SUMS OF PRIMES AND SQUARES OF PRIMES IN SHORT INTERVALS

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Abstract. Let $\mathcal{H}_2$ denote the set of even integers $n \not\equiv 1 \pmod 3$. We prove that when $H \geq X^{0.33}$, almost all integers $n \in \mathcal{H}_2 \cap (X, X+H]$ can be represented as the sum of a prime and the square of a prime. We also prove a similar result for sums of three squares of primes.

1. Introduction

Additive prime number theory was ushered in by two seminal papers: I.M. Vinogradov's celebrated proof of the three primes theorem [24] and L.K. Hua's work [10]. In the latter, Hua posed several questions that have represented the central problems in the field ever since. This note is concerned with two of those questions. Let

$$\mathcal{H}_2 = \big\{ n \in \mathbb{N} \bigm| n \not\equiv 1 \pmod 3,\ 2 \mid n \big\},$$
$$\mathcal{H}_3 = \big\{ n \in \mathbb{N} \bigm| n \equiv 3 \pmod{24},\ 5 \nmid n \big\}.$$

It is conjectured that every sufficiently large $n \in \mathcal{H}_2$ can be represented as the sum of a prime and the square of another prime, and that every sufficiently large integer $n \in \mathcal{H}_3$ can be represented as the sum of three squares of primes. However, both these conjectures are still wide open. Let $E_j(X)$ denote the number of integers $n \in \mathcal{H}_j$, with $n \leq X$, which cannot be represented in the desired form. Hua [10] proved that

$$E_j(X) \ll X(\log X)^{-A} \qquad (j = 2, 3), \tag{1}$$

for some $A > 0$. Later, Schwarz [22] showed that (1) holds for any fixed $A > 0$. Bauer [2] and Leung and Liu [13] used the method of Montgomery and Vaughan [20] to prove that $E_j(X) \ll X^{1-\delta_j}$ for some (very small) absolute constants $\delta_j > 0$. In the case of sums of three squares, there have been also a series of recent advances [3, 8, 12, 15, 16, 17], culminating in the result of Harman and the first author [8] that $E_3(X) \ll X^{6/7+\varepsilon}$ for any fixed $\varepsilon > 0$.

Zhan and the second author [14] considered short interval versions of (1). They obtained the following result.

Theorem. *Let $A > 0$ and $\varepsilon > 0$ be fixed. If $X^{7/16+\varepsilon} \leq H \leq X$, then*

$$E_2(X+H) - E_2(X) \ll H(\log X)^{-A}. \tag{2}$$

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Also, if $X^{3/4+\varepsilon} \le H \le X$*, then*

$$E_3(X+H) - E_3(X) \ll H(\log X)^{-A}. \tag{3}$$

The implied constants in (2) *and* (3) *depend at most on* A *and* ε*.*

The admissible range for H in the second part of this theorem was extended to $H \ge X^{1/2+\varepsilon}$ by Mikawa [18], and then recently to $H \ge X^{7/16+\varepsilon}$ by Mikawa and Peneva [19].

The proofs in [14, 18, 19] use the Hardy–Littlewood circle method to count representations of the desired form on average over n. For example, let $Y = X^{7/12+\varepsilon/2}$ and write

$$R_2(n) = \sum_{\substack{p_1+p_2^2=n \\ p_j \in \mathbf{I}_j}} 1, \tag{4}$$

where p_1 and p_2 denote primes and

$$\mathbf{I}_1 = [X-Y, X), \quad \mathbf{I}_2 = \big[\tfrac{1}{2}Y^{1/2}, Y^{1/2}\big). \tag{5}$$

Deferring some standard notation to the end of this Introduction, we now define

$$r_2(n) = r_2(n; X, Y) = \sum_{\substack{m_1+m_2^2=n \\ m_j \in \mathbf{I}_j}} \frac{1}{(\log m_1)(\log m_2)}, \tag{6}$$

$$\mathfrak{S}_2(n, P) = \sum_{q \le P} \frac{\mu(q)}{\phi(q)^2} \sum_{\substack{1 \le a \le q \\ (a,q)=1}} S(q,a) e(-an/q), \tag{7}$$

where m_1 and m_2 denote integers and

$$S(q,a) = \sum_{\substack{1 \le x \le q \\ (x,q)=1}} e\big(ax^2/q\big). \tag{8}$$

The estimate (2) was established in [14] by showing that when $H \ge Y^{3/4+\varepsilon/2}$, the asymptotic formula

$$R_2(n) = r_2(n)\mathfrak{S}_2(n,P)\big(1 + O\big((\log X)^{-A}\big)\big)$$

holds for all but $O\big(H(\log X)^{-A}\big)$ integers $n \in \mathcal{H}_2 \cap (X, X+H]$. Here, $P = (\log X)^B$ for some $B = B(A) > 0$. In the present paper, we demonstrate how a rather simple sieve idea yields a similar result for $H \ge Y^{2/3+\varepsilon/2}$. This leads to the following theorem.

Theorem 1. *Let* $A > 0$*,* $\delta > 0$ *and* $\varepsilon > 0$ *be fixed, and suppose that* $X^{7/18+\varepsilon} \le H \le X$*. There exists a* $B = B(A) > 0$ *such that when* $P = (\log X)^B$*, the asymptotic formula*

$$R_2(n) = r_2(n)\mathfrak{S}_2(n,P)\big(1 + O\big((\log X)^{-1+\delta}\big)\big) \tag{9}$$

holds for all but $O\big(H(\log X)^{-A}\big)$ *integers* $n \in \mathcal{H}_2 \cap (X, X+H]$*. The implied constants depend at most on* A, δ *and* ε*.*

In particular, it follows from this theorem that (2) holds when $H \geq X^{7/18+\varepsilon}$. The error term in (9) is somewhat weaker than the error term in the analogous result in [14], but that is a small price to pay for the longer range for H.

It appears very difficult to improve further on Theorem 1, if an asymptotic formula similar to (9) is required. On the other hand, if one is content merely with the existence of representations of n as the sum of a prime and a square of a prime, then further progress is possible. Indeed, combining the circle method with Harman's sieve method (see [6, 7]), we obtain the following result.

Theorem 2. *Let $A > 0$ be fixed and suppose that $X^{0.33} \leq H \leq X$. Then* (2) *holds.*

The exponent 0.33 is not the exact limit of the method but just a reasonably close upper bound for that limit. It can be easily "improved" to 0.3275 by choosing $\theta_2 = 0.595$ in the calculations in §5. However, it appears that in order to replace 0.33 by even 0.325, one needs a substantially new idea.

The methods used in the proofs of Theorems 1 and 2 can be easily adapted to improve on the result of Mikawa and Peneva on sums of three squares of primes. In particular, when $X^{7/18+\varepsilon} \leq H \leq X$, we obtain an asymptotic result similar to Theorem 1. The application of the sieve method to this problem, on the other hand, is somewhat less successful. We obtain the following analogue of Theorem 2.

Theorem 3. *Let $A > 0$ be fixed and suppose that $X^{7/20} \leq H \leq X$. Then* (3) *holds.*

One can use Theorem 3 to estimate the number of exceptions in a short interval for representations as sums of four squares of primes. Let $E_4(X)$ denote the number of integers n, with $n \leq X$ and $n \equiv 4 \pmod{24}$, which cannot be represented as the sum of squares of primes. Combining Theorem 3 with known results on the difference between two consecutive primes, we obtain the following result.

Corollary 1. *Let $A > 0$ be fixed and suppose that $X^{0.27} \leq H \leq X$. Then*

$$E_4(X + H) - E_4(X) \ll H(\log X)^{-A}.$$

Notation. Throughout the paper, the letter p, with or without indices, is reserved for prime numbers; c denotes an absolute constant, not necessarily the same in all occurrences. As usual in number theory, $\mu(n)$, $\phi(n)$ and $\tau(n)$ denote, respectively, the Möbius function, Euler's totient function and the number of divisors function; $\|x\|$ denotes the distance from x to the nearest integer. We write $e(x) = \exp(2\pi i x)$, $e_q(x) = e(x/q)$, and $(a, b) = \gcd(a, b)$. Also, we use $m \sim M$ and $m \asymp M$ as abbreviations for the conditions $M \leq m < 2M$ and

$c_1M \le m < c_2M$. Finally, if $z \ge 2$, we define $\Pi(z) = \prod_{p \le z} p$ and introduce the functions

$$\Phi(n,z) = \begin{cases} 1 & \text{if } p \mid n \implies p \ge z, \\ 0 & \text{otherwise}; \end{cases} \tag{10}$$

$$\Psi(n,z) = \begin{cases} 1 & \text{if } p \mid n \implies p \le z, \\ 0 & \text{otherwise}. \end{cases} \tag{11}$$

2. Outline of the method

In this section, we outline the proofs of Theorems 1 and 2. The details of those proofs are presented in §4 and §5. The proof of Theorem 3 and its corollary are given in §6.

2.1. **The circle method.** Suppose that X is a large real, and let $L = \log X$, $Y = X^{\theta_1}$, $H = Y^{\theta_2}$, where θ_1 and θ_2 are positive constants to be specified later. Also, let $\mathbf{I}_1$ and $\mathbf{I}_2$ be the intervals (5) with $Y = X^{\theta_1}$. For any pair of arithmetic functions λ_1, λ_2, put

$$R(n; \lambda_1, \lambda_2) = \sum_{\substack{m_1 + m_2^2 = n \\ m_i \in \mathbf{I}_i}} \lambda_1(m_1)\lambda_2(m_2). \tag{12}$$

In particular, we have $R_2(n) = R(n; \varpi, \varpi)$, where ϖ is the characteristic function of the primes. In the proofs of Theorems 1 and 2, we apply the circle method to $R(n; \lambda_1, \lambda_2)$ with different choices of λ_1 and λ_2.

The application of the circle method starts with the identity

$$R(n; \lambda_1, \lambda_2) = \int_0^1 S_1(\alpha) S_2(\alpha) e(-\alpha n)\, d\alpha, \tag{13}$$

where

$$S_j(\alpha) = \sum_{m \in \mathbf{I}_j} \lambda_j(m) e\big(\alpha m^j\big) \qquad (j = 1, 2).$$

Suppose that $A > 0$ is a fixed real, which we assume to be larger than some absolute constant. We set

$$P = L^B, \quad Q_0 = YP^{-3}, \quad Q = HP^{-1}, \tag{14}$$

where B is a parameter to be chosen later in terms of A. We define the sets of major and minor arcs as follows:

$$\mathfrak{M} = \bigcup_{q \le P} \bigcup_{\substack{1 \le a \le q \\ (a,q)=1}} \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ} \right], \quad \mathfrak{m} = \big[Q^{-1}, 1 + Q^{-1}\big] \setminus \mathfrak{M}. \tag{15}$$

We also write $\mathfrak{M}(q,a) = \big\{\alpha \in \mathbb{R} \bigm| |q\alpha - a| < Q^{-1}\big\}$.

In order to proceed further, we need to make some assumptions regarding λ_1 and λ_2. We require the following hypotheses:

($A_{j.1}$) We have $\lambda_j(m) \ll 1$ and $\lambda_j(m) = 0$ when $\Phi(m, P) = 0$.

(A$_{1.2}$) There exists a smooth function f_1 such that the inequality

$$\sup_{\mathbf{J}\subseteq\mathbf{I}_1}\left|\sum_{m\in\mathbf{J}}\big(\lambda_1(m)-D(\chi)f_1(m)\big)\chi(m)\right|\ll YP^{-5}$$

holds for all Dirichlet characters χ with moduli $q\le P$. Here, the supremum is over all subintervals of $\mathbf{I}_1$, and $D(\chi)=1$ or 0 according as χ is principal or not.

(A$_{2.2}$) There exists a smooth function f_2 such that the inequality

$$\int_{Y/4}^{Y}\left|\sum_{t<m^2\le t+\delta t}\big(\lambda_2(m)-D(\chi)f_2(m)\big)\chi(m)\right|^2 dt\ll (qQ)^2P^{-4}$$

holds for all Dirichlet characters χ with moduli $q\le P$ and all real δ with $0<\delta\ll qQY^{-1}$.

When $\alpha\in\mathfrak{M}(q,a)$, we define the functions

$$S_1^*(\alpha)=\frac{\mu(q)}{\phi(q)}T_1(\alpha-a/q),\quad S_2^*(\alpha)=\frac{S(q,a)}{\phi(q)}T_2(\alpha-a/q),$$

where $S(q,a)$ is defined in (8) and

$$T_j(\beta)=\sum_{m\in\mathbf{I}_j}f_j(m)e\big(\beta m^j\big)\qquad(j=1,2).$$

Since the intervals $\mathfrak{M}(q,a)$ are disjoint, this defines $S_j^*(\alpha)$ on $\mathfrak{M}$. The analysis of the major arcs aims to prove that one can approximate $S_j(\alpha)$ by $S_j^*(\alpha)$ on average over $\alpha\in\mathfrak{M}$. By Cauchy's inequality,

$$\int_{\mathfrak{M}}|S_1(S_2-S_2^*)|\,d\alpha\le PI_1^{1/2}\Bigg(\max_{\substack{1\le a\le q\le P\\(a,q)=1}}\int_{\mathfrak{M}(q,a)}|S_2-S_2^*|^2\,d\alpha\Bigg)^{1/2},\tag{16}$$

where

$$I_1=\int_0^1|S_1|^2\,d\alpha=\sum_{m\in\mathbf{I}_1}\lambda_1(m)^2\ll Y.\tag{17}$$

Let $\alpha\in\mathfrak{M}(q,a)$ and note that (A$_{2.1}$) implies that $\lambda_2(m)=0$ when $(m,q)>1$. Using the orthogonality of the characters modulo q, we obtain

$$|S_2(\alpha)-S_2^*(\alpha)|^2\le\sum_{\chi\bmod q}|W_2(\alpha-a/q,\chi)|^2,\tag{18}$$

where

$$W_j(\beta,\chi)=\sum_{m\in\mathbf{I}_j}(\lambda_j(m)-D(\chi)f_j(m))\chi(m)e\big(\beta m^j\big)\qquad(j=1,2).$$

Inserting (17) and (18) into the right side of (16), we get

$$\int_{\mathfrak{M}}|S_1(S_2-S_2^*)|\,d\alpha\ll PY^{1/2}\Bigg(\max_{q\le P}\sum_{\chi\bmod q}\int_{-1/(qQ)}^{1/(qQ)}|W_2(\beta,\chi)|^2\,d\beta\Bigg)^{1/2}.\tag{19}$$

Combining Gallagher's lemma [5, Lemma 1]) with a device of Saffari and Vaughan [21, p. 25], we find that

$$\int_{-1/(qQ)}^{1/(qQ)} |W_2(\beta,\chi)|^2 \, d\beta \ll \frac{1}{(qQ)^2} \int_{Y/4}^{Y} \Big| \sum_{t<m^2\le t+\delta t} \big(\lambda_2(m) - D(\chi) f_2(m)\big)\chi(m) \Big|^2 \, dt + \delta,$$

for some $\delta \asymp (qQ)Y^{-1} \ll HY^{-1}$. Thus, by (14), (19) and hypothesis ($\mathrm{A}_{2.2}$) above,

$$\int_{\mathfrak{M}} |S_1(S_2 - S_2^*)| \, d\alpha \ll (Y/P)^{1/2}. \tag{20}$$

Before proceeding further, we make an assumption regarding the smooth functions f_1 and f_2 appearing in hypotheses ($\mathrm{A}_{j.2}$): we suppose that

$$|f_j(m)| \ll 1, \quad |f_j'(m)| \ll (1+|m|)^{-1} \qquad (j=1,2). \tag{21}$$

These simple conditions suffice to deduce the bounds

$$T_j(\beta) \ll Y^{1/j}(1+Y|\beta|)^{-1} \qquad (j=1,2). \tag{22}$$

Let

$$\mathfrak{M}_0 = \bigcup_{q\le P} \bigcup_{\substack{1\le a\le q \\ (a,q)=1}} \left[\frac{a}{q} - \frac{1}{qQ_0}, \frac{a}{q} + \frac{1}{qQ_0}\right], \quad \mathfrak{m}_0 = \mathfrak{M} \setminus \mathfrak{M}_0. \tag{23}$$

By (14), (17) and (22),

$$\begin{aligned}\left(\int_{\mathfrak{m}_0} |S_1 S_2^*| \, d\alpha\right)^2 &\ll I_1 \sum_{q\le P} \sum_{1\le a\le q} \frac{|S(q,a)|^2}{\phi(q)^2} \int_{1/(qQ_0)}^{1/2} |T_2(\beta)|^2 \, d\beta \\ &\ll Y^2 \sum_{q\le P} q^{\eta} \int_{1/(qQ_0)}^{\infty} \frac{d\beta}{(1+Y|\beta|)^2} \ll YP^{-1+\eta}. \end{aligned} \tag{24}$$

Here and through the remainder of this section, $\eta > 0$ is a fixed real that can be taken arbitrarily small.

Now, if $\alpha \in \mathfrak{M}(q,a) \cap \mathfrak{M}_0$, we have (similarly to (18))

$$|S_1(\alpha) - S_1^*(\alpha)| \ll q^{-1/2+\eta} \sum_{\chi \bmod q} |W_1(\alpha - a/q, \chi)|. \tag{25}$$

Using partial summation, we deduce from (25) and ($\mathrm{A}_{1.2}$) that

$$|S_1(\alpha) - S_1^*(\alpha)| \ll q^{1/2+\eta/2} YP^{-10}(1+Y|\beta|).$$

From this inequality and (22), we obtain

$$\int_{\mathfrak{M}_0} |(S_1 - S_1^*)S_2^*| \, d\alpha \ll \sum_{q\le P} q^{\eta} \sum_{1\le a\le q} \int_{-1/(qQ_0)}^{1/(qQ_0)} Y^{3/2} P^{-5} \, d\beta \ll Y^{1/2} P^{-1+\eta}. \tag{26}$$

Finally, by (14) and (22),

$$\int_{1/(qQ_0)}^{1/2} T_1(\beta)T_2(\beta)\,d\beta \ll Y^{1/2}P^{-2},$$

whence

$$\int_{\mathfrak{M}_0} S_1^*(\alpha)S_2^*(\alpha)e(-\alpha n)\,d\alpha = \mathfrak{S}_2(n,P)\mathfrak{I}(n;\lambda_1,\lambda_2) + O\big(Y^{1/2}P^{-1}\big), \tag{27}$$

where $\mathfrak{S}_2(n,P)$ is defined in (7) and

$$\mathfrak{I}(n;\lambda_1,\lambda_2) = \int_{-1/2}^{1/2} T_1(\beta)T_2(\beta)e(-\beta n)\,d\beta = \sum_{\substack{m_1+m_2^2=n\\ m_i\in\mathbf{I}_i}} f_1(m_1)f_2(m_2).$$

Combining (20), (24), (26) and (27), we get

$$\begin{aligned}\int_{\mathfrak{M}} S_1S_2e(-\alpha n)\,d\alpha &= \int_{\mathfrak{M}_0} S_1^*S_2^*e(-\alpha n)\,d\alpha + \int_{\mathfrak{M}} S_1(S_2-S_2^*)e(-\alpha n)\,d\alpha\\ &\quad + \int_{\mathfrak{m}_0} S_1S_2^*e(-\alpha n)\,d\alpha + \int_{\mathfrak{M}_0}(S_1-S_1^*)S_2^*e(-\alpha n)\,d\alpha\\ &= \mathfrak{S}_2(n,P)\mathfrak{I}(n;\lambda_1,\lambda_2) + O\big(Y^{1/2}P^{-1/2+\eta}\big). \end{aligned}\tag{28}$$

In order to estimate the contribution from the minor arcs, we now make another hypothesis regarding λ_2:

($A_{2.3}$) Given any $A>0$, there exists a $B_0=B_0(A)>0$ such that when $B\ge B_0$, the inequality

$$\int_{Y/4}^{Y}\bigg|\sum_{t<m^2\le t+H}\lambda_2(m)e\big(\alpha m^2\big)\bigg|^2 dt \ll H^2L^{-A}$$

holds for all $\alpha\in\mathfrak{m}$.

Using the well-known bound

$$\sum_{X<n\le X+H} e(\alpha n) \ll \min\big(H, \|\alpha\|^{-1}\big),$$

we obtain

$$\begin{aligned}&\sum_{X<n\le X+H}\bigg|\int_{\mathfrak{m}} S_1(\alpha)S_2(\alpha)e(-\alpha n)\,d\alpha\bigg|^2\\ &\ll \int_{\mathfrak{m}}\int_{\mathfrak{m}} |S_1(\alpha)S_2(\alpha)S_1(\beta)S_2(\beta)|\big(H,\|\alpha-\beta\|^{-1}\big)\,d\alpha d\beta\\ &\ll \int_{\mathfrak{m}}\int_{\mathfrak{m}} |S_1(\beta)S_2(\alpha)|^2\big(H,\|\alpha-\beta\|^{-1}\big)\,d\alpha d\beta\\ &\ll I_1 \max_{\beta\in[0,1]}\int_{\mathfrak{m}} |S_2(\alpha)|^2\big(H,\|\alpha-\beta\|^{-1}\big)\,d\alpha. \end{aligned}\tag{29}$$

Moreover, a simple subdivision argument yields

$$\int_{\mathfrak{m}} |S_2(\alpha)|^2 \big(H, \|\alpha-\beta\|^{-1}\big)\, d\alpha \ll HL \int_{\mathbf{J}_\gamma} |S_2(\alpha)|^2\, d\alpha, \tag{30}$$

for some $\gamma \in [0,1]$ and $\mathbf{J}_\gamma = \mathfrak{m} \cap \big[\gamma - H^{-1}, \gamma + H^{-1}\big]$. Since an interval of length $2H^{-1}$ can intersect at most one major arc, $\mathbf{J}_\gamma$ is either an interval or the union of two intervals. Hence,

$$\int_{\mathbf{J}_\gamma} |S_2(\alpha)|^2\, d\alpha \ll \int_{-1/H}^{1/H} |S_2(\alpha+\beta)|^2\, d\beta, \tag{31}$$

for some $\alpha \in \mathfrak{m}$. By Gallagher's lemma and hypothesis ($\mathrm{A}_{2.3}$), the last integral is $O\big(L^{-A-1}\big)$, which together with (29)–(31) gives

$$\sum_{X<n\le X+H} \bigg| \int_{\mathfrak{m}} S_1(\alpha) S_2(\alpha) e(-\alpha n)\, d\alpha \bigg|^2 \ll HYL^{-A}. \tag{32}$$

Combining (13), (15), (28) and (32), we obtain the following result.

Proposition 1. *Let $A > 0$ be fixed and $P = L^B$, with $B \ge B_0(A) > 0$. Suppose that λ_1 is an arithmetic function satisfying hypotheses ($\mathrm{A}_{1.1}$) and ($\mathrm{A}_{1.2}$), and that λ_2 is an arithmetic functions satisfying hypotheses ($\mathrm{A}_{2.1}$)–($\mathrm{A}_{2.3}$). Furthermore, suppose that the functions f_1 and f_2 appearing in hypotheses ($\mathrm{A}_{j.2}$) satisfy (21). Then*

$$\sum_{X<n\le X+H} \big| R(n;\lambda_1,\lambda_2) - \mathfrak{S}_2(n,P)\mathfrak{I}(n;\lambda_1,\lambda_2) \big|^2 \ll HYL^{-A}. \tag{33}$$

2.2. **The sieve method.** One can use Proposition 1 with $\lambda_1 = \lambda_2 = \varpi$, the characteristic function of the primes, to obtain an asymptotic formula for $R_2(n)$ for almost all $n \in \mathcal{H}_2 \cap (X, X+H]$ (that is, for all but $O\big(HL^{-A}\big)$ such n). However, when one tries to verify the hypotheses of the proposition, one is forced to choose $\theta_1 > \frac{7}{12}$ and $\theta_2 > \frac{3}{4}$, and so one recovers the result of Zhan and the second author mentioned in the Introduction. Thus, in the proofs of the theorems, we use different choices for λ_1 and λ_2.

First, let

$$\lambda_0(m) = \Phi(m, z_0), \quad z_0 = Y^{1/4} P^{-2}. \tag{34}$$

We note that

$$R_2(n) = R(n;\varpi,\lambda_0) - R_0(n), \tag{35}$$

where $R_0(n)$ is the number of solutions of the equation

$$p_1 + (p_2p_3)^2 = n$$

in primes p_1, p_2, p_3 subject to

$$p_1 \in \mathbf{I}_1, \quad z_0 < p_2 \le Y^{1/4}, \quad p_2 \le p_3, \quad p_2p_3 \in \mathbf{I}_2.$$

It turns out that Proposition 1 can be applied to $R(n;\varpi,\lambda_0)$ when $\theta_1 > \frac{7}{12}$ and $\theta_2 > \frac{2}{3}$. This yields the asymptotic formula

$$R(n;\varpi,\lambda_0) = \mathfrak{S}_2(n,P)\mathfrak{I}(n) + O\big(Y^{1/2}L^{-A}\big)$$

for almost all $n \in \mathcal{H}_2 \cap (X, X+H]$. Here, we have $\mathfrak{I}(n) \sim r_2(n)$, so this asymptotic formula is very close to the conjectured asymptotic formula for $R_2(n)$. In order to complete the proof of Theorem 1, we shall use an upper-bound sieve to show that

$$R_0(n) \ll \mathfrak{S}_2(n,P)r_2(n)L^{-1+\delta} \tag{36}$$

for almost all $n \in \mathcal{H}_2 \cap (X, X+H]$.

We now proceed to outline the proof of Theorem 2. We introduce two pairs of arithmetic functions: $\lambda_1^{\pm}$ such that

$$\lambda_1^-(m) \le \varpi(m) \le \lambda_1^+(m) \qquad (m \in \mathbf{I}_1), \tag{37}$$

and $\lambda_2^{\pm}$ such that

$$\lambda_2^-(m) \le \lambda_0(m) \le \lambda_2^+(m) \qquad (m \in \mathbf{I}_2). \tag{38}$$

Then

$$R(n;\varpi,\lambda_0) \ge R(n;\lambda_1^+,\lambda_2^-) + R(n;\lambda_1^-,\lambda_2^+) - R(n;\lambda_1^+,\lambda_2^+). \tag{39}$$

We remark that this inequality is a variant of the vector sieve of Brüdern and Fouvry [4]. We shall use Harman's sieve to construct the functions $\lambda_i^{\pm}$ so that Proposition 1 can be applied to each of the three terms on the right side of (39). It will then follow from (39) that

$$R(n;\varpi,\lambda_0) \ge (\sigma(\theta_1,\theta_2) + o(1))r_2(n)\mathfrak{S}_2(n,P) \tag{40}$$

for almost all $n \in \mathcal{H}_2 \cap (X, X+H]$. Here, $\sigma(\theta_1,\theta_2)$ is independent of any parameters other than the exponents θ_1 and θ_2. Moreover, as a function of θ_1 and θ_2, σ is continuous and non-decreasing with respect to each variable. Since $\sigma(0.55+\varepsilon, 0.6) \ge 0.17$, Theorem 2 follows readily from (35), (36) and (40).

3. Lemmas

In this section, we collect various auxiliary results required in the proofs of the theorems. These lemmas fall in three major categories: bounds for exponential sums; results from elementary number theory and sieve theory; and results concerning the singular series.

3.1. **Bounds for exponential sums.** The first two lemmas are essentially restatements of Lemmas 3.1 and 3.2 in [14]. We omit the proofs, since they are identical to the proofs in [14].

Lemma 3.1. *Let $A > 0$, $B > 0$, and $x^{1/2} \le y \le x$, with x sufficiently large. Suppose that $\alpha \in \mathbb{R}$ and $a, q \in \mathbb{Z}$ are such that*

$$L^B \le q \le yL^{-B}, \quad (a,q) = 1, \quad |\alpha - a/q| < q^{-2}, \tag{41}$$

where $L = \log x$. Suppose also that (a_m) is a sequence of complex numbers with $|a_m| \le \tau(m)^c$, and that

$$1 \le M \le x^{1/4} L^{-B}.$$

Then, for $B \ge B_0(A) > 0$, one has

$$\int_x^{2x} \Big| \sum_{\substack{m \sim M \\ t < m^2k^2 \le t+y}} \sum_{k} a_m e\big(\alpha m^2 k^2\big) \Big|^2 dt \ll y^2 L^{-A}.$$

Lemma 3.2. *Let $A > 0$, $B > 0$, and $x^{1/2} \le y \le x$, with x sufficiently large. Suppose that $\alpha \in \mathbb{R}$ and $a, q \in \mathbb{Z}$ satisfy (41). Suppose also that (a_m) and (b_k) are sequences of complex numbers with $|a_m| \le \tau(m)^c$ and $|b_k| \le \tau(k)^c$, and that*

$$L^B \le M \le yx^{-1/2} L^{-B}.$$

Then, for $B \ge B_0(A) > 0$, one has

$$\int_x^{2x} \Big| \sum_{\substack{m \sim M \\ t < m^2k^2 \le t+y}} \sum_{k} a_m b_k e\big(\alpha m^2 k^2\big) \Big|^2 dt \ll y^2 L^{-A}.$$

The next lemma is a simple tool for reducing the estimation of a bilinear sum to the estimation of a similar sum subject to 'nicer' summation conditions. The proof is a standard application of Perron's integral formula, so we omit it and refer the reader to Kumchev [12, Lemma 2.7].

Lemma 3.3. *Let $F : \mathbb{N} \to \mathbb{C}$ satisfy $|F(x)| \le X$, let $M, K \ge 2$, and define the bilinear form*

$$\mathcal{B}(M,K) = \sum_{\substack{m \sim M \\ m < k}} \sum_{k \sim K} a_m b_k F(mk),$$

where $|a_m| \le 1$, $|b_k| \le 1$. Then

$$\mathcal{B}(M,K) \ll L \Big| \sum_{m \sim M} \sum_{k \sim K} a'_m b'_k F(mk) \Big| + (XMK)^{-1},$$

where $|a'_m| \le |a_m|$, $|b'_k| \le |b_k|$ and $L = \log(2MKX)$. The same estimate holds, if we replace the summation condition $m < k$ in the definition of $\mathcal{B}(M,K)$ with $U \le mk < U'$.

Lemma 3.4. *Let $A > 0$, $B > 0$, and $x^{1/2} \le y \le x$, with x sufficiently large. Suppose that $\alpha \in \mathbb{R}$ and $a, q \in \mathbb{Z}$ satisfy (41). Suppose also that (a_m) is a sequence of complex numbers with $|a_m| \le \tau(m)^c$, and that*

$$1 \le M \le x^{1/4} L^{-2B}, \quad 2 \le z \le yx^{-1/2} L^{-2B}.$$

Then, for $B \ge B_0(A) > 0$, one has

$$\int_x^{2x} \Bigg| \sum_{\substack{m\sim M \\ t<m^2k^2\le t+y}} \sum_{k} a_m \Phi(k,z) e\big(\alpha m^2 k^2\big) \Bigg|^2 dt \ll y^2 L^{-A},$$

where $\Phi(k,z)$ is the function defined in (10).

Proof. Let g_t denote the indicator function of the interval $\big(t^{1/2}, (t+y)^{1/2}\big]$. We have

$$\sum_{m\sim M} \sum_{k} a_m \Phi(k,z) g_t(mk) e\big(\alpha m^2 k^2\big) = \sum_{d|\Pi(z)} \sum_{m\sim M} \sum_{k} a_m \mu(d) g_t(mkd) e\big(\alpha m^2 k^2 d^2\big).$$

It thus suffices to show that

$$\int_x^{2x} \Bigg| \sum_{\substack{d|\Pi(z) \\ d\sim D}} \sum_{m\sim M} \sum_{k} a_m \mu(d) g_t(mkd) e\big(\alpha m^2 k^2 d^2\big) \Bigg|^2 dt \ll y^2 L^{-A}, \tag{42}$$

where $1 \le D \ll x^{1/2} M^{-1}$. We distinguish three cases depending on the size of D.
Case 1: $D \le L^B$. Upon defining the convolution

$$b_r = \sum_{\substack{dm=r \\ d\sim D, m\sim M \\ d|\Pi(z)}} a_m \mu(d),$$

we can rewrite the left side of (42) as

$$\int_x^{2x} \Bigg| \sum_{r\sim R} \sum_{k} b_r g_t(rk) e\big(\alpha r^2 k^2\big) \Bigg|^2 dt,$$

where $|b_r| \le \tau(r)^c$ and $R = MD \le x^{1/4} L^{-B}$. Therefore, (42) follows from Lemma 3.1.
Case 2: $L^B \le D \le yx^{-1/2} L^{-B}$. Upon defining the convolution

$$b_r = \sum_{\substack{mk=r \\ m\sim M}} a_m,$$

we can rewrite the left side of (42) as

$$\int_x^{2x} \Bigg| \sum_{d\sim D, d|\Pi(z)} \sum_{r} b_r \mu(d) g_t(rd) e\big(\alpha r^2 d^2\big) \Bigg|^2 dt,$$

where $|b_r| \le \tau(r)^c$. Therefore, (42) follows from Lemma 3.2.

Case 3: $D \geq yx^{-1/2}L^{-B}$. Set $V = yx^{-1/2}L^{-B}$. Each d appearing in the summation has a factorization $d = p_1 \cdots p_r$ subject to

$$p_r < \cdots < p_1 < z, \quad p_1 \cdots p_r \geq V.$$

Therefore, there is a unique integer s, $1 \leq s < r$, such that

$$L^B \leq z^{-1}V \leq p_1 \cdots p_s \leq V \leq p_1 \cdots p_{s+1}.$$

On writing $p = p_s$, $p' = p_{s+1}$, $d_1 = p_1 \cdots p_{s-1}$, $d_2 = p_{s+2} \cdots p_r$, we can express the left side of (42) as

$$\int_x^{2x} \Big| \sum_{p,p'} \sum_{d_1,d_2} \sum_{m \sim M} \sum_k a_m \mu(d_1)\mu(d_2)\psi(d_1,p) g_t(mkpp'd_1d_2) e\big(\alpha(mkpp'd_1d_2)^2\big) \Big|^2 dt,$$

where p, p', d_1, d_2 are subject to

$$p' < p < z, \quad pp'd_1d_2 \sim D, \quad d_1 \mid \Pi(z), \quad d_2 \mid \Pi(p'), \quad L^B \leq d_1 p < V \leq d_1pp'.$$

Hence, using Lemma 3.3 to remove the summation conditions

$$p' < p, \quad pp'd_1d_2 \sim D, \quad \text{and} \quad d_1pp' \geq V,$$

we can show that the left side of (42) is bounded by

$$L^c \int_x^{2x} \Big| \sum_{L^B \leq v \leq V} \sum_u \tilde{a}_v b_u g_t(uv) e\big(\alpha u^2 v^2\big) \Big|^2 dt + L^c,$$

with coefficients $|\tilde{a}_v| \leq 1$ and $|b_u| \leq \tau(u)^c$ (the new variables being $u = mkp'd_2$ and $v = pd_1$). Thus, (42) follows from Lemma 3.2. □

3.2. **Some lemmas from sieve theory.** Let $\Phi(m,z)$ and $\Psi(m,z)$ be the functions defined in (10) and (11). Lemma 3.5 below is Theorem 1 in Tenenbaum [23, §III.5]. Lemma 3.6 is a variant of Theorem 3 in Tenenbaum [23, §III.6].

Lemma 3.5. *If x and z are large real numbers, then*

$$\sum_{m \leq x} \Psi(m,z) \ll x \exp\big(-(\log x)/(2\log z)\big).$$

Lemma 3.6. *Let $2 \leq z \leq x \leq z^c$, and let w be the continuous solution of the differential delay equation*

$$\begin{cases} (tw(t))' = w(t-1) & \text{if } t > 2, \\ w(t) = t^{-1} & \text{if } 1 < t \leq 2. \end{cases}$$

Then for any fixed $A > 0$,

$$\sum_{m \leq x} \Phi(m,z) = \frac{1}{\log z} \sum_{z < m \leq x} w\left(\frac{\log m}{\log z}\right) + O\left(x(\log x)^{-A}\right),$$

the implied constant depending at most on A.

We now introduce some standard sieve-theoretic notation. If $\mathcal{A}$ is an integer sequence, we define

$$\mathcal{A}_d = \big\{ a \in \mathcal{A} \mid m \equiv 0 \ (\mathrm{mod}\ d) \big\}.$$

Suppose that when d is squarefree, we have

$$|\mathcal{A}_d| = g(d)N + r(d), \tag{43}$$

where N is a large parameter independent of d and g is a multiplicative function such that $0 \le g(p) < 1$ for all p. We assume that there exist constants $\kappa \ge 0$ and $K \ge 2$ such that

$$\prod_{w \le p < z} \big(1 - g(p)\big)^{-1} \le \left(\frac{\log z}{\log w} \right)^{\kappa} \left(1 + \frac{K}{\log w} \right) \tag{44}$$

whenever $2 \le w < z$. The next lemma is a version of the upper-bound Rosser–Iwaniec sieve: see Iwaniec [11, Theorem 1].

Lemma 3.7. *Let $z \ge 2$, $s \ge 1$, and let $\mathcal{A}$ be an integer sequence. Suppose that N, the arithmetic function g and the remainders $r(d)$ are defined by* (43)*, and that* (44) *holds for some absolute constants $\kappa \ge 0$ and $K \ge 2$. Then*

$$\sum_{a \in \mathcal{A}} \Phi(a, z) \le NV(z)\big(1 + O\big(e^{-s}\big)\big) + \sum_{d \le z^s} \mu(d)^2 |r(d)|,$$

where $V(z) = \prod_{p \le z} \big(1 - g(p)\big)$. The implied constant depends at most on κ and K.

3.3. **The singular series.** In this section, we collect the necessary information about the singular series for sums of a prime and a square of a prime and for sums of three squares of primes. Let $S(q, a)$ be given by (8). We define

$$A_2(n, q) = \frac{\mu(q)}{\phi(q)^2} \sum_{\substack{1 \le a \le q \\ (a,q)=1}} S(q, a) e_q(-an), \quad A_3(n, q) = \frac{1}{\phi(q)^3} \sum_{\substack{1 \le a \le q \\ (a,q)=1}} S(q, a)^3 e_q(-an),$$

$$\mathfrak{S}_j(n, P) = \sum_{q \le P} A_j(n, q), \quad \mathfrak{P}_j(n, P) = \prod_{p \le P} \big(1 + A_j(n, p) + A_j(n, p^2) + \cdots\big). \tag{45}$$

Note that $\mathfrak{S}_2(n, P)$ is the sum defined earlier in (7). These sums and products were studied in great detail by Schwarz [22, §§2–3]. Here is a list of some facts that can be found there:

i) $A_j(n, q)$ is multiplicative in q.
ii) $A_3(n, p^k) = 0$ when $p \ge 3, k \ge 2$ or $p = 2, k \ge 4$.
iii) If $n \in \mathcal{H}_2$, then $A_2(n, p) > -1$ for all p.
iv) If $n \in \mathcal{H}_3$, then $A_3(n, 2^j) \ge 0$ and $A_3(n, p) > -1$ for all $p \ge 3$.
v) $\sum_{n=1}^{q_1 q_2} A_j(n, q_1) A_j(n, q_2) = 0$ when $q_1 \ne q_2$.

Furthermore, it is not difficult to show that

$$A_2(n,p) = \begin{cases} (p-1)^{-1} & \text{if } p \mid n, \\ \left(\frac{n}{p}\right)p^{-1} + O\big(p^{-2}\big) & \text{if } p \nmid n, \end{cases} \tag{46}$$

where $\left(\frac{n}{p}\right)$ is the Legendre symbol modulo p. There is also a similar expression for $A_3(n,p)$ (see Mikawa [18, (4.1)]), from which we can deduce that $|A_3(n,p)| \le 3p^{-1} + O\big(p^{-2}\big)$. Hence,

$$|A_j(n,q)| \ll q^{-1} \prod_{p \mid q} \big(1 + p^{-1}\big)^c \ll q^{-1}(\log\log q)^c. \tag{47}$$

We also have

$$(\log P)^{-1} \ll \mathfrak{P}_2(n,P) \ll \log P, \quad (\log P)^{-3} \ll \mathfrak{P}_3(n,P) \ll (\log P)^3. \tag{48}$$

Finally, we state and prove a lemma, which allows us to approximate $\mathfrak{S}_j(n,P)$ by $\mathfrak{P}_j(n,P)$ on average over n, provided that P is small compared to n. The lemma is essentially a generalization of a result of Schwarz [22, Satz 1], but our proof is considerably shorter.

Lemma 3.8. *Let $A \ge 2$ and $\varepsilon > 0$ be fixed. Suppose that $x^\varepsilon \le y \le x$ and $P \le Q \le \exp\big((\log x)^{1-\varepsilon}\big)$. Then*

$$\sum_{x<n\le x+y} \big(\mathfrak{S}_j(n,P) - \mathfrak{P}_j(n,Q)\big)^2 \ll yP^{-1}\log x + y(\log x)^{-A}. \tag{49}$$

Proof. We may assume that $\varepsilon < \frac{1}{4}$. Put $Q_2 = \prod_{p\le Q} p$, $Q_3 = 4Q_2$, and $Q_0 = y^{1/3}$. We have

$$\mathfrak{P}_j(n,Q) = \sum_{q\le Q_j} A_j(n,q)\Psi(q,Q) = \sum_{q\le Q_0} A_j(n,q)\Psi(q,Q) + \Sigma, \tag{50}$$

where

$$\Sigma = \sum_{Q_0<q\le Q_j} A_j(n,q)\Psi(q,Q) \ll \sum_{Q_0<q\le Q_j} q^{-1}(\log\log q)^c\Psi(q,Q).$$

An appeal to Lemma 3.5 then yields

$$\Sigma \ll (\log\log Q_j)^c \sum_{Q_0<q\le Q_j} q^{-1}\Psi(q,Q) \ll (\log Q)^c \exp\big(-(\log Q_0)/(2\log Q)\big).$$

Since $\Psi(q,Q) = 1$ when $1 \le q \le P$, we deduce from this inequality and (50) that

$$\mathfrak{S}_j(n,P) - \mathfrak{P}_j(n,Q) = \sum_{P<q\le Q_0} \theta_q A_j(n,q) + O\big((\log x)^{-A-2}\big),$$

where $\theta_q = 1 - \Psi(q,Q)$. Since the sum over q does not exceed $(\log x)^2$ (recall (47)), the desired conclusion then follows from the bound

$$\sum_{x<n\le x+y} \sum_{P<q_1,q_2\le Q_0} \theta_{q_1}\theta_{q_2} A_j(n,q_1)A_j(n,q_2) \ll yP^{-1}\log x + y(\log x)^{-A}. \tag{51}$$

By (47),

$$\sum_{x<n\le x+y}\sum_{P<q\le Q_0}\theta_q^2 A_j(n,q)^2 \ll \sum_{x<n\le x+y}\sum_{q>P} q^{-2}(\log\log q)^c \ll yP^{-1}\log x.$$

On the other hand, when $q_1\ne q_2$, (47) and v) above yield

$$\Bigg|\sum_{x<n\le x+y} A_j(n,q_1)A_j(n,q_2)\Bigg| \le 2\sum_{n=1}^{q_1q_2}|A_j(n,q_1)A_j(n,q_2)| \ll (q_1q_2)^{\varepsilon/2},$$

whence

$$\sum_{x<n\le x+y}\sum_{\substack{P<q_1<q_2\le Q_0\\ q_1\ne q_2}}\theta_{q_1}\theta_{q_2}A_j(n,q_1)A_j(n,q_2) \ll Q_0^{2+\varepsilon} \ll y^{3/4}.$$

This establishes (51). □

4. Proof of Theorem 1

We first verify the hypotheses of Proposition 1 for $R(n;\varpi,\lambda_0)$, where ϖ is the indicator function of the primes and λ_0 is defined by (34). These functions clearly satisfy hypotheses ($A_{j.1}$) in §2. When $\theta_1 > \frac{7}{12}$, ϖ satisfies hypothesis ($A_{1.2}$) with $f_1(u) = (\log u)^{-1}$ ($u \ge 2$). This is a short interval form of the Siegel–Walfisz theorem that can be established by the same methods as Huxley's theorem on primes in short intervals. The same methods establish also hypothesis ($A_{2.2}$) for λ_0 with

$$f_2(u) = \frac{1}{\log u} + \int_{z_0}^{\sqrt{u}} \frac{dt}{t(\log t)(\log(u/t))} \qquad (u\in \mathbf{I}_2),$$

provided that $\theta_2 > \frac{7}{12}$. The first term in the above sum accounts for the primes in the support of λ_0, and the second term accounts for products p_1p_2 with $z_0 < p_1 \le p_2$. (The reader can find a justification of hypothesis ($A_{2.2}$) in the case when $\lambda_2 = \varpi$ in [14, Lemma 5.1] or in Mikawa and Peneva [19, Lemma 2].)

Finally, we consider hypothesis ($A_{2.3}$). We set $z_1 = Y^{1/6+\varepsilon/2}$ and note that

$$\lambda_0(m) = \Phi(m,z_0) = \Phi(m,z_1) - \sum_{\substack{z_1<p\le z_0\\ p\mid m}} \Phi\big(mp^{-1},p\big).$$

Since $mp^{-1} \le Y^{1/3-\varepsilon/2}$ in the sum above, we have $\Phi\big(mp^{-1},p\big) = \varpi\big(mp^{-1}\big) = \Phi\big(mp^{-1},z_1\big)$. Hence,

$$\lambda_0(m) = \Phi(m,z_1) - \sum_{\substack{z_1<p\le z_0\\ p\mid m}} \Phi\big(mp^{-1},z_1\big) = \lambda_0'(m) - \lambda_0''(m), \quad \text{say.} \tag{52}$$

Suppose that $\alpha\in\mathfrak{m}$ and $\theta_2 \ge \frac{2}{3}+\varepsilon$ (note that the latter condition ensures that $z_1 \le HY^{-1/2}L^{-2B}$). Then Lemma 3.4 with $x = Y$, $y = H$, $(m,k) = (1,m)$ and $z = z_1$ establishes hypothesis ($A_{2.3}$) for λ_0'; the same lemma with $x = Y$, $y = H$, $(m,k) = (p, mp^{-1})$ and $z = z_1$ establishes hypothesis ($A_{2.3}$) for λ_0''. The hypothesis ($A_{2.3}$) for λ_0 then follows from (52). We remark that the choice of z_0 in (34) is determined by the hypothesis on M in the application of Lemma 3.4 to λ_0''.

Suppose now that $\theta_1 \geq \frac{7}{12} + \frac{1}{2}\varepsilon$ and $\theta_2 \geq \frac{2}{3} + \varepsilon$. Having verified all the hypotheses of Proposition 1, we can then apply that proposition to get

$$R(n; \varpi, \lambda_0) = \mathfrak{S}_2(n, P)\mathfrak{I}(n; \varpi, \lambda_0) + O\big(Y^{1/2}L^{-A}\big) \tag{53}$$

for almost all $n \in \mathcal{H}_2 \cap (X, X+H]$. Note that with the above choices of f_1 and f_2, we have

$$\mathfrak{I}(n; \varpi, \lambda_0) = r_2(n)\big(1 + O\big(L^{-1}\log L\big)\big). \tag{54}$$

Combining (35), (48), (49), (53) and (54), we obtain the asymptotic formula

$$R_2(n) = \mathfrak{S}_2(n, P)r_2(n)\big(1 + O\big(L^{-1}\log L\big)\big) - R_0(n) \tag{55}$$

for almost all $n \in \mathcal{H}_2 \cap (X, X+H]$, provided that $X^{7/18+\varepsilon} \leq H \leq X^{1-\varepsilon}$ and $P = (\log X)^B$ with B sufficiently large in terms of A. Therefore, Theorem 1 follows from the following proposition.

Proposition 2. *Let $A > 0$, $\delta > 0$ and $\varepsilon > 0$ be fixed, and suppose that $Y^\varepsilon \leq H \leq Y^{1-\varepsilon}$. There exists a $B_0 = B_0(A) > 0$ such that when $B \geq B_0$, one has*

$$R_0(n) \ll r_2(n)\mathfrak{S}_2(n, P)L^{-1+\delta}$$

for all but $O\big(HL^{-A}\big)$ integers $n \in \mathcal{H}_2 \cap (X, X+H]$.

Proof. We estimate $R_0(n)$ by means of an upper-bound sieve. Observe that $R_0(n)$ is the number of primes in the sequence

$$\mathcal{A} = \big\{m \in I_1 \;\big|\; m = n - (p_1p_2)^2 \text{ with } \; p_1 \in \mathbf{I}_3,\; p_1 \leq p_2,\; p_1p_2 \in \mathbf{I}_2\big\},$$

where $\mathbf{I}_3 = \big[z_0, Y^{1/4}\big)$. Hence,

$$R_0(n) \leq \sum_{m \in \mathcal{A}} \Phi(m, z), \tag{56}$$

where z is any parameter with $2 \leq z \leq X^{1/2}$. We now proceed to apply Lemma 3.7 to the right side of (56).

When $X < n \leq X + H$, $|\mathcal{A}|$ is the number of products p_1p_2, where

$$p_1 \in \mathbf{I}_3, \quad p_1p_2 \in \mathbf{I}_2, \quad p_1 \leq p_2.$$

Thus, upon writing $\mathbf{J}(p)$ for the interval defined by the conditions $px \in \mathbf{I}_2$ and $x \geq p$, we deduce from the Prime Number Theorem that

$$|\mathcal{A}| = N + O\big(Y^{1/2}\exp\big(-L^{1/2}\big)\big),$$

where

$$N = \sum_{p \in \mathbf{I}_3} \int_{\mathbf{J}(p)} \frac{du}{\log u} \ll Y^{1/2}L^{-2}\log L. \tag{57}$$

Suppose that d is a squarefree integer, with $d \le Y^{1/8}$. Then

$$|\mathcal{A}_d| = \sum_{h \in \mathcal{R}_d} \sum_{p_1 \in \mathbf{I}_3} \sum_{\substack{p_2 \in \mathbf{J}(p_1) \\ p_1 p_2 \equiv h \pmod d}} 1,$$

where $\mathcal{R}_d$ represents a maximal set of incongruent solutions of $x^2 \equiv n \pmod d$. In particular, we have $|\mathcal{A}_d| = 0$ when $(d, n) > 1$. We now define

$$r(d) = |\mathcal{A}_d| - g(d)N, \qquad g(d) = \begin{cases} \phi(d)^{-1}|\mathcal{R}_d| & \text{if } (n,d) = 1, \\ 0 & \text{if } (n,d) > 1, \end{cases}$$

and note that

$$|\mathcal{R}_d| = \prod_{p \mid d} \left(1 + \left(\frac{n}{p}\right)\right),$$

$\left(\frac{n}{p}\right)$ being the Legendre symbol modulo p. We note that when $n \in \mathcal{H}_2$, g satisfies the hypothesis (44) of Lemma 3.7 with $\kappa = 2$. Furthermore, it follows from the above definitions that if $D \le Y^{1/8}$, we have

$$\sum_{d \le D} \mu(d)^2 |r(d)| \le \sum_{p \in \mathbf{I}_3} \sum_{d \le D} \tau(d) \max_{(a,d)=1} \max_{x \in \mathbf{J}(p)} \left| \pi(x; d, a) - \frac{1}{\phi(d)} \int_2^x \frac{dt}{\log t} \right|,$$

where $\pi(x; d, a)$ is the number of primes $p \equiv a \pmod d$ with $p \le x$. The sum over d can be estimated by means of the Bombieri–Vinogradov theorem and Cauchy's inequality. Thus, for any fixed $A > 0$ and any $D \le Y^{1/8} L^{-B(A)}$, we obtain the bound

$$\sum_{d \le D} \mu(d)^2 |r(d)| \ll \sum_{p \in I_3} Y^{1/2} p^{-1} L^{-A} \ll Y^{1/2} L^{-A}.$$

We now apply Lemma 3.7 with $D = Y^{1/9}$ and $z = \exp\big(L^{1-\delta/2}\big)$ to the sequence $\mathcal{A}$. We get

$$\sum_{m \in \mathcal{A}} \Phi(m, z) \ll N \prod_{p \le z} \big(1 - g(p)\big) + Y^{1/2} L^{-A}, \tag{58}$$

where $A > 0$ can be taken arbitrarily large. Comparing the definition of g and (46), we find that when $n \in \mathcal{H}_2$,

$$\prod_{p \le z} \big(1 - g(p)\big) \ll \prod_{p \le z} \left(1 - \frac{1}{p}\right) \cdot \prod_{p \le z} \big(1 + A_2(n, p)\big) \ll \frac{\mathfrak{P}_2(n, z)}{\log z}. \tag{59}$$

Here, $\mathfrak{P}_2(n, z)$ is the partial singular product defined in (45). Combining the lower bound (48) and inequalities (56)–(59), we conclude that

$$R_0(n) \ll Y^{1/2} L^{-3+\delta} \mathfrak{P}_2(n, z).$$

Finally, by (47) and Lemma 3.8 with $x = X$, $y = H$ and $Q = z$, the asymptotic formula

$$\mathfrak{P}_2(n, z) = \mathfrak{S}_2(n, P)\big(1 + O\big(L^{-1}\big)\big)$$

holds for almost all integers $n \in \mathcal{H}_2 \cap (X, X+H]$, provided that $P \geq L^{A+5}$. $\square$

5. Proof of Theorem 2

As we stated already in §2, the proof of Theorem 2 makes use of two pairs of functions, $\lambda_1^{\pm}$ and $\lambda_2^{\pm}$ satisfying (37) and (38), respectively. We borrow the functions $\lambda_1^{\pm}$ from Baker, Harman and Pintz [1]: we choose $\lambda_1^-(m) = a_0(m)$ and $\lambda_1^+(m) = a_1(m)$, where a_0 and a_1 are the functions constructed in [1] (see [1, §4] for details). We note that, by construction, these functions satisfy hypotheses ($A_{1.1}$) and ($A_{1.2}$) of Proposition 1 when $\theta_1 \geq 0.55 + \varepsilon$.

Next, we turn to the construction of $\lambda_2^{\pm}$. As hypothesis ($A_{2.3}$) is the most demanding among the requirements imposed on λ_2 in Proposition 1, our construction focuses on satisfying that hypothesis. Let

$$U = Y^{\varepsilon/2}, \quad V = HY^{-1/2-\varepsilon/2}, \quad W = Y^{1/2}V^{-1}. \tag{60}$$

Recall also the definition of z_0 in (34). We apply twice Buchstab's identity

$$\Phi(m, z) = \Phi(m, w) - \sum_{\substack{w<p\leq z \\ p|m}} \Phi\big(mp^{-1}, p\big) \qquad (2 \leq w < z) \tag{61}$$

to decompose λ_0 as follows:

$$\begin{aligned} \lambda_0(m) &= \Phi(m, V) - \sum_{\substack{V<p\leq z_0 \\ p|m}} \Phi\big(mp^{-1}, V\big) + \sum_{\substack{V<p_2<p_1\leq z(p_2) \\ p_1p_2|m}} \Phi\big(m(p_1p_2)^{-1}, p_2\big) \\ &= \gamma_1(m) - \gamma_2(m) + \gamma_3(m), \quad \text{say.} \end{aligned} \tag{62}$$

Here, we have $z(p) = \min\big(z_0, Y^{1/2}p^{-2}\big)$. In particular, when $\theta_2 \geq \frac{2}{3} + \varepsilon$, the sum γ_3 is empty and (62) turns into (52). We now split $\gamma_3(m)$ into two subsums. We have

$$\gamma_3(m) = \Bigg\{ \sum_{\substack{\cdots \\ p_1p_2<W}} + \sum_{\substack{\cdots \\ p_1p_2\geq W}} \Bigg\} \Phi\big(m(p_1p_2)^{-1}, p_2\big) = \gamma_4(m) + \gamma_5(m), \quad \text{say,} \tag{63}$$

where the $\cdots$ represent the summation conditions $V < p_2 < p_1 \leq z(p_2)$ and $p_1p_2 \mid m$. We are now in position to define λ_2^-. We set

$$\lambda_2^-(m) = \gamma_1(m) - \gamma_2(m) + \gamma_5(m). \tag{64}$$

Note that, by (62) and (63), we have $\lambda_2^-(m) = \lambda_0(m) - \gamma_4(m)$, so λ_2^- satisfies (38). Furthermore, by virtue of (34) and (60), we can use Lemma 3.4 to verify hypothesis ($A_{2.3}$) for γ_1 and γ_2. Finally, in γ_5, we have $U \leq m(p_1p_2)^{-1} \leq V$, so we can apply Lemma 3.2 with $(m, k) = \big(m(p_1p_2)^{-1}, p_1p_2\big)$ to verify hypothesis ($A_{2.3}$) for γ_5. We conclude that λ_2^- satisfies both (38) and hypotheses ($A_{2.1}$) and ($A_{2.3}$) of Proposition 1. When $\theta_2 > \frac{7}{12}$, λ_2^- satisfies also hypothesis ($A_{2.2}$), though this may require some explanation.

As we mentioned earlier, hypothesis ($A_{2.2}$) holds for $\lambda_2 = \varpi$ when $\theta_2 > \frac{7}{12}$. One way to prove this is to use (61) to decompose ϖ into a linear combination of functions similar to our γ_i's and then to establish hypothesis ($A_{2.2}$) for each function in that decomposition. Applying that same decomposition to λ_2^- instead to ϖ is equivalent to taking the intersection of two partitions of a set. Therefore, such a decomposition of λ_2^- will produce more terms than the respective decomposition of ϖ, but every such term will be a subsum of a sum appearing in the decomposition of ϖ. Thus, the same results, which establish ($A_{2.2}$) for all terms in the decomposition of ϖ, will establish ($A_{2.2}$) for all terms in the decomposition of λ_2^-.

We now proceed with the construction of λ_2^+. By (61),

$$\begin{aligned}\lambda_0(m) &= \Phi(m, V) - \sum_{\substack{V < p \leq z_1 \\ p \mid m}} \Phi\big(mp^{-1}, p\big) - \sum_{\substack{z_1 < p \leq z_0 \\ p \mid m}} \Phi\big(mp^{-1}, p\big) \\ &= \beta_1(m) - \beta_2(m) - \beta_3(m), \quad \text{say.}\end{aligned} \tag{65}$$

Here, $z_1 = \max\big(V, z_0^{1/2}\big)$. Note that when $\theta_2 \geq \frac{5}{8}$, $z_1 = V$ and the sum β_2 is empty. Suppose now that $\theta_2 < \frac{5}{8}$ (and hence, $z_1 = z_0^{1/2}$). We apply (61) two more times to β_2:

$$\begin{aligned}\beta_2(m) &= \sum_{\substack{V < p \leq z_1 \\ p \mid m}} \Phi\big(mp^{-1}, V\big) - \sum_{\substack{V < p_2 < p_1 \leq z_1 \\ p_1 p_2 \mid m}} \Phi\big(m(p_1p_2)^{-1}, V\big) \\ &\qquad + \sum_{\substack{V < p_3 < p_2 < p_1 \leq z_1 \\ p_1 p_2 p_3 \mid m}} \Phi\big(m(p_1p_2p_3)^{-1}, p_3\big) \\ &= \beta_4(m) - \beta_5(m) + \beta_6(m), \quad \text{say.}\end{aligned} \tag{66}$$

We define

$$\lambda_2^+(m) = \beta_1(m) - \beta_4(m) + \beta_5(m). \tag{67}$$

By (65) and (66), we have $\lambda_2^+(m) = \lambda_0(m) + \beta_3(m) + \beta_6(m)$, so λ_2^+ satisfies (38) and hypothesis ($A_{2.1}$) of Proposition 1. Hypothesis ($A_{2.2}$) holds when $\theta_2 > \frac{7}{12}$ for the same reasons as in the case of λ_2^-. Finally, λ_2^+ satisfies hypothesis ($A_{2.3}$), because Lemma 3.4 can be applied to each of the three terms on the right side of (67).

Suppose now that $\lambda_i^\pm$ are the above functions and that $\theta_1 \geq 0.55 + \varepsilon$ and $\frac{7}{12} < \theta_2 \leq \frac{2}{3}$. With these choices, we can apply Proposition 1 to each of the three terms on the right side of (39). We deduce that

$$R(n; \varpi, \lambda_0) \geq \mathfrak{S}_2(n, P)\mathfrak{I}(n)(1 + o(1)) \tag{68}$$

for almost all $n \in \mathcal{H}_2 \cap (X, X + H]$. Here,

$$\mathfrak{I}(n) = \sum_{\substack{m_1 + m_2^2 = n \\ m_j \in \mathbf{I}_j}} \big(f_1^+(m_1)f_2^-(m_2) + f_1^-(m_1)f_2^+(m_2) - f_1^+(m_1)f_2^+(m_2)\big),$$

$f_j^\pm$ being the smooth functions appearing in hypotheses ($A_{j.2}$).

The functions $f_j^{\pm}$ arise via applications of Lemma 3.6. For example, when $\theta_2 > \frac{5}{8}$, we have $\lambda_2^+(m) = \Phi(m, V)$, and Lemma 3.6 gives

$$\sum_{m \le x} \lambda_2^+(m) = \frac{1}{\log V} \sum_{z < m \le x} w\left(\frac{\log m}{\log V}\right) + O\big(Y^{1/2} L^{-A}\big)$$

for any fixed $A > 0$ and any $x \le Y^{1/2}$. Hence, in this case, we have

$$f_2^+(m) = \frac{1}{\log V} w\left(\frac{\log m}{\log V}\right) \qquad (m \ge V).$$

Furthermore, the functions $f_j^{\pm}$ satisfy asymptotic formulas of the form

$$\sum_{\substack{m \in \mathbf{I}_j \\ m \le x}} f_j^{\pm}(m) = \big(\sigma_j^{\pm} + O\big(L^{-1}\big)\big) \sum_{\substack{m \in \mathbf{I}_j \\ m \le x}} \frac{1}{\log m}, \tag{69}$$

where $\sigma_j^{\pm} = \sigma_j^{\pm}(\theta_j)$ are numbers depending only on θ_1 and θ_2. The values of $\sigma_1^{\pm}$ are estimated in [1]: when $\theta_1 \ge 0.55 + \varepsilon$, we have

$$\sigma_1^+ < 1.01, \quad \sigma_1^- > 0.99. \tag{70}$$

On the other hand, the values of $\sigma_2^{\pm}$ arising from the above construction of $\lambda_2^{\pm}$ are

$$\sigma_2^- = 1 - \iint_{\mathcal{D}_2^-} w\left(\frac{1 - u_1 - u_2}{u_2}\right) \frac{du_1 du_2}{u_1 u_2^2} + O(\varepsilon),$$
$$\sigma_2^+ = 1 + \int_{1/4}^{1/2} w\left(\frac{1-u}{u}\right) \frac{du}{u^2} + \iiint_{\mathcal{D}_2^+} w\left(\frac{1 - u_1 - u_2 - u_3}{u_3}\right) \frac{du_1 du_2 du_3}{u_1 u_2 u_3^2} + O(\varepsilon),$$

where

$$\mathcal{D}_2^- : \quad 2\theta_2 - 1 < u_2 < u_1 < \tfrac{1}{2},\ u_1 + 2u_2 < 1,\ u_1 + u_2 < 2 - 2\theta_2,$$
$$\mathcal{D}_2^+ : \quad 2\theta_2 - 1 < u_3 < u_2 < u_1 < \tfrac{1}{4}.$$

A computer calculation then yields

$$\sigma_2^-(\tfrac{3}{5}) > 0.22, \quad \sigma_2^+(\tfrac{3}{5}) < 2.26. \tag{71}$$

Combining (69)–(71), we get

$$\mathfrak{I}(n) \ge r_2(n)\big(0.17 + O\big(L^{-1}\big)\big)$$

Inserting this bound into (68), we obtain

$$R(n; \varpi, \lambda_0) \ge \mathfrak{S}_2(n, P) r_2(n)(0.17 + o(1)) \tag{72}$$

for almost all $n \in \mathcal{H}_2 \cap (X, X + H]$.

Finally, we choose $\theta_1 = 0.55 + \varepsilon$ and $\theta_2 = \frac{3}{5} - 2\varepsilon$. Theorem 2 is a direct consequence of (35), (72) and Proposition 2. □

6. Sums of three and four squares

6.1. **Proof of Theorem 3.** The argument is similar to the proof of Theorem 2, so we only outline the differences between the two proofs. Let $R_3(n)$ denote the number of representations of n in the form

$$R_3(n) = \sum_{\substack{p_1^2+p_2^2+p_3^2=n \\ p_1^2+p_2^2\in\mathbf{I}_1, p_3\in\mathbf{I}_2}} 1.$$

In place of the quantity defined in (12), we use

$$R(n;\lambda_1,\lambda_2) = \sum_{\substack{m_1^2+m_2^2+m_3^2=n \\ m_1^2+m_2^2\in\mathbf{I}_1, m_3\in\mathbf{I}_2,}} \lambda_1(m_1,m_2)\lambda_2(m_3). \tag{73}$$

We set $\lambda_1(m,k) = \varpi(m)\varpi(k)$ and $\lambda_2(m) = \lambda_2^-(m)$, where λ_2^- is the function defined in (64). Similarly to (35) and (39), we have

$$R_3(n) \geq R(n;\lambda_1,\lambda_2) - R_0(n), \tag{74}$$

where $R_0(n)$ is the number of solutions of the equation

$$p_1^2 + p_2^2 + (p_3p_4)^2 = n$$

in primes $p_1, \dots, p_4$ subject to

$$p_1^2 + p_2^2 \in \mathbf{I}_1, \quad z_0 < p_3 \leq Y^{1/4}, \quad p_3 \leq p_4, \quad p_3p_4 \in \mathbf{I}_2.$$

Suppose again that $A > 0$ is a fixed (large) real and set

$$P = L^B, \quad Q_0 = YP^{-3}, \quad Q = HP^{-1}, \tag{75}$$

where B is a parameter to be chosen later in terms of A. Similarly to Proposition 2, one can show that

$$R_0(n) \ll \mathfrak{S}_3(n,P)Y^{1/2}L^{-4+\delta} \tag{76}$$

for almost all $n \in \mathcal{H}_3 \cap (X, X+H]$. Here, $\mathfrak{S}_3(n,P)$ is defined by (45).

Next, we use the circle method to evaluate the quantity $R(n;\lambda_1,\lambda_2)$ in (74). The orthogonality relation (13) holds with $S_1(\alpha)$ replaced by the sum

$$S_1(\alpha) = \sum_{p_1^2+p_2^2\in\mathbf{I}_1} e\big(\alpha(p_1^2+p_2^2)\big).$$

We define the sets of major and minor arcs as before. By the discussion in §5, λ_2 satisfies hypotheses ($\mathrm{A}_{2.j}$) in §2. Since

$$I_1 = \int_0^1 |S_1|^2\,d\alpha = \sum_{m\in\mathbf{I}_1}\bigg(\sum_{p_1^2+p_2^2=m} 1\bigg)^2 \ll YL^3,$$

we obtain similarly to (32) that

$$\sum_{X<n\le X+H}\left|\int_{\mathfrak{m}} S_1(\alpha)S_2(\alpha)e(-\alpha n)\,d\alpha\right|^2 \ll HYL^{-A}. \tag{77}$$

Furthermore, similarly to (20) and (24), we have

$$\int_{\mathfrak{M}} |S_1(S_2-S_2^*)|\,d\alpha \ll Y^{1/2}P^{-1/2+\eta} \tag{78}$$

and (recall (23))

$$\int_{\mathfrak{m}_0} |S_1S_2^*|\,d\alpha \ll Y^{1/2}P^{-1/2+\eta}. \tag{79}$$

Define

$$T_1(\beta)=\sum_{m\in\mathbf{I}_1} f_1(m)e(\beta m), \quad f_1(m)=\int_0^1 \frac{du}{\sqrt{u(1-u)}(\log mu)(\log m(1-u))}.$$

When $\alpha\in\mathfrak{M}(q,a)\cap\mathfrak{M}_0$ and $\theta_1>\frac{7}{12}$, a variant of Mikawa and Peneva [19, Lemma 3] yields

$$|S_1(\alpha)-S_1^*(\alpha)| \ll q^\eta YP^{-10}(1+Y|\alpha-a/q|),$$

where

$$S_1^*(\alpha)=\frac{\pi}{4}\frac{S(q,a)^2}{\phi(q)^2}T_1(\alpha-a/q).$$

Hence,

$$\int_{\mathfrak{M}_0} |(S_1-S_1^*)S_2^*|\,d\alpha \ll Y^{1/2}P^{-1+\eta}. \tag{80}$$

Finally, we have

$$\int_{\mathfrak{M}_0} S_1^*(\alpha)S_2^*(\alpha)e(-\alpha n)\,d\alpha = \frac{\pi}{4}\mathfrak{S}_3(n,P)\mathfrak{I}(n;\lambda_2)+O\big(Y^{1/2}P^{-1}\big), \tag{81}$$

where $\mathfrak{S}_3(n,P)$ is defined in (45) and

$$\mathfrak{I}(n;\lambda_2)=\int_{-1/2}^{1/2} T_1(\beta)T_2(\beta)e(-\beta n)\,d\beta = \sum_{\substack{m_1+m_2^2=n\\ m_i\in\mathbf{I}_i}} f_1(m_1)f_2^-(m_2).$$

Combining (78)–(81), we conclude that

$$\int_{\mathfrak{M}} S_1(\alpha)S_2(\alpha)e(-\alpha n)\,d\alpha = \frac{\pi}{4}\mathfrak{S}_3(n,P)\mathfrak{I}(n;\lambda_2)+O\big(Y^{1/2}P^{-1/2+\eta}\big). \tag{82}$$

From (74), (76), (77) and (82), we obtain that

$$R_3(n) \gg \mathfrak{S}_3(n,P)Y^{1/2}L^{-3}$$

for almost all $n\in\mathcal{H}_3\cap(X,X+H]$, provided that $\theta_1>\frac{7}{12}$ and the value of $\sigma_2^-(\theta_2)$ in §5 is positive. In particular, upon choosing $\theta_1=\frac{7}{12}+\varepsilon$ and $\theta_2=\frac{3}{5}-2\varepsilon$, we deduce Theorem 3. □

6.2. **Proof of Corollary 1.** Let $E_4'(X)$ be the number of exceptional integers n counted by $E_4(X)$ with $n \not\equiv 1 \pmod 5$, and let $E_4''(X) = E_4(X) - E_4'(X)$. By a result of Harman, Watt and Wong [9, Theorem 3], there exist prime numbers q_1 and q_2 such that

$$X^{1/2} - X^{0.2625} < q_j \le X^{1/2} - \tfrac{1}{2}X^{0.2625}, \quad q_j \equiv j \pmod 5.$$

If n is counted by $E_4'(X+H) - E_4'(X)$, then $n - q_1^2$ is counted by $E_3(X_1 + H) - E_3(X_1)$, where $X_1 \asymp X^{0.7625}$. Since $H \ge X_1^{7/20}$, Theorem 3 yields

$$E_4'(X+H) - E_4'(X) \le E_3(X_1 + H) - E_3(X_1) \ll HL^{-A}, \tag{83}$$

for any fixed $A > 0$. Similarly, if n is counted by $E_4''(X+H) - E_4''(X)$, then the integer $n - q_2^2$ is counted by $E_3(X_2 + H) - E_3(X_2)$, where $X_2 \asymp X^{0.7625}$. Hence, Theorem 3 yields

$$E_4''(X+H) - E_4''(X) \le E_3(X_2 + H) - E_3(X_2) \ll HL^{-A}, \tag{84}$$

for any fixed $A > 0$. The result follows from (83) and (84). □

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