A BINARY ADDITIVE EQUATION INVOLVING FRACTIONAL POWERS

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1. Introduction

It is well-known that the number of integers $n \le x$ that can be expressed as sums of two squares is $O\big(x(\log x)^{-1/2}\big)$. On the other hand, Deshouillers [2] showed that when $1 < c < \frac{4}{3}$, every sufficiently large integer n can be represented in the form

$$[m_1^c] + [m_2^c] = n, \tag{1}$$

with integers m_1, m_2; henceforth, $[\theta]$ denotes the integral part of θ. Subsequently, the range for c in this result was extended by Gritsenko [3] and Konyagin [5]. In particular, the latter author showed that (1) has solutions in integers m_1, m_2 for $1 < c < \frac{3}{2}$ and n sufficiently large.

The analogous problem with prime variables is considerably more difficult, possibly at least as difficult as the binary Goldbach problem. The only progress in that direction is a result of Laporta [6], which states that if $1 < c < \frac{17}{16}$, then almost all n (in the sense usually used in analytic number theory) can be represented in the form (1) with primes m_1, m_2. Recently, Balanzario, Garaev and Zuazua [1] considered the equation

$$[m^c] + [p^c] = n, \tag{2}$$

where p is a prime number and m is an integer. They showed that when $1 < c < \frac{17}{11}$, this hybrid problem can be solved for almost all n. While the range of c in this result is even longer than Konyagin's, one may ask whether, when c is close to 1, it is not possible to solve (2) for all sufficiently large n. Indeed, such a result would fit perfectly with the case $c = 1$, when the problem is trivial. The main purpose of the present note is to address this issue. We establish the following theorem.

Theorem 1. *Suppose that $1 < c < \frac{16}{15}$. Then every sufficiently large integer n can be represented in the form (2).*

To prove this theorem we borrow an idea from Deshouillers [2]. In order to prove the existence of solutions of (1), he translated the additive equation (1) into a question about Diophantine approximation by fractional powers. We reduce (2) to a similar problem on Diophantine approximation with a prime variable. The same idea leads to a simple proof of a slightly weaker version of the result of Balanzario, Garaev and Zuazua. For $x \ge 2$, let $E_c(x)$ denote the number of integers $n \le x$ that cannot be represented in the form (2). We prove the following theorem.

Theorem 2. *Suppose that $1 < c < \frac{3}{2}$ and $\varepsilon > 0$. Then*

$$E_c(x) \ll x^{3(1-1/c)+\varepsilon}.$$

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We remark that Theorem 1 is hardly best possible. It is likely that more sophisticated exponential sum estimates and/or sieve techniques would have allowed us to extend the range of c. The resulting improvement, however, would have been minuscule; thus, we decided not to pursue such ideas.

Acknowledgement. After this work was completed, the author discovered that J.-M. Deshouillers had remarked in his Ph.D. thesis that the method in his work [2] can yield a result along the lines of Theorem 1.

Notation. Most of our notation is standard. We use Landau's O-notation, Vinogradov's $\ll$-symbol, and occasionally, we write $A \asymp B$ instead of $A \ll B \ll A$. We also write $\{\theta\}$ for the fractional part of θ and $\|\theta\|$ for the distance from θ to the nearest integer. Finally, we define $e(\theta) = \exp(2\pi i\theta)$.

2. Proof of Theorem 1: initial stage

In this section, we only assume that $1 < c < 2$. We write $\gamma = 1/c$ and set

$$X = \left(\tfrac{1}{2}n\right)^{\gamma}, \quad X_1 = \tfrac{5}{4}X, \quad \delta = \gamma X^{1-c}. \tag{3}$$

If n is sufficiently large, it has at most one representation of the form (2) with $X < p \le X_1$. Furthermore, such a representation exists if and only if there is an integer m satisfying the inequality

$$\big(n - [p^c]\big)^{\gamma} \le m < \big(n + 1 - [p^c]\big)^{\gamma}. \tag{4}$$

We now proceed to show that such an integer exists, if p satisfies the conditions

$$X < p \le X_1, \quad \{p^c\} < \tfrac{1}{2}, \quad 1 - \tfrac{5}{6}\delta < \left\{\big(n - p^c\big)^{\gamma}\right\} < 1 - \tfrac{2}{3}\delta. \tag{5}$$

Under these assumptions, one has

$$X^{1-c} = (n - X^c)^{\gamma-1} < \big(n - p^c\big)^{\gamma-1} \le (n - X_1^c)^{\gamma-1} < 1.1X^{1-c}.$$

Hence,

$$\begin{aligned}
\big(n - [p^c]\big)^{\gamma} &= \big(n - p^c\big)^{\gamma}\left(1 + \gamma\{p^c\}\big(n - p^c\big)^{-1} + O\big(n^{-2}\big)\right) \\
&< \big(n - p^c\big)^{\gamma} + \tfrac{1}{2}\gamma\big(n - p^c\big)^{\gamma-1} + O\big(n^{\gamma-2}\big) \\
&< \big(n - p^c\big)^{\gamma} + 0.55\delta + O\big(\delta n^{-1}\big) \\
&< \left[\big(n - p^c\big)^{\gamma}\right] + 1 - 0.1\delta,
\end{aligned}$$

and

$$\begin{aligned}
\big(n + 1 - [p^c]\big)^{\gamma} &= \big(n - p^c\big)^{\gamma}\left(1 + \gamma(1 + \{p^c\})\big(n - p^c\big)^{-1} + O\big(n^{-2}\big)\right) \\
&\ge \big(n - p^c\big)^{\gamma} + \gamma\big(n - p^c\big)^{\gamma-1} + O\big(n^{\gamma-2}\big) \\
&> \big(n - p^c\big)^{\gamma} + \delta + O\big(\delta n^{-1}\big) \\
&> \left[\big(n - p^c\big)^{\gamma}\right] + 1 + 0.1\delta.
\end{aligned}$$

Consequently, conditions (5) are indeed sufficient for the existence of an integer m satisfying (4). It remains to show that there exist primes satisfying the inequalities in (5). To this end, it suffices to show that

$$\sum_{X < p \le X_1} \Phi\big(p^c\big)\Psi\big((n - p^c)^{\gamma}\big) > 0 \tag{6}$$

for some smooth, non-negative, 1-periodic functions Φ and Ψ such that Φ is supported in $(0, 1/2)$ and Ψ is supported in $(1 - \frac{5}{6}\delta, 1 - \frac{2}{3}\delta)$.

Let ψ_0 be a non-negative C^∞-function that is supported in $[0, 1]$ and is normalized in L^1: $\|\psi_0\|_1 = 1$. We choose Φ and Ψ to be the 1-periodic extensions of the functions

$$\Phi_0(t) = \psi_0(2t) \qquad \text{and} \qquad \Psi_0(t) = \psi_0\big(6\delta^{-1}(t-1)+5\big),$$

respectively. Writing $\hat{\Phi}(m)$ and $\hat{\Psi}(m)$ for the mth Fourier coefficients of Φ and Ψ, we can report that

$$\begin{aligned} \hat{\Phi}(0) &= \tfrac{1}{2}, \quad |\hat{\Phi}(m)| \ll_r (1+|m|)^{-r} \quad \text{for all } r \in \mathbb{Z}, \\ \hat{\Psi}(0) &= \tfrac{1}{6}\delta, \quad |\hat{\Psi}(m)| \ll_r \delta(1+\delta|m|)^{-r} \quad \text{for all } r \in \mathbb{Z}. \end{aligned} \tag{7}$$

Replacing $\Phi(p^c)$ and $\Psi((n-p^c)^\gamma)$ on the left side of (6) by their Fourier expansions, we obtain

$$\sum_{X<p\le X_1} \Phi\big(p^c\big)\Psi\big((n-p^c)^\gamma\big) = \sum_{h\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\sum_{X<p\le X_1} \hat{\Phi}(h)\hat{\Psi}(j)\, e\big(hp^c + j(n-p^c)^\gamma\big). \tag{8}$$

Set $H = X^\varepsilon$ and $J = X^{c-1+\varepsilon}$, where $\varepsilon > 0$ is fixed. By (7) with $r = [\varepsilon^{-1}] + 2$, the contribution to the the right side of (8) from the terms with $|h| > H$ or $|j| > J$ is bounded above by a constant depending on ε. Thus,

$$\sum_{X<p\le X_1} \Phi\big(p^c\big)\Psi\big((n-p^c)^\gamma\big) = \tfrac{1}{12}\delta\big(\pi(X_1) - \pi(X)\big) + O\big(\delta\mathcal{R} + 1\big),$$

where $\pi(X)$ is the number of primes $\le X$ and

$$\mathcal{R} = \sum_{\substack{|h|\le H \\ (h,j)\neq(0,0)}}\sum_{|j|\le J}\bigg|\sum_{X<p\le X_1} e\big(hp^c + j(n-p^c)^\gamma\big)\bigg|.$$

Thus, it suffices to show that

$$\sum_{X<p\le X_1} e\big(hp^c + j(n-p^c)^\gamma\big) \ll X^{2-c-3\varepsilon} \tag{9}$$

for all pairs of integers (h, j) such that $|h| \le H$, $|j| \le J$, and $(h, j) \neq (0, 0)$.

3. Bounds on exponential sums

In this section, we establish estimates for bilinear exponential sums, which we shall need in the proof of (9). Our first lemma is a variant of van der Corput's third-derivative estimate (see [4, Corollary 8.19]).

Lemma 3. *Suppose that $2 \le F \le N^{3/2}$, $N < N_1 \le 2N$, and $0 < \delta < 1$. Let $f \in C^3[N, N_1]$ and suppose that we can partition $[N, N_1]$ into $O(1)$ subintervals so that on each subinterval one of the following sets of conditions holds:*

i) $\delta F N^{-2} \ll |f''(t)| \ll F N^{-2}$;
ii) $\delta F N^{-3} \ll |f'''(t)| \ll F N^{-3}$, $|f''(t)| \ll \delta F N^{-2}$.

Then

$$\sum_{N<n\le N_1} e(f(n)) \ll \delta^{-1/2}\big(F^{1/6}N^{1/2} + F^{-1/3}N\big).$$

Proof. Let η be a parameter to be chosen later so that $0 < \eta \le \delta$ and let $\mathbf{I}$ be one of the subintervals of $[N, N_1]$ mentioned in the hypotheses. If i) holds in $\mathbf{I}$, then by [4, Corollary 8.13],

$$\sum_{n \in \mathbf{I}} e(f(n)) \ll \delta^{-1/2}\big(F^{1/2} + NF^{-1/2}\big). \tag{10}$$

Now suppose that ii) holds in $\mathbf{I}$. We subdivide $\mathbf{I}$ into two subsets:

$$\mathbf{I}_1 = \big\{ t \in \mathbf{I} : \eta F N^{-2} \le |f''(t)| \ll \delta F N^{-2} \big\}, \quad \mathbf{I}_2 = \mathbf{I} \setminus \mathbf{I}_1.$$

Since f'' is monotone on $\mathbf{I}$, the set $\mathbf{I}_1$ consists of at most two intervals and $\mathbf{I}_2$ is a (possibly empty) subinterval of $\mathbf{I}$. If $\mathbf{I}_2 = [a, b]$, then there is a $\xi \in (a, b)$ such that

$$f''(b) - f''(a) = (b - a) f'''(\xi) \qquad \Longrightarrow \qquad b - a \ll \eta\delta^{-1} N.$$

Thus, by [4, Corollary 8.13] and [4, Corollary 8.19],

$$\sum_{n \in \mathbf{I}_1} e(f(n)) \ll \eta^{-1/2}\big(F^{1/2} + NF^{-1/2}\big), \tag{11}$$

$$\sum_{n \in \mathbf{I}_2} e(f(n)) \ll \eta\delta^{-4/3} F^{1/6} N^{1/2} + \eta^{1/2}\delta^{-2/3} F^{-1/6} N. \tag{12}$$

Combining (10)–(12), we get

$$\sum_{N < n \le N_1} e(f(n)) \ll \eta^{-1/2}\big(F^{1/2} + NF^{-1/2}\big) + \eta\delta^{-4/3} N^{1/2} F^{1/6} + \eta^{1/2}\delta^{-2/3} N F^{-1/6}. \tag{13}$$

We now choose

$$\eta = \delta \max\big(F^{-1/3}, F^{2/3} N^{-1}\big).$$

With this choice, (13) yields

$$\sum_{N < n \le N_1} e(f(n)) \ll \delta^{-1/2}\big(F^{1/6} N^{1/2} + F^{-1/3} N\big) + \delta^{-1/3}\big(F^{5/6} N^{-1/2} + F^{-1/6} N^{1/2}\big),$$

and the lemma follows on noting that, when $F \ll N^{3/2}$,

$$F^{-1/6} N^{1/2} \ll F^{-1/3} N, \quad F^{5/6} N^{-1/2} \ll F^{1/6} N^{1/2}.$$

□

Next, we turn to the bilinear sums needed in the proof of (9). From now on, X, X_1, N, H, J have the same meaning as in §2 and ε is subject to $0 < \varepsilon < \frac{1}{2}\big(\frac{16}{15} - c\big)$.

Lemma 4. *Suppose that* $1 < c < \frac{6}{5} - 6\varepsilon$, $M < M_1 \le 2M$, $2 \le K < K_1 \le 2K$, *and*

$$M \ll X^{1 - 2c/3 - \varepsilon}. \tag{14}$$

Further, suppose that h, j *are integers with* $|h| \le H$, $|j| \le J$, $(h, j) \ne (0, 0)$, *and that the coefficients* a_m *satisfy* $|a_m| \le 1$. *Then*

$$\sum_{M < m \le M_1} \sum_{\substack{K < k \le K_1 \\ X < mk \le X_1}} a_m e\big(hm^c k^c + j(n - m^c k^c)^\gamma\big) \ll X^{2 - c - 4\varepsilon}.$$

Proof. We shall focus on the case $j \neq 0$, the case $j = 0$ being similar and easier. We set

$$y = jn^\gamma, \quad x = y^{-1}hn, \quad T = T_m = n^\gamma m^{-1} \asymp K.$$

With this notation, we have

$$f(k) = f_m(k) = hm^c k^c + j(n - m^c k^c)^\gamma = y\alpha(kT_m^{-1}),$$

where

$$\alpha(t) = \alpha(t; x) = xt^c + (1 - t^c)^\gamma. \tag{15}$$

We have

$$f''(k) = yT^{-2}\alpha''(kT^{-1}), \quad f'''(k) = yT^{-3}\alpha'''(kT^{-1}), \tag{16}$$

and

$$\alpha''(t) = (c-1)t^{c-2}\big(cx - (1 - t^c)^{\gamma-2}\big), \tag{17}$$

$$\alpha'''(t) = -(c-1)(2c-1)t^{2c-3}(1 - t^c)^{\gamma-3} + (c-2)t^{-1}\alpha''(t). \tag{18}$$

Moreover, by virtue of (3),

$$\tfrac{1}{2} < (kT^{-1})^c \leq \tfrac{1}{2}(1.25)^c < \tfrac{4}{5} \tag{19}$$

whenever $X < mk \leq X_1$.

Let $\delta_0 = X^{-\varepsilon/10}$. If $|x| \geq \delta_0^{-1}$, then by (16), (17), and (19),

$$|f''(k)| \asymp |xy|K^{-2} \asymp |h|nK^{-2} \qquad \Longrightarrow \qquad JX^{1-\varepsilon}K^{-2} \ll |f''(k)| \ll JXK^{-2}.$$

Thus, by Lemma 3 with $\delta = X^{-\varepsilon}$, $F = JX$ and $N = K$,

$$\sum_{\substack{M<m\leq M_1 \\ X<mk\leq X_1}} \sum_{K<k\leq K_1} a_m e\big(f_m(k)\big) \ll MX^{\varepsilon/2}\big(X^{(c+\varepsilon)/6}K^{1/2} + KX^{-c/3}\big). \tag{20}$$

Note that we need also to verify that $JX \leq K^{3/2}$. This is a consequence of (14).

Suppose now that $|x| \leq \delta_0^{-1}$. The set where $|\alpha''(kT^{-1})| \geq \delta_0$ consists of at most two intervals. Consequently, we can partition $[K, K_1]$ into at most three subintervals such that on each of them we have one of the following sets of conditions:

i) $\delta_0|y|K^{-2} \ll |f''(k)| \ll \delta_0^{-1}|y|K^{-2}$;
ii) $|y|K^{-3} \ll |f'''(k)| \ll |y|K^{-3}$, $|f''(k)| \ll \delta_0|y|K^{-2}$.

Thus, by Lemma 3 with $\delta = \delta_0^2$, $F = \delta_0^{-1}|y| \asymp \delta_0^{-1}|j|X$, and $N = K$,

$$\sum_{\substack{M<m\leq M_1 \\ X<mk\leq X_1}} \sum_{K<k\leq K_1} a_m e\big(f_m(k)\big) \ll MX^{\varepsilon/10}\big(X^{(c+2\varepsilon)/6}K^{1/2} + KX^{-1/3}\big). \tag{21}$$

Again, we have $\delta_0^{-1}|j|X \leq JX^{1+\varepsilon/10} \leq K^{3/2}$, by virtue of (14).

Combining (20) and (21), we obtain the conclusion of the lemma, provided that $c < \frac{4}{3} - 5\varepsilon$ and

$$M \ll X^{3-7c/3-10\varepsilon}.$$

Once again, the latter inequality is a consequence of (14). □

Lemma 5. *Suppose that* $1 < c < \frac{16}{15} - 2\varepsilon$, $M < M_1 \leq 2M$, $K < K_1 \leq 2K$, *and*

$$X^{2c-2+9\varepsilon} \ll M \ll X^{3-2c-9\varepsilon}. \tag{22}$$

Further, suppose that h, j are integers with $|h| \le H$, $|j| \le J$, $(h, j) \ne (0, 0)$, and that the coefficients a_m, b_k satisfy $|a_m| \le 1$, $|b_k| \le 1$. Then

$$\sum_{\substack{M<m\le M_1 \\ X<mk\le X_1}} \sum_{K<k\le K_1} a_m b_k e\big(hm^c k^c + j(n - m^c k^c)^\gamma\big) \ll X^{2-c-4\varepsilon}.$$

Proof. As in the proof of Lemma 4, we shall focus on the case $j \ne 0$. By symmetry, we may assume that $M \ge X^{1/2}$. We set

$$y = jn^\gamma, \quad x = y^{-1}hn, \quad T = n^\gamma.$$

With this notation, we have

$$f(k, m) = hm^c k^c + j(n - m^c k^c)^\gamma = y\alpha(mkT^{-1}),$$

where $\alpha(t)$ is the function defined in (15).

By Cauchy's inequality and [4, Lemma 8.17],

$$\Bigg| \sum_{\substack{M<m\le M_1 \\ X<mk\le X_1}} \sum_{K<k\le K_1} a_m b_k e\big(f(k, m)\big)\Bigg|^2 \ll \frac{X}{Q} \sum_{|q|\le Q} \sum_{K<k\le 2K} \Bigg| \sum_{m\in \mathbf{I}(k,q)} e\big(g(m; k, q)\big)\Bigg|$$

$$\ll \frac{X^2}{Q} + \frac{X}{Q} \sum_{0<|q|\le Q} \sum_{K<k\le 2K} \Bigg| \sum_{m\in \mathbf{I}(k,q)} e\big(g(m; k, q)\big)\Bigg|, \tag{23}$$

where $g(m; k, q) = f(k + q, m) - f(k, m)$, $Q = J^2 X^{6\varepsilon}$, and $\mathbf{I}(k, q)$ is a subinterval of $[M, M_1]$ such that

$$X < mk, m(k + q) \le X_1$$

for all $m \in \mathbf{I}(k, q)$. We remark that the right inequality in (22) ensures that $Q \ll KX^{-\varepsilon}$. When $q \ne 0$, we write

$$g(m; k, q) = yT^{-1} \int_{mk}^{m(k+q)} \alpha'(tT^{-1})\, dt = qy \int_0^1 \beta(m(k + \theta q)T^{-1})\, \frac{d\theta}{k + \theta q},$$

where $\beta(t) = t\alpha'(t)$. Introducing the notation

$$z_\theta = z_\theta(k, q) = yq(k + \theta q)^{-1}, \quad U_\theta = U_\theta(k, q) = T(k + \theta q)^{-1} \asymp M,$$

we find that

$$g''(m) = \int_0^1 z_\theta U_\theta^{-2} \beta''(mU_\theta^{-1})\, d\theta, \quad g'''(m) = \int_0^1 z_\theta U_\theta^{-3} \beta'''(mU_\theta^{-1})\, d\theta,$$

and

$$\beta''(t) = (c - 1)t^{c-2}\big(c^2 x + (1 - t^c)^{\gamma-3}(c + (c - 1)t^c)\big), \tag{24}$$

$$\beta'''(t) = (c - 1)(2c - 1)t^{2c-3}(1 - t^c)^{\gamma-4}\big((c - 1)t^c + 2c\big) + (c - 2)t^{-1}\beta''(t). \tag{25}$$

Let $\delta_0 = X^{-\varepsilon/10}$. If $|x| \ge \delta_0^{-1}$, then by (24) and a variant of (19),

$$|g''(m)| \asymp |qxy|(XM)^{-1} \qquad \Longrightarrow \qquad |q|JX^{-\varepsilon}M^{-1} \ll |g''(m)| \ll |q|JM^{-1}.$$

Thus, by Lemma 3 with $\delta = X^{-\varepsilon}$, $F = |q|JM$ and $N = M$,

$$\sum_{m\in \mathbf{I}(k,q)} e\big(g(m; k, q)\big) \ll (|q|J)^{1/6} M^{2/3} X^{\varepsilon/2}. \tag{26}$$

Note that we need also to verify that $F \le M^{3/2}$, which holds if

$$M \gg X^{6(c-1)+12\varepsilon}. \tag{27}$$

Suppose now that $|x| \le \delta_0^{-1}$. We then deduce from (24) and (25) that

$$|\beta''(mU_\theta^{-1})| \ll \delta_0^{-1}, \quad |\beta'''(mU_\theta^{-1})| \ll \delta_0^{-1},$$

whence

$$|\beta''(mU_\theta^{-1})| = |\beta''(mU_0^{-1})| + O\big(|q|K^{-1}\delta_0^{-1}\big) = |\beta''(mU_0^{-1})| + O\big(\delta_0^2\big).$$

We now note that the subset of $[M, M_1]$ where $|\beta''(mU_0^{-1})| \ge \delta_0$ consists of at most two intervals. Consequently, we can partition $[M, M_1]$ into at most three subintervals such that on each of them we have one of the following sets of conditions:

i) $\delta_0|qy|(XM)^{-1} \ll |g''(m)| \ll \delta_0^{-1}|qy|(XM)^{-1}$;

ii) $|qy|X^{-1}M^{-2} \ll |g'''(m)| \ll |qy|X^{-1}M^{-2}$, $|g''(m)| \ll \delta_0|qy|(XM)^{-1}$.

Thus, Lemma 3 with $\delta = \delta_0^2$, $F = \delta_0^{-1}|qj|M$, and $N = M$ yields (26), provided that (27) holds.

Combining (23) and (26), we get

$$\Bigg| \sum_{\substack{M<m\le M_1\\ X<mk\le X_1}} \sum_{K<k\le K_1} a_m b_k e\big(f(k,m)\big)\Bigg|^2 \ll X^2Q^{-1} + X^{2+\varepsilon/2}(QJ)^{1/6}M^{-1/3}. \tag{28}$$

In view of our choice of Q, the conclusion of the lemma follows from (28), provided that

$$M \gg X^{7.5(c-1)+10\varepsilon}.$$

Both (27) and the last inequality follow from the assumption that $M \ge X^{1/2}$ and the hypothesis $c < \frac{16}{15} - 2\varepsilon$. □

We close this section with a lemma that will be needed in the proof of Theorem 2.

Lemma 6. *Suppose that $1 < c < 2$, $2 \le X < X_1 \le 2X$, and $0 < \delta < \frac{1}{4}$. Let $\mathcal{S}_\delta$ denote the number of integers n such that $X < n \le X_1$ and $\|n^c\| < \delta$. Then*

$$\mathcal{S}_\delta \ll \delta(X_1 - X) + \delta^{-1/2}X^{c/2}.$$

Proof. Let Φ be the 1-periodic extension of a smooth function that majorizes the characteristic function of the interval $[-\delta, \delta]$ and is majorized by the characteristic function of $[-2\delta, 2\delta]$. Then

$$\mathcal{S}_\delta \le \sum_{X<n\le X_1} \Phi\big(n^c\big) = \sum_{X<n\le X_1} \hat\Phi(0) + \sum_{h\ne 0} \hat\Phi(h) \sum_{X<n\le X_1} e\big(hn^c\big). \tag{29}$$

If $h \ne 0$, [4, Corollary 8.13] yields

$$\sum_{X<n\le X_1} e\big(hn^c\big) \ll |h|^{1/2}X^{c/2},$$

whence

$$\begin{aligned}\sum_{h\ne 0} \hat\Phi(h) \sum_{X<n\le X_1} e\big(hn^c\big) &\ll X^{c/2} \sum_{h\ne 0} |\hat\Phi(h)||h|^{1/2}\\ &\ll X^{c/2} \sum_{h\ne 0} \frac{\delta|h|^{1/2}}{(1+\delta|h|)^2} \ll \delta^{-1/2}X^{c/2}.\end{aligned} \tag{30}$$

Since $\hat{\Phi}(0) \le 4\delta$, the lemma follows from (29) and (30). □

4. Proof of Theorem 1: conclusion

Suppose that $1 < c < \frac{16}{15}$ and $0 < \varepsilon < \frac{1}{2}\big(\frac{16}{15} - c\big)$. To prove (9), we recall Vaughan's identity in the form of [4, Proposition 13.4]. We can use it to express the sum in (9) as a linear combination of $O(\log^2 X)$ sums of the form

$$\sum_{M<m\le M_1} \sum_{\substack{K<k\le K_1 \\ X<mk\le X_1}} a_m b_k e\big(hm^c k^c + j(n - m^c k^c)^\gamma\big),$$

where either

i) $|a_m| \ll m^{\varepsilon/2}$, $b_k = 1$, and $M \ll X^{2/3}$; or
ii) $|a_m| \ll m^{\varepsilon/2}$, $|b_k| \ll k^{\varepsilon/2}$, and $X^{1/3} \ll M \ll X^{2/3}$.

A sum subject to conditions ii) is $\ll X^{2-c-3.5\varepsilon}$ by Lemma 5. A sum subject to conditions i) can be bounded using Lemma 4 if (14) holds and using Lemma 5 if (14) fails. In either case, the resulting bound is $\ll X^{2-c-3.5\varepsilon}$. Therefore, each of the $O(\log^2 X)$ terms in the decomposition of (9) is $\ll X^{2-c-3.5\varepsilon}$. This establishes (9) and completes the proof of the theorem.

5. Proof of Theorem 2

We can cover the interval $(x^{1/2}, x]$ by $O((\log x)^3)$ subintervals of the form $(N, N_1]$, with $N_1 = N\big(1 + (\log N)^{-2}\big)$. Thus, it suffices to show that

$$Z_c(N) \ll N^{3-3/c+5\varepsilon/6}, \tag{31}$$

where $Z_c(N)$ is the number of integers n in the range

$$N < n \le N\big(1 + (\log N)^{-2}\big)$$

that cannot be represented in the form (2).

As in the proof of Theorem 1, we derive solutions of (2) from solutions of (4). We set $\gamma = 1/c$, $\eta = (\log N)^{-2}$, and write

$$N_1 = (1+\eta)N, \quad X = \big(\tfrac{1}{2}N\big)^\gamma, \quad X_1 = (1+\eta)X, \quad \delta = \gamma X^{1-c}.$$

Suppose that $N < n \le N_1$ and $X < p \le X_1$. Then

$$(1-\eta)\delta < \gamma\big(n - p^c\big)^{\gamma-1} < (1+2\eta)\delta.$$

Assuming that p satisfies the inequalities

$$4\eta < \{p^c\} < 1 - 4\eta, \quad 1 - \delta - \eta\delta < \big\{\big(n - p^c\big)^\gamma\big\} < 1 - \delta + \eta\delta, \tag{32}$$

we deduce that

$$\begin{aligned}
\big(n - [p^c]\big)^\gamma &< \big(n - p^c\big)^\gamma + (1-4\eta)(1+2\eta)\delta + O\big(\delta n^{-1}\big) \\
&< \big[\big(n - p^c\big)^\gamma\big] + 1 - \eta\delta, \\
\big(n + 1 - [p^c]\big)^\gamma &> \big(n - p^c\big)^\gamma + (1+4\eta)(1-\eta)\delta + O\big(\delta n^{-1}\big) \\
&> \big[\big(n - p^c\big)^\gamma\big] + 1 + \eta\delta.
\end{aligned}$$

In particular, a prime p, $X < p \le X_1$, that satisfies (32) yields a solution m of (4) and a representation of n in the form (2).

Let Φ be the 1-periodic extension of a smooth function Φ_0 that majorizes the characteristic function of $[6\eta, 1-6\eta]$ and is majorized by the characteristic function of $[4\eta, 1-4\eta]$. Further, let Ψ be the 1-periodic extension of

$$\Psi_0(t) = \psi_0\big((2\eta\delta)^{-1}(t-1+\delta)+\tfrac{1}{2}\big),$$

where ψ_0 is the function appearing in the proof of Theorem 1. Then Ψ_0 is supported inside $[1-\delta-\eta\delta, 1-\delta+\eta\delta]$ and the Fourier coefficients of Ψ satisfy

$$\hat{\Psi}(0) = 2\eta\delta, \quad |\hat{\Psi}(h)| \ll_r \eta\delta(1+\eta\delta|h|)^{-r} \quad \text{for all } r \in \mathbb{Z}. \tag{33}$$

Hence,

$$\begin{aligned}\sum_{X<p\le X_1} \Phi\big(p^c\big)\Psi\big((n-p^c)^\gamma\big) &= \sum_{h\in\mathbb{Z}}\sum_{X<p\le X_1} \Phi\big(p^c\big)\hat{\Psi}(h)\, e\big(h(n-p^c)^\gamma\big) \\ &= \hat{\Psi}(0)\sum_{X<p\le X_1}\Phi\big(p^c\big) + \mathcal{R}(n) \\ &= 2\eta\delta\big(\pi(X_1)-\pi(X)+O(\mathcal{S})\big) + \mathcal{R}(n). \end{aligned} \tag{34}$$

Here,

$$\mathcal{R}(n) = \sum_{h\ne 0}\hat{\Psi}(h)\sum_{X<p\le X_1}\Phi\big(p^c\big)\, e\big(h(n-p^c)^\gamma\big)$$

and $\mathcal{S}$ is the number of integers m such that $X < m \le X_1$ and $\|m^c\| < 6\eta$. By Lemma 6,

$$\mathcal{S} \ll \eta(X_1 - X) + \eta^{-1/2}X^{c/2} \ll \eta^2 X. \tag{35}$$

Combining (34), (35) and the Prime Number Theorem, we find that

$$\sum_{X<p\le X_1}\Phi\big(p^c\big)\Psi\big((n-p^c)^\gamma\big) \gg X^{2-c}(\log X)^{-5} \tag{36}$$

for any n, $N < n \le N_1$, for which we have

$$\mathcal{R}(n) \ll X^{2-c-\varepsilon/12}. \tag{37}$$

Since the sum on the right side of (36) is supported on the primes p satisfying (32), (31) will follow if we show that (37) holds for all but $O\big(N^{3-3\gamma+5\varepsilon/6}\big)$ integers $n \in (N, N_1]$.

Set $H = X^{c-1+\varepsilon/6}$. By (33) with $r = 2 + [2\varepsilon^{-1}]$, the contribution to $\mathcal{R}(n)$ from terms with $|h| > H$ is bounded. Consequently,

$$Z_c(N) \ll X^{-2+\varepsilon/6}\sum_{N<n\le N_1}\mathcal{R}_1(n)^2,$$

where

$$\mathcal{R}_1(n) = \sum_{0<|h|\le H}\Bigg|\sum_{X<p\le X_1}\Phi\big(p^c\big)e\big(h(n-p^c)^\gamma\big)\Bigg|.$$

Appealing to Cauchy's inequality and the Weyl–van der Corput lemma [4, Lemma 8.17], we obtain

$$\begin{aligned} Z_c(N) &\ll X^{c-3+\varepsilon/3}\sum_{0<|h|\le H}\sum_{N<n\le N_1}\Bigg|\sum_{X<p\le X_1}\Phi\big(p^c\big)e\big(h(n-p^c)^\gamma\big)\Bigg|^2 \\ &\ll X^{c-2+\varepsilon/3}Q^{-1}\sum_{0<|h|\le H}\sum_{|q|\le Q}\sum_{X<p\le X_1}\Bigg|\sum_{N<n\le N_1} e\big(f(n)\big)\Bigg|, \end{aligned}$$

where $Q \le \eta X$ is a parameter at our disposal and

$$f(n) = h\big((n - p^c)^\gamma - (n - (p+q)^c)^\gamma\big).$$

We choose $Q = \eta X^{1-\varepsilon/6}$. Then

$$|qh|N^{-1} \ll |f'(n)| \ll |qh|N^{-1} \ll \eta < \tfrac{1}{2},$$

so [4, Corollary 8.11] and the trivial bound yield

$$\sum_{N<n\le N_1} e(f(n)) \ll N(1+|qh|)^{-1}.$$

We conclude that

$$Z_c(N) \ll NX^{c-2+2\varepsilon/3} \sum_{0<|h|\le H} \sum_{|q|\le Q} (1+|qh|)^{-1} \ll NX^{2c-3+5\varepsilon/6}.$$

This establishes (31) and completes the proof of the theorem.

References

[1] E. P. Balanzario, M. Z. Garaev, and R. Zuazua, *Exceptional set of a representation with fractional powers*, Acta Math. Hungar. **114** (2007), 103–115.

[2] J.-M. Deshouillers, *Un problème binaire en théorie additive*, Acta Arith. **25** (1973/74), 393–403.

[3] S. A. Gritsenko, *Three additive problems*, Izv. Ross. Akad. Nauk **41** (1992), 447–464, in Russian.

[4] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society, 2004.

[5] S. V. Konyagin, *An additive problem with fractional powers*, Mat. Zametki **73** (2003), 633–636, in Russian.

[6] M. B. S. Laporta, *On a binary problem with prime numbers*, Math. Balkanica (N.S.) **13** (1999), 119–123.

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