X^{1-\delta}$  and  $\delta < 1/2$ , we conclude that the integers  $\Delta_0 M_1, \dots, \Delta_0 M_k$  satisfy conditions (b) and (c) in §2.3. Therefore, if  $s$  is sufficiently large, there exist positive integers  $y_1, \dots, y_s$  such that

$$\Delta_0 M_j = y_1^j + \dots + y_s^j \quad (1 \leq j \leq k).$$

Substituting these into (4.2), we obtain

$$\begin{aligned} N &= sf(X) + \sum_{j=1}^k \frac{f^{(j)}(X)}{j!} (y_1^j + \dots + y_s^j) + O(1) \\ &= f(X + y_1) + \dots + f(X + y_s) + O(1). \end{aligned}$$

Thus, as desired,  $N$  lies within a bounded distance of a sum of the form  $f(x_1) + \dots + f(x_s)$ .  $\square$

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