ON WEYL SUMS OVER PRIMES AND ALMOST PRIMES

ANGEL V. KUMCHEV

1. Introduction

In this paper, we pursue estimates for the exponential sum

$$f(\alpha) = \sum_{P \le p < 2P} e\left(\alpha p^k\right), \tag{1.1}$$

where α is a real number, k is a positive integer, $e(z) = \exp(2\pi i z)$, and the summation is over prime numbers. This sum was introduced as a tool in analytic number theory by I. M. Vinogradov in the late 1930's. In 1937, Vinogradov developed an ingenious new method for estimating sums over primes and applied that method to obtain the first unconditional estimate for $f(\alpha)$ with $k = 1$. That estimate is the main novelty in his celebrated proof [25] that every sufficiently large odd integer is the sum of three primes. In the sharper form given in [27, Chapter 6], Vinogradov's result states (essentially) that if a and q are integers satisfying

$$q \ge 1, \quad (a, q) = 1, \quad |q\alpha - a| < q^{-1}, \tag{1.2}$$

one has

$$f(\alpha) \ll q^{\varepsilon} P \left(q^{-1} + P^{-2/5} + qP^{-1}\right)^{1/2} \tag{1.3}$$

for any fixed $\varepsilon > 0$. Vinogradov also obtained estimates for $f(\alpha)$ with $k \ge 2$ and used them to give the first unconditional results concerning the Waring–Goldbach problem. When $k \ge 2$, the sharpest estimates for $f(\alpha)$ obtained by Vinogradov's method were proven by Harman [3, 4]. In particular, he showed in [3] that if (1.2) holds, one has

$$f(\alpha) \ll P^{1+\varepsilon} \left(q^{-1} + P^{-1/2} + qP^{-k}\right)^{4^{1-k}}. \tag{1.4}$$

Vinogradov's approach does not rely heavily on the particular form of the phases in (1.1) and can be applied to more general sums (see [3, 28]). In 1991, Baker and Harman [1] demonstrated that, using the diophantine properties of the sequence am^k/q, one can derive sharper bounds for $f(a/q)$ with $k \ge 2$. They proved (essentially) that if q is near $P^{k/2}$ and $(a, q) = 1$, one has

$$f(a/q) \ll P^{1-\rho(k)+\varepsilon},$$

where $\rho(2) = 1/7$ and $\rho(k) = \frac{2}{3} \times 2^{-k}$ for $k \ge 3$. They applied this bound to obtain new results on the distribution of αp^k modulo one. On the other hand, research on topics related to the Waring–Goldbach problem prompted several authors to give improvements on (1.4)

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valid for all real α. The sharpest result of that kind asserts (in a slightly stronger form) that if $k \geq 3$ and a and q are integers with

$$1 \leq q \leq P^{k/2}, \quad (a, q) = 1, \quad |q\alpha - a| < P^{-k/2},$$

one has

$$f(\alpha) \ll P^{1-\rho(k)+\varepsilon} + \frac{q^{-(1/2k)+\varepsilon} P (\log P)^4}{(1 + P^k |\alpha - a/q|)^{1/2}}, \tag{1.5}$$

where $\rho(3) = 1/24$ and $\rho(k) = 2^{-k-1}$ if $k \geq 4$. This was proven by Kawada and Wooley [11] in the case $k \geq 4$ and by Wooley [29] in the case $k = 3$. In the present paper, we combine the Kawada–Wooley and the Baker–Harman methods and obtain the following improvement on the first term on the right side of (1.5).

Theorem 1. *Let $k \geq 3$ and define*

$$\rho(k) = \begin{cases} 1/14, & \text{if } k = 3, \\ \frac{2}{3} \times 2^{-k}, & \text{if } k \geq 4. \end{cases} \tag{1.6}$$

Suppose that $\alpha \in \mathbb{R}$ and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying

$$1 \leq q \leq Q, \quad (a, q) = 1, \quad |q\alpha - a| < Q^{-1} \tag{1.7}$$

with

$$Q = P^{(k^2 - 2k\rho(k))/(2k-1)}. \tag{1.8}$$

Then for any fixed $\varepsilon > 0$ one has

$$f(\alpha) \ll P^{1-\rho(k)+\varepsilon} + \frac{q^{-1/2k} P^{1+\varepsilon}}{(1 + P^k |\alpha - a/q|)^{1/2}}, \tag{1.9}$$

where the implied constant depends at most on k and ε.

The proof of Theorem 1 uses machinery from additive number theory and diophantine approximation. If α is close to a rational a/q with a small denominator, we are able to obtain a substantially sharper result using methods from multiplicative number theory. Developing an approach introduced by Linnik [15] and applied by several authors to derive versions of Vinogradov's bound (1.3) for the linear sum $f(\alpha)$, we prove the following 'major arc' estimate.

Theorem 2. *Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, and suppose that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying*

$$1 \leq q \leq Q, \quad (a, q) = 1, \quad |q\alpha - a| < QP^{-k} \tag{1.10}$$

with $Q \leq P$. Then for any fixed $\varepsilon > 0$ one has

$$f(\alpha) \ll Q^{1/2} P^{11/20+\varepsilon} + \frac{q^{\varepsilon} P (\log P)^c}{(q + P^k |q\alpha - a|)^{1/2}}, \tag{1.11}$$

where $c > 0$ is an absolute constant and the constant implied in $\ll$ depends at most on k and ε.

When $k = 1$, Theorem 2 can be used in two ways. Choosing $Q = P^{1/2}$, we essentially recover (1.3), a result that has previously been inaccessible via multiplicative methods. Alternatively, applying (1.11) with $Q \le P^{1/2-\varepsilon}$, we obtain an estimate that is sharper than (1.3), but is not applicable for all $\alpha \in \mathbb{R}$. When $k \ge 2$, only the latter scenario occurs. However, in this case, the resulting estimate—when applicable—is quite sharp. For example, if $q \le P^{9/20}$, we obtain

$$f(a/q) \ll Pq^{-1/2+\varepsilon},$$

which is also the estimate one obtains for $k \ge 2$ on the assumption of the Generalized Riemann Hypothesis, albeit in the slightly longer range $q \le P^{1/2}$.

Combining Theorem 2 with Theorem 1 and (1.4), we obtain the following result.

Theorem 3. *Let $k \ge 2$, let $\rho(k)$ be defined by (1.6) for $k \ge 3$, and let $\rho(2) = 1/8$. Suppose that $\alpha \in \mathbb{R}$ and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying (1.7) with*

$$Q = \begin{cases} P^{3/2}, & \text{if } k = 2, \\ P^{(k^2-2k\rho(k))/(2k-1)}, & \text{if } k \ge 3. \end{cases}$$

Then for any fixed $\varepsilon > 0$ one has

$$f(\alpha) \ll P^{1-\rho(k)+\varepsilon} + \frac{q^{\varepsilon} P(\log P)^c}{(q + P^k|q\alpha - a|)^{1/2}}, \tag{1.12}$$

where $c > 0$ is an absolute constant and the constant implied in $\ll$ depends at most on k and ε.

Theorems 2 and 3 enable us to make progress in several questions related to the Waring–Goldbach problem. Using Theorem 3, we can deduce new estimates for cardinalities of exceptional sets for sums of powers of primes. For example, replacing (1.4) by (1.12) in a recent work by Liu and Zhan [18] on sums of three squares of primes, we obtain the following result.

Theorem 4. *Let*

$$\mathcal{N} = \{n \in \mathbb{N} : n \equiv 3 \pmod{24},\ n \not\equiv 0 \pmod{5}\}.$$

Then for any fixed $\varepsilon > 0$ all but $O\left(x^{7/8+\varepsilon}\right)$ integers $n \in \mathcal{N} \cap (1, x]$ can be expressed as the sum of three squares of prime numbers.

Theorem 4 improves on [18, Theorem 1], in which the bound for the number of possible exceptions is $O\left(x^{11/12+\varepsilon}\right)$. Using our bounds for cubic Weyl sums, one can also sharpen the estimates of Wooley [29] for exceptional sets for sums of cubes of primes. The author [13] has proved the following theorem.

Theorem 5. *Let $5 \le s \le 8$ be an integer. Define θ_s and the sets $\mathcal{N}_s$ by*

$$\theta_5 = 79/84, \quad \theta_6 = 31/35, \quad \theta_7 = 51/84, \quad \theta_8 = 23/84;$$

$$\mathcal{N}_5 = \{n \in \mathbb{N} :\ n \equiv 1 \pmod{2},\ n \not\equiv 0, \pm 2 \pmod{9},\ n \not\equiv 0 \pmod{7}\},$$

$$\mathcal{N}_6 = \{n \in \mathbb{N} :\ n \equiv 0 \pmod{2},\ n \not\equiv \pm 1 \pmod{9}\},$$

$$\mathcal{N}_7 = \{n \in \mathbb{N} :\ n \equiv 1 \pmod{2},\ n \not\equiv 0 \pmod{9}\},$$

$$\mathcal{N}_8 = \{n \in \mathbb{N} :\ n \equiv 0 \pmod{2}\}.$$

Then all but $O\left(x^{\theta_s}\right)$ integers $n \in \mathcal{N}_s \cap (1, x]$ can be represented as the sum of s cubes of prime numbers.

The respective exponents θ_s in Wooley [29] are as follows:

$$\theta_5 = 35/36, \quad \theta_6 = 17/18, \quad \theta_7 = 23/36, \quad \theta_8 = 11/36.$$

Estimates for exceptional sets of the above type depend on one's ability to apply the Hardy–Littlewood circle method with a set of major arcs that is significantly larger than the 'standard' set of major arcs in the Waring–Goldbach problem. Let

$$\mathfrak{M} = \mathfrak{M}(Q, P) = \bigcup_{1 \le q \le Q} \bigcup_{\substack{1 \le a \le q \\ (a,q)=1}} \left\{ \alpha \in \mathbb{R} : |q\alpha - a| < QP^{-k} \right\}, \tag{1.13}$$

and define

$$S^*(q, a) = \sum_{\substack{x=1 \\ (x,q)=1}}^{q} e\left(\frac{ax^k}{q}\right), \quad v(\beta) = \int_P^{2P} \frac{e\left(\beta y^k\right)}{\log y} dy,$$

$$\mathfrak{S}_{k,s}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{S^*(q,a)^s}{\phi(q)^s} e\left(-\frac{an}{q}\right), \quad J_{k,s}(n) = \int_{\mathbb{R}} v(\beta)^s e(-n\beta) d\beta.$$

Applications of the circle method to the Waring–Goldbach problem require approximations of the form

$$\int_{\mathfrak{M}} f(\alpha)^s e(-n\alpha) d\alpha \approx \mathfrak{S}_{k,s}(n) J_{k,s}(n),$$

with Q as large as possible. The standard approach toward such approximations (see Hua [10, Chapter 7]) works when $Q \le (\log P)^A$ for some fixed $A > 0$. Starting with celebrated works by Vaughan [22] and Montgomery and Vaughan [20], this traditional barrier has been broken in some special cases (see [16, 17, 20, 21]), but so far the general result has withstood improvement. Using Theorem 2, we can change that. The author [13, Proposition 1] has established the following general theorem.

Theorem 6. *Let k, s and n be integers with $k \ge 2$, $s \ge 5$ and $P^k \ll n \ll P^k$. Let $\varepsilon > 0$ be fixed, and let $\mathfrak{M}$ be defined by (1.13) with $Q \le P^{1/2-\varepsilon}$. Then for any $A > 0$ one has*

$$\int_{\mathfrak{M}} f(\alpha)^s e(-n\alpha) d\alpha = \mathfrak{S}_{k,s}(n) J_{k,s}(n) + O\left(P^{s-k} (\log P)^{-A}\right), \tag{1.14}$$

where the implied constant depends at most on A, k, s and ε.

Remark 1.1. A comment is in order regarding the proofs of Theorems 1 and 2. As usual in such matters, we reduce the estimation of sums over primes to the estimation of multiple sums. However, instead of applying combinatorial identities such as Vaughan's [23] or Heath-Brown's [9], we use a sieve argument due to Harman [6]. This makes the proofs of the theorems a little longer, but has the added benefit that, in the process, we obtain also estimates for certain Weyl sums over almost primes free of small prime divisors (see Lemmas 3.3 and 5.6 below). Such estimates are of independent interest, since Weyl sums over almost primes arise naturally in applications in which we want to combine the circle method with sieve methods. For example, the proof of Theorem 5 uses sieve ideas and Lemma 3.3, whereas the respective 'sievefree' result relying on Theorem 3 provides the somewhat weaker exponents

$$\theta_5 = 20/21, \quad \theta_6 = 19/21, \quad \theta_7 = 13/21, \quad \theta_8 = 2/7.$$

Estimates for Weyl sums over almost primes were also crucial in the author's work [14] on the Waring–Goldbach for seventh powers.

Remark 1.2. After the work on this paper was completed, the author learned that Professor Harman [7] has obtained independently Theorem 1 for $k \geq 5$. His proof also depends on an interaction between the methods in [1] and [11], but there are some differences in the details. Furthermore, Harman and the author [8] have obtained a further improvement on Theorem 4: by [8, Theorem 1], the number of exceptional integers counted in Theorem 4 is $O\left(x^{6/7+\varepsilon}\right)$.

Notation. Throughout the paper, the letter ε denotes a sufficiently small positive real number. Any statement in which ε occurs holds for each positive ε, and any implied constant in such a statement is allowed to depend on ε. Implicit constants are also allowed to depend on k. Any additional dependence will be mentioned explicitly. The letter p, with or without indices, is reserved for prime numbers; c denotes an absolute constant, not necessarily the same in all occurrences. Also, we use P to denote the 'main parameter' and write $L = \log P$.

As usual in number theory, $\mu(n)$, $\phi(n)$ and $\tau(n)$ denote, respectively, the Möbius function, the Euler totient function and the number of divisors function. We write $e(x) = \exp(2\pi i x)$ and $(a,b) = \gcd(a,b)$ and use $\chi(n)$ to denote Dirichlet characters, sometimes referring to the function χ^0, defined by taking $\chi^0(n) = 1$ for all $n \in \mathbb{N}$, as the 'trivial character'. Also, we use $m \sim M$ and $m \asymp M$ as abbreviations for the conditions $M \leq m < 2M$ and $c_1 M \leq m < c_2 M$, and $\sum_{\chi \bmod q}$ to denote summation over the Dirichlet characters mod q. Finally, if $z \geq 2$, we define

$$\psi(n,z) = \begin{cases} 1, & \text{if } (n, \mathcal{P}(z)) = 1, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{where} \quad \mathcal{P}(z) = \prod_{p<z} p. \tag{1.15}$$

2. Auxiliary results

When $k \geq 3$, we define the multiplicative function $w_k(q)$ by

$$w_k\left(p^{ku+v}\right) = \begin{cases} kp^{-u-1/2}, & \text{if } u \geq 0, v = 1, \\ p^{-u-1}, & \text{if } u \geq 0, v = 2, \ldots, k. \end{cases}$$

This function enters our analysis through applications of the following result.

Lemma 2.1. *Let k be an integer with $k \geq 3$ and let $0 < \rho \leq 2^{1-k}$. Also, let $X \geq 2$ and let $\mathcal{I}$ be any subinterval of $[X, 2X)$. Then either*

$$\sum_{x \in \mathcal{I}} e\left(\alpha x^k\right) \ll X^{1-\rho+\varepsilon}, \tag{2.1}$$

or there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that

$$1 \leq q \leq X^{k\rho}, \quad (a,q) = 1, \quad |q\alpha - a| \leq X^{k(\rho-1)}, \tag{2.2}$$

and

$$\sum_{x\in\mathcal{I}} e\left(\alpha x^k\right) \ll \frac{q^\varepsilon w_k(q)X}{1+X^k|\alpha-a/q|} + X^{1/2+\varepsilon}. \tag{2.3}$$

Proof. By Dirichlet's theorem on diophantine approximation, there exist $a\in\mathbb{Z}$, $q\in\mathbb{N}$ with

$$1\le q\le X^{k-1},\quad (a,q)=1,\quad |q\alpha-a|\le X^{1-k}.$$

If $q>X$, Weyl's inequality [24, Lemma 2.4] yields (2.1) with $\rho=2^{1-k}$. By the argument of [24, Theorem 4.2],

$$\frac{1}{q}\sum_{x=1}^{q} e\left(\frac{ax^k}{q}\right) \ll w_k(q), \tag{2.4}$$

whenever $(a,q)=1$. If $q\le X$, we deduce (2.3) from (2.4) and [24, Lemmas 6.1 and 6.2]. Thus, at least one of (2.1) and (2.3) holds, and the lemma follows on noting that when conditions (2.2) fail, then (2.3) implies (2.1). □

The following lemma is a slight variation of [1, Lemma 6]. The proof is the same.

Lemma 2.2. *Let q and X be positive integers exceeding 1 and let $0<\delta<\frac{1}{2}$. Suppose that $q\nmid a$ and denote by $\mathcal{S}$ the number of integers x such that*

$$X\le x<2X,\quad (x,q)=1,\quad \left\|ax^k/q\right\|<\delta,$$

where $\|\alpha\|=\min_{n\in\mathbb{Z}}|\alpha-n|$. Then

$$\mathcal{S}\ll\delta q^\varepsilon(q+X).$$

Next, we list some mean-value estimates for Dirichlet polynomials. We define the Dirichlet polynomials

$$M(s,\chi)=\sum_{m\asymp M}\xi_m\chi(m)m^{-s},\quad N(s,\chi)=\sum_{n\asymp N}\eta_n\chi(n)n^{-s}, \tag{2.5}$$

$$R(s,\chi)=\sum_{r\asymp R}\delta_r\chi(r)r^{-s},\quad K(s,\chi)=\sum_{r\asymp K}\chi(r)r^{-s}, \tag{2.6}$$

where the coefficients ξ_m,η_n,δ_r are complex numbers such that

$$|\xi_m|\le\tau(m)^c,\quad |\eta_n|\le\tau(n)^c,\quad |\delta_r|\le\tau(r)^c.$$

Lemma 2.3. *Suppose that $M\ge N\ge 2$ and $M(s,\chi),N(s,\chi)$ are defined by (2.5). Further, set $P=MN$ and suppose that $1\le q,T\le P^c$. Then*

$$\sum_{\chi \bmod q}\int_{-T}^{T}\left|MN\left(\tfrac12+it,\chi\right)\right|dt \ll L^c\left(P^{1/2}+(qTM)^{1/2}+qT\right).$$

Proof. This follows from the mean-value theorem for Dirichlet polynomials [19, Theorem 6.4] and Cauchy's inequality. □

Lemma 2.4. *Suppose that $M\ge N\ge 2$, $R\ge 2$, and $M(s,\chi),N(s,\chi),R(s,\chi)$ are defined by (2.5) and (2.6). Further, set $P=MNR$ and suppose that $1\le q,T\le P^c$ and $R\le P^{8/35}$. Then*

$$\sum_{\chi \bmod q}\int_{-T}^{T}\left|MNR\left(\tfrac12+it,\chi\right)\right|dt \ll L^c\left(P^{1/2}+(qTM)^{1/2}+qTP^{1/20}\right). \tag{2.7}$$

Proof. This is a variant of [2, Lemma 4]. If $M \geq P^{9/20}$, the upper bound (2.7) follows from Lemma 2.3, so we may assume that $M \leq P^{9/20}$. Let Δ denote the right side of (2.7). Following the proof of [2, Lemma 4] without referring to (3.2), (3.23) or (3.30) in [2], we obtain

$$\text{(2.8)}\quad L^{-c} \sum_{\chi \bmod q} \int_{-T}^{T} \left|MNR\left(\tfrac{1}{2}+it,\chi\right)\right| dt \ll \Delta_1 + \Delta_2 + (qT)^{1/2}(PM^5)^{1/12} + (qT)^{3/4}(PR^7)^{1/16} + (qT)^{1/4}(P(MR)^3)^{1/8},$$

where $\Delta_1 = P^{1/2}$ and $\Delta_2 = qTP^{11/20}$. By the hypotheses $M \leq P^{9/20}$ and $R \leq P^{8/35}$,

$$(qT)^{1/2}(PM^5)^{1/12} \leq (qT)^{1/2}P^{13/48} \leq \Delta_1^{1/2}\Delta_2^{1/2} \leq \Delta,$$
$$(qT)^{3/4}(PR^7)^{1/16} \leq (qT)^{3/4}P^{13/80} = \Delta_1^{1/4}\Delta_2^{3/4} \leq \Delta,$$

and

$$(qT)^{1/4}(P(MR)^3)^{1/8} \leq (qT)^{1/4}P^{85/224} \leq \Delta_1^{3/4}\Delta_2^{1/4} \leq \Delta.$$

Thus, (2.7) follows from (2.8). □

Lemma 2.5. *Suppose that $K, T \geq 2$ and $K(s,\chi)$ is defined by (2.6). Then*

$$\sideset{}{'}\sum_{\chi \bmod q} \int_{-T}^{T} \left|K\left(\tfrac{1}{2}+it,\chi\right)\right|^4 dt \ll qTL^c,$$

where $L = \log(2qTK)$ and $\sum'$ denotes summation over the non-principal characters mod q. *Also, if χ^0 is the trivial character and $K \leq T^2$, we have*

$$\int_{T}^{2T} \left|K\left(\tfrac{1}{2}+it,\chi^0\right)\right|^4 dt \ll TL^c.$$

Proof. The proof of this result is similar to the proof of [12, Lemma 5]. □

Lemma 2.6. *Suppose that $M \geq N \geq 2$, $K \geq 2$, and $M(s,\chi), N(s,\chi), K(s,\chi)$ are defined by (2.5) and (2.6). Further, set $P = MNK$ and suppose that $1 \leq q, T \leq P^c$. Then*

$$\sideset{}{'}\sum_{\chi \bmod q} \int_{-T}^{T} \left|MNK\left(\tfrac{1}{2}+it,\chi\right)\right| dt \ll L^c\left(P^{1/2} + (qTM)^{1/2} + qTP^{1/20}\right),$$

where $\sum'$ denotes summation over the non-principal characters mod q *or over $\chi = \chi^0$, the trivial character, according as $q > 1$ or $q = 1$.*

Proof. The proof is similar to the proof of [2, Lemma 10] under hypothesis (3.39) in [2], with Lemmas 2.4 and 2.5 playing the roles of Lemmas 4 and 9 in [2]. □

The next lemma is a simple tool that reduces the estimation of a bilinear sum to the estimation of a similar sum subject to 'nicer' summation conditions.

Lemma 2.7. *Let $\Phi : \mathbb{N} \to \mathbb{C}$ satisfy $|\Phi(x)| \leq X$, let $M, N \geq 2$, and define the bilinear form*

$$\mathcal{B}(M,N) = \sum_{\substack{m\sim M \\ m<n}} \sum_{n\sim N} \xi_m \eta_n \Phi(mn),$$

where $|\xi_m| \le 1$, $|\eta_n| \le 1$. *Then*

$$\mathcal{B}(M,N) \ll L \left| \sum_{m\sim M} \sum_{n\sim N} \xi'_m \eta'_n \Phi(mn) \right| + 1, \tag{2.9}$$

where $|\xi'_m| \le |\xi_m|$, $|\eta'_n| \le |\eta_n|$ *and* $L = \log(2MNX)$. *The same estimate holds, if we replace the summation condition* $m < n$ *in the definition of* $\mathcal{B}(M,N)$ *with* $U \le mn < U'$.

Proof. Suppose that $\mathcal{B}(M,N)$ is subject to the condition $m < n$ (the alternative case can be dealt with in a similar fashion). We recall the truncated Perron formula

$$\frac{1}{2\pi i} \int_{b-iT_1}^{b+iT_1} \frac{u^w}{w} dw = E(u) + O\left(\frac{u^b}{T_1 |\log u|} \right), \tag{2.10}$$

where $b > 0$ and $E(u)$ is 0 or 1 according as $0 < u < 1$ or $u > 1$. By (2.10) with $b = L^{-1}$ and $T_1 = (MNX)^2$,

$$\mathcal{B}(M,N) = \frac{1}{2\pi i} \int_{b-iT_1}^{b+iT_1} \sum_{m\sim M} \sum_{n\sim N} \xi_m \eta_n \Phi(mn) \left(\frac{n}{m+1/2} \right)^w \frac{dw}{w} + O(1),$$

whence (2.9) follows upon choosing

$$\xi'_m = \xi_m (m+1/2)^{-w_0} \quad \text{and} \quad \eta'_n = \tfrac{1}{2} \eta_n n^{w_0}$$

for a suitable w_0 with $\operatorname{Re} w_0 = b$. □

3. Multilinear Weyl sums, I

In this section, we obtain upper bounds for the multilinear Weyl sums appearing in the proof of Theorem 1. Our first result—a *Type II sum* estimate—is a variant of [11, Lemma 3.1] and [29, Lemma 2.1].

Lemma 3.1. *Let* $k \ge 3$ *and* $0 < \rho < \left(2^k + 2\right)^{-1}$. *Suppose that* $\alpha \in \mathbb{R}$ *and that there exist* $a \in \mathbb{Z}$ *and* $q \in \mathbb{N}$ *such that* (1.7) *holds with* Q *subject to*

$$P^{4k\rho} \le Q \le P^{k-2k\rho}. \tag{3.1}$$

Let $M \ge N \ge 2$, $|\xi_m| \le 1$, $|\eta_n| \le 1$, *and define*

$$g(\alpha) = \sum_{\substack{m\sim M \\ mn\sim P}} \sum_{n\sim N} \xi_m \eta_n e\left(\alpha (mn)^k\right).$$

Then

$$g(\alpha) \ll P^{1-\rho+\varepsilon} + \frac{w_k(q)^{1/2} P^{1+\varepsilon}}{\left(1 + P^k |\alpha - a/q|\right)^{1/2}},$$

provided that

$$\max\left(P^{2^k\rho}, P^{(k-1+4k\rho)/(2k-1)} \right) \le M \le P^{1-2\rho}. \tag{3.2}$$

Proof. We follow the proof of [11, Lemma 3.1]. Let $\mathcal{I}(n_1, n_2)$ be the (possibly empty) interval $[M, 2M) \cap [P/n_1, 2P/n_2)$ and define

$$T_1(\alpha) = \sum_{N \le n_1 < n_2 < 2N} \left| \sum_{m \in \mathcal{I}(n_1,n_2)} e\left(\alpha\left(n_2^k - n_1^k\right) m^k\right) \right|.$$

By Cauchy's inequality and an interchange of the order of summation,

$$|g(\alpha)|^2 \ll PM + MT_1(\alpha). \tag{3.3}$$

Define σ by $M^\sigma = P^{2\rho} L^{-1}$ and denote by $\mathcal{N}$ the set of pairs (n_1, n_2) with $n_j \sim N$ for which there exist $b \in \mathbb{Z}$, $r \in \mathbb{N}$ such that

$$1 \le r \le M^{k\sigma}, \quad (b, r) = 1, \quad \left| r\left(n_2^k - n_1^k\right)\alpha - b \right| \le M^{k(\sigma-1)}. \tag{3.4}$$

By (3.2), we may suppose that $\sigma < 2^{1-k}$. We can then apply Lemma 2.1 with $\rho = \sigma$ to the inner summation in $T_1(\alpha)$. We get

$$T_1(\alpha) \ll NP^{1-2\rho+\varepsilon} + T_2(\alpha), \tag{3.5}$$

where

$$T_2(\alpha) = \sum_{(n_1,n_2) \in \mathcal{N}} \frac{w_k(r)M}{1 + M^k \left|\left(n_2^k - n_1^k\right)\alpha - b/r\right|}.$$

We now change the summation variables in $T_2(\alpha)$ to

$$d = (n_1, n_2), \quad n = n_1/d, \quad h = (n_2 - n_1)/d.$$

We obtain

$$T_2(\alpha) \ll \sum_{dh \le N} \sum_{n} \frac{w_k(r)M}{1 + M^k \left|hd^k R(n,h)\alpha - b/r\right|}, \tag{3.6}$$

where $R(n, h) = \left((n + h)^k - n^k\right)/h$ and the inner summation is over n satisfying

$$n \sim Nd^{-1}, \quad (n, h) = 1, \quad (nd, (n + h)d) \in \mathcal{N}.$$

For each pair (d, h) appearing in the summation on the right side of (3.6), Dirichlet's theorem on diophantine approximation yields $b_1 \in \mathbb{Z}$ and $r_1 \in \mathbb{N}$ with

$$1 \le r_1 \le M^k P^{-2k\rho}, \quad (b_1, r_1) = 1, \quad \left|r_1 hd^k\alpha - b_1\right| \le P^{2k\rho} M^{-k}. \tag{3.7}$$

As $R(n, h) \le 3^k N^{k-1}$, combining (3.2), (3.4) and (3.7), we get

$$\begin{aligned} |b_1 r R(n,h) - br_1| &\le r_1 M^{k(\sigma-1)} + rR(n,h) P^{2k\rho} M^{-k} \\ &\le L^{-k} + 3^{3k} P^{k-1+4k\rho} M^{-(2k-1)} L^{-k} < 1. \end{aligned}$$

Hence,

$$\frac{b}{r} = \frac{b_1 R(n,h)}{r_1}, \quad r = \frac{r_1}{(r_1, R(n,h))}. \tag{3.8}$$

Combining (3.6) and (3.8), we obtain

$$T_2(\alpha) \ll \sum_{dh \le N} \frac{M}{1 + M^k N_1^{k-1} \left|hd^k\alpha - b_1/r_1\right|} \sum_{\substack{n \sim N_1 \\ (n,h)=1}} w_k\left(\frac{r_1}{(r_1, R(n,h))}\right), \tag{3.9}$$

where $N_1 = Nd^{-1}$. By [11, eq. (3.11)],

$$\sum_{\substack{n\sim N_1\\(n,h)=1}} w_k\left(\frac{r_1}{(r_1,R(n,h))}\right) \ll r_1^{\varepsilon} w_k(r_1)N_1 + r_1^{\varepsilon},$$

so we deduce from (3.9) that

$$T_2(\alpha) \ll T_3(\alpha) + P^{1+\varepsilon}, \tag{3.10}$$

where

$$T_3(\alpha) = \sum_{dh\le N} \frac{r_1^{\varepsilon} w_k(r_1)MN_1}{1+M^kN_1^{k-1}\left|hd^k\alpha - b_1/r_1\right|}.$$

We now write $\mathcal{H}$ for the set of pairs (d,h) with $dh \le N$ for which there exist $b_1 \in \mathbb{Z}$ and $r_1 \in \mathbb{N}$ subject to

$$1 \le r_1 \le P^{2k\rho}, \quad (b_1,r_1)=1, \quad \left|r_1hd^k\alpha - b_1\right| \le NP^{k(2\rho-1)}. \tag{3.11}$$

We have

$$T_3(\alpha) \ll T_4(\alpha) + NP^{1-2\rho+\varepsilon}, \tag{3.12}$$

where

$$T_4(\alpha) = \sum_{(d,h)\in\mathcal{H}} \frac{r_1^{\varepsilon} w_k(r_1)MN_1}{1+M^kN_1^{k-1}\left|hd^k\alpha - b_1/r_1\right|}.$$

For each $d \le N$, Dirichlet's theorem on diophantine approximation provides $b_2 \in \mathbb{Z}$ and $r_2 \in \mathbb{N}$ with

$$1 \le r_2 \le \tfrac{1}{2}P^{k(1-2\rho)}N^{-1}, \quad (b_2,r_2)=1, \quad \left|r_2d^k\alpha - b_2\right| \le 2NP^{k(2\rho-1)}. \tag{3.13}$$

Combining (3.11) and (3.13), we obtain

$$\begin{aligned}|b_2r_1h - b_1r_2| &\le r_2NP^{k(2\rho-1)} + 2r_1hNP^{k(2\rho-1)}\\ &\le \tfrac{1}{2} + 2N^2P^{k(4\rho-1)} < 1,\end{aligned}$$

whence

$$\frac{b_1}{r_1} = \frac{hb_2}{r_2}, \quad r_1 = \frac{r_2}{(r_2,h)}.$$

Therefore, on writing $Z = M^kN_1^{k-1}\left|d^k\alpha - b_2/r_2\right|$, we deduce that

$$T_4(\alpha) \le \sum_{d\le N}\sum_{h<N_1} \frac{r_2^{\varepsilon}MN_1}{1+Zh} w_k\left(\frac{r_2}{(r_2,h)}\right) \ll \sum_{d\le N} \frac{r_2^{\varepsilon}w_k(r_2)MN^2L}{d^2(1+ZNd^{-1})}.$$

Here we have used the estimate

$$\sum_{d\sim D} w_k\left(r/\left(r,d^j\right)\right) \ll r^{\varepsilon}w_k(r)D \qquad (1\le j\le k), \tag{3.14}$$

which can be established similarly to [11, Lemma 2.3]. Hence,

$$T_4(\alpha) \ll T_5(\alpha) + NP^{1-2\rho+\varepsilon}, \tag{3.15}$$

where

$$T_5(\alpha) = \sum_{d\in\mathcal{D}} \frac{w_k(r_2)NP^{1+\varepsilon}}{d^2\left(1+P^kd^{-k}\left|d^k\alpha - b_2/r_2\right|\right)}$$

and $\mathcal{D}$ is the set of integers $d \le P^{2\rho}$ for which there exist $b_2 \in \mathbb{Z}$ and $r_2 \in \mathbb{N}$ with

$$1 \le r_2 \le P^{2k\rho}L^{-1}, \quad (b_2, r_2) = 1, \quad \left|r_2 d^k\alpha - b_2\right| \le P^{k(2\rho-1)}L^{-1}. \tag{3.16}$$

Combining (3.16) and the hypotheses (1.7) and (3.1), we deduce that

$$\begin{aligned}\left|r_2 d^k a - b_2 q\right| &\le r_2 d^k Q^{-1} + qP^{k(2\rho-1)}L^{-1} \\ &\le P^{4k\rho}Q^{-1}L^{-1} + QP^{k(2\rho-1)}L^{-1} \le 2L^{-1} < 1,\end{aligned}$$

whence

$$\frac{b_2}{r_2} = \frac{d^k a}{q}, \quad r_2 = \frac{q}{(q, d^k)}.$$

Thus, recalling (3.14), we get

$$T_5(\alpha) \ll \frac{NP^{1+\varepsilon}}{1 + P^k\left|\alpha - a/q\right|} \sum_{d \le P^{2\rho}} w_k\left(q/(q, d^k)\right) d^{-2} \ll \frac{w_k(q)NP^{1+\varepsilon}}{1 + P^k\left|\alpha - a/q\right|}. \tag{3.17}$$

The lemma follows from (3.2), (3.3), (3.5), (3.10), (3.12), (3.15) and (3.17). □

The next lemma provides an estimate for trilinear sums usually referred to as *Type I/II sums*.

Lemma 3.2. *Let $k \ge 3$ and $0 < \rho < 2^{1-k}$. Suppose that $\alpha \in \mathbb{R}$ and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that* (1.7) *holds with Q given by* (1.8)*. Let $M, N, X \ge 2$, $|\xi_m| \le 1$, $|\eta_n| \le 1$, and define*

$$g(\alpha) = \sum_{\substack{m\sim M \\ mnx\sim P}} \sum_{n\sim N} \sum_{x\sim X} \xi_m\eta_n e\left(\alpha(mnx)^k\right).$$

Then

$$g(\alpha) \ll P^{1-\rho+\varepsilon} + \frac{w_k(q)P^{1+\varepsilon}}{1 + P^k|\alpha - a/q|},$$

provided that

$$M \le P^{(k-(2k+1)\rho)/(2k-1)}, \quad MN \le P^{1-2^{k-1}\rho}, \quad MN^2 \le P^{1-2\rho}. \tag{3.18}$$

Proof. Define σ by $X^\sigma = P^\rho L^{-1}$ and denote by $\mathcal{M}$ the set of pairs (m, n) with $m \sim M$ and $n \sim N$ for which there exist $b_1 \in \mathbb{Z}$ and $r_1 \in \mathbb{N}$ with

$$1 \le r_1 \le X^{k\sigma}, \quad (b_1, r_1) = 1, \quad \left|r_1(mn)^k\alpha - b_1\right| \le X^{k(\sigma-1)}. \tag{3.19}$$

Noting that (3.18) implies $\sigma < 2^{1-k}$, we apply Lemma 2.1 to the summation over x and get

$$g(\alpha) \ll T_1(\alpha) + P^{1-\rho+\varepsilon}, \tag{3.20}$$

where

$$T_1(\alpha) = \sum_{(m,n)\in\mathcal{M}} \frac{w_k(r_1)X}{1 + X^k\left|(mn)^k\alpha - b_1/r_1\right|}.$$

For each $m \sim M$, we apply Dirichlet's theorem on diophantine approximation to find $b \in \mathbb{Z}$ and $r \in \mathbb{N}$ with

$$1 \le r \le X^kP^{-k\rho}, \quad (b, r) = 1, \quad \left|rm^k\alpha - b\right| \le P^{k\rho}X^{-k}. \tag{3.21}$$

By (3.18), (3.19) and (3.21),

$$\begin{aligned}\left|b_1 r - bn^k r_1\right| &\le rX^{k(\sigma-1)} + r_1 n^k P^{k\rho} X^{-k} \\ &\le L^{-k} + 2^{4k} P^{k(2\rho-1)} (MN^2)^k L^{-k} \le 2^{4k+1} L^{-k} < 1,\end{aligned}$$

whence

$$\frac{b_1}{r_1} = \frac{n^k b}{r}, \quad r_1 = \frac{r}{(r, n^k)}.$$

Thus, by (3.14),

$$\begin{aligned}T_1(\alpha) &\ll \sum_{m\sim M} \frac{X}{1 + (NX)^k \left|m^k\alpha - b/r\right|} \sum_{n\sim N} w_k\left(\frac{r}{(r,n^k)}\right) \\ &\ll \sum_{m\sim M} \frac{r^\varepsilon w_k(r) NX}{1 + (NX)^k \left|m^k\alpha - b/r\right|}.\end{aligned} \tag{3.22}$$

Let $\mathcal{M}'$ be the set of integers $m \sim M$ for which there exist $b \in \mathbb{Z}$ and $r \in \mathbb{N}$ with

$$1 \le r \le P^{k\rho} L^{-1}, \quad (b, r) = 1, \quad \left|rm^k\alpha - b\right| \le P^{k(\rho-1)} M^k L^{-1}. \tag{3.23}$$

By (3.22),

$$T_1(\alpha) \ll T_2(\alpha) + P^{1-\rho+\varepsilon}, \tag{3.24}$$

where

$$T_2(\alpha) = \sum_{m\in\mathcal{M}'} \frac{r^\varepsilon w_k(r) NX}{1 + (NX)^k \left|m^k\alpha - b/r\right|}.$$

We now consider two cases depending on the size of q in (1.7).

Case 1. Suppose that $q \le P^{k(1-\rho)} M^{-k}$. In this case, we estimate $T_2(\alpha)$ as in the proof of Lemma 3.1. Combining (1.7), (1.8), (3.18) and (3.23), we obtain

$$\left|rm^k a - bq\right| \le qP^{k(\rho-1)} M^k L^{-1} + rm^k Q^{-1} \le 3^k L^{-1} < 1.$$

Therefore,

$$\frac{b}{r} = \frac{m^k a}{q}, \quad r = \frac{q}{(q, m^k)},$$

and by (3.14),

$$T_2(\alpha) \ll \frac{q^\varepsilon NX}{1 + P^k|\alpha - a/q|} \sum_{m\sim M} w_k\left(\frac{q}{(q,m^k)}\right) \ll \frac{w_k(q) P^{1+\varepsilon}}{1 + P^k|\alpha - a/q|}. \tag{3.25}$$

Case 2. Suppose that $q > P^{k(1-\rho)} M^{-k}$. In this case, we estimate $T_2(\alpha)$ by the method of [1, Lemma 10]. By a standard splitting argument,

$$T_2(\alpha) \ll \sum_{d|q} \sum_{m\in\mathcal{M}_d(R,Z)} \frac{w_k(r) NX^{1+\varepsilon}}{1 + (NX)^k (RZ)^{-1}}, \tag{3.26}$$

where

$$1 \le R \le P^{k\rho} L^{-1}, \quad P^{k(1-\rho)} M^{-k} L \le Z \le P^k M^{-k}, \tag{3.27}$$

and $\mathcal{M}_d(R, Z)$ is the subset of $\mathcal{M}'$ containing integers m subject to

$$(m, q) = d, \quad r \sim R, \quad \left|rm^k\alpha - b\right| < Z^{-1}.$$

We now estimate the inner sum on the right side of (3.26). We have

$$\sum_{m\in\mathcal{M}_d(R,Z)} w_k(r) \ll \sum_{r\sim R} w_k(r)\mathcal{S}_0(r), \tag{3.28}$$

where $\mathcal{S}_0(r)$ is the number of integers $m \sim M$ with $(m,q)=d$ for which there exists $b\in\mathbb{Z}$ such that

$$(b,r)=1 \quad \text{and} \quad \left|rm^k\alpha - b\right| < Z^{-1}. \tag{3.29}$$

By (1.7), (3.27) and (3.29),

$$\mathcal{S}_0(r) \le \mathcal{S}(r), \tag{3.30}$$

where we write $\mathcal{S}(r)$ for the number of integers m subject to

$$m\sim Md^{-1}, \quad (m,q')=1, \quad \left\|ard^{k-1}m^k/q'\right\| < \delta,$$

with $q'=qd^{-1}$, $\delta = Z^{-1}+2^{k+1}RM^k(qQ)^{-1}$ and $\|\theta\| = \min_{n\in\mathbb{Z}}|\theta-n|$. When $(q,rd^k)<q$, we appeal to Lemma 2.2 and, on noting that (3.18) implies $M\le P^{k(\rho-1)}M^{-k}<q$, we obtain

$$\mathcal{S}(r) \ll \delta q^{\varepsilon} d^{-1}(M+q) \ll \delta q^{1+\varepsilon}. \tag{3.31}$$

Combining (3.30) and (3.31), we get

$$\mathcal{S}_0(r) \ll \delta q^{1+\varepsilon}. \tag{3.32}$$

Since for each $m\sim M$ there is at most one pair (b,r) satisfying (3.29) and $r\sim R$, we have

$$\sum_{r\sim R}\mathcal{S}_0(r) \le \sum_{\substack{m\sim M\\(m,q)=d}} 1 \ll Md^{-1}+1, \tag{3.33}$$

and we also have the bounds

$$\sum_{r\sim R} w_k(r)^j \ll \begin{cases} R^{-1+\varepsilon}, & \text{if } k=3, j=4,\\ R^{-1+1/k}, & \text{if } k\ge 4, j=k,\end{cases} \tag{3.34}$$

which follow from [11, Lemma 2.4]. We now apply Hölder's inequality and then appeal to (3.32), (3.33) and (3.34). We obtain

$$\begin{aligned}\sum_{\substack{r\sim R\\(q,rd^3)<q}} w_3(r)\mathcal{S}_0(r) &\ll \left(\delta q^{1+\varepsilon}\right)^{1/4}\left(\sum_{r\sim R} w_3(r)^4\right)^{1/4}\left(\sum_{r\sim R}\mathcal{S}_0(r)\right)^{3/4}\\ &\ll \delta^{1/4}q^{1/4+\varepsilon}R^{-1/4}M^{3/4}.\end{aligned} \tag{3.35}$$

Similarly, if $k\ge 4$, we have

$$\begin{aligned}\sum_{\substack{r\sim R\\(q,rd^k)<q}} w_k(r)\mathcal{S}_0(r) &\ll \left(\delta q^{1+\varepsilon}\right)^{1/k}\left(\sum_{r\sim R} w_k(r)^k\right)^{1/k}\left(\sum_{r\sim R}\mathcal{S}_0(r)\right)^{1-1/k}\\ &\ll \delta^{1/k}q^{1/k+\varepsilon}R^{(1-k)/k^2}M^{(k-1)/k}.\end{aligned} \tag{3.36}$$

On the other hand, by (3.33),

$$\sum_{\substack{r\sim R\\(q,rd^k)=q}} w_k(r)\mathcal{S}_0(r) \ll R^{-1/k}\left(Md^{-1}+1\right) \ll Mq^{-1/k}+1, \tag{3.37}$$

on noting that the sum on the left side is empty unless $Rd^k \gg q$. Combining (3.28) and (3.35)–(3.37), we deduce

$$\sum_{m\in\mathcal{M}_d(R,Z)} w_3(r) \ll \delta^{1/4}q^{1/4+\varepsilon}R^{-1/4}M^{3/4}+Mq^{-1/3}+1 \tag{3.38}$$

and

$$\sum_{m\in\mathcal{M}_d(R,Z)} w_k(r) \ll \delta^{1/k}q^{1/k+\varepsilon}R^{(1-k)/k^2}M^{(k-1)/k}+Mq^{-1/k}+1 \tag{3.39}$$

for $k \geq 4$.

Substituting (3.38) into (3.26), we get

$$\begin{aligned} T_2(\alpha) &\ll \frac{M^{3/4}NX^{1+\varepsilon}}{1+(NX)^3(RZ)^{-1}}\left(\frac{Q}{RZ}+\frac{M^3}{Q}\right)^{1/4}+P^{1+\varepsilon}q^{-1/3}+NX^{1+\varepsilon}\\ &\ll \left(PQM^2\right)^{1/4+\varepsilon}+P^{1+\varepsilon}\left(M^2Q^{-1}\right)^{1/4}+MP^{\rho+\varepsilon}+NX^{1+\varepsilon}. \end{aligned}$$

The choice of Q and the hypothesis (3.18) of the lemma ensure that the first three terms on the right side of the last inequality are $\ll P^{1-\rho+\varepsilon}$; furthermore, in conjunction with the hypothesis $q > P^{3-3\rho}M^{-3}$ of the present case, the definition of Q in (1.8) implies $NX \ll P^{1-\rho}$. Therefore, if $k=3$,

$$T_2(\alpha) \ll P^{1-\rho+\varepsilon}. \tag{3.40}$$

If $k \geq 4$, by (3.26) and (3.39),

$$\begin{aligned} T_2(\alpha) &\ll \frac{M^{(k-1)/k}NX^{1+\varepsilon}R^{1/k^2}}{1+(NX)^k(RZ)^{-1}}\left(\frac{Q}{RZ}+\frac{M^k}{Q}\right)^{1/k}+P^{1+\varepsilon}q^{-1/k}+NX^{1+\varepsilon}\\ &\ll \left(P^\rho QM^{k-1}\right)^{1/k+\varepsilon}+P^{1+\varepsilon}\left(P^\rho M^{k-1}Q^{-1}\right)^{1/k}+MP^{\rho+\varepsilon}+NX^{1+\varepsilon}, \end{aligned}$$

and using (1.8) and (3.18), we find that (3.40) holds in this case as well.

The desired estimate follows from (3.20), (3.24), (3.25) and (3.40). □

The following lemma uses the sieve of Eratosthenes–Legendre and Lemmas 3.1 and 3.2 to derive an upper bound for a bilinear Weyl sum with coefficients supported on numbers not divisible by small primes.

Lemma 3.3. *Let $k \geq 3$ and $0 < \rho < \left(2^k+2\right)^{-1}$. Suppose that $\alpha \in \mathbb{R}$ and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that (1.7) holds with Q given by (1.8). Let $z, M, N \geq 2$, let $|\xi_m| \leq 1$, and let $\psi(n,z)$ be defined by (1.15). Also, write*

$$g(\alpha) = \sum_{\substack{m\sim M\\mn\sim P}}\sum_{n\sim N} \xi_m\psi(n,z)e\left(\alpha(mn)^k\right).$$

Then

$$g(\alpha) \ll P^{1-\rho+\varepsilon} + \frac{w_k(q)^{1/2} P^{1+\varepsilon}}{\left(1 + P^k|\alpha - a/q|\right)^{1/2}},$$

provided that

$$z \le z_0 = \min\left(P^{(k-(8k-2)\rho)/(2k-1)}, P^{1-(2^k+2)\rho}\right) \tag{3.41}$$

and

$$M \le \min\left(P^{(k-(2k+1)\rho)/(2k-1)}, P^{1-(2^{k-1}+2)\rho}\right). \tag{3.42}$$

Proof. Let $\mathcal{I}(m,d)$ denote the interval

$$\left[Nd^{-1}, 2Nd^{-1}\right) \cap \left[P(md)^{-1}, 2P(md)^{-1}\right).$$

Using the properties of the Möbius function, we can write $g(\alpha)$ in the form

$$\begin{aligned} g(\alpha) &= \sum_{d|\mathcal{P}(z)} \sum_{m\sim M} \sum_{n\in\mathcal{I}(m,d)} \xi_m \mu(d) e\left(\alpha(mnd)^k\right) \\ &= \left\{ \sum_{d\le P^{2\rho}} + \sum_{d>P^{2\rho}} \right\} \cdots = g_1(\alpha) + g_2(\alpha), \quad \text{say.} \end{aligned}$$

Note that the hypothesis (3.42) of the lemma implies hypothesis (3.18) of Lemma 3.2 with $(m,n,x) = (m,d,n)$, so a simple splitting-up argument yields

$$g_1(\alpha) \ll P^{1-\rho+\varepsilon} + \frac{w_k(q) P^{1+\varepsilon}}{1 + P^k|\alpha - a/q|}.$$

Therefore, it suffices to show that

$$g_2(\alpha) \ll P^{1-\rho+\varepsilon} + \frac{w_k(q)^{1/2} P^{1+\varepsilon}}{\left(1 + P^k|\alpha - a/q|\right)^{1/2}}. \tag{3.43}$$

We write

$$g_2(\alpha) = \left\{ \sum_{d\le z_0 P^{2\rho}} + \sum_{d>z_0 P^{2\rho}} \right\} \cdots = g_{2,1}(\alpha) + g_{2,2}(\alpha), \quad \text{say.} \tag{3.44}$$

By Lemma 2.7,

$$g_{2,1}(\alpha) \ll L \left| \sum_{d|\mathcal{P}(z)} \sum_{m\sim M} \sum_{n\sim P/(md)} \xi_m \eta_n \delta_d e\left(\alpha(mnd)^k\right) \right| + 1,$$

with $|\eta_n| \le 1$, $|\delta_d| \le 1$. Thus, Lemma 3.1 with $(m,n) = (mn,d)$ yields

$$g_{2,1}(\alpha) \ll P^{1-\rho+\varepsilon} + \frac{w_k(q)^{1/2} P^{1+\varepsilon}}{\left(1 + P^k|\alpha - a/q|\right)^{1/2}}. \tag{3.45}$$

We now turn our attention to $g_{2,2}(\alpha)$. Each d appearing in the summation has a factorization $d = p_1 \cdots p_r$ subject to

$$p_r < \cdots < p_1 < z, \quad p_1 \cdots p_r > z_0 P^{2\rho}.$$

Therefore, there is a unique integer r_1, $1 \le r_1 < r$, such that

$$P^{2\rho} \le p_1 \cdots p_{r_1} \le z_0 P^{2\rho} < p_1 \cdots p_{r_1+1}.$$

On writing $p = p_{r_1}$, $p' = p_{r_1+1}$, $d_1 = p_1 \cdots p_{r_1-1}$, $d_2 = p_{r_1+2} \cdots p_r$, we obtain

$$g_{2,2}(\alpha) = \sum_{p,p'} \sum_{d_1,d_2} \sum_{m,n} \xi_m \mu(d_1)\mu(d_2)\psi(d_1,p) e\left(\alpha(mnpp'd_1d_2)^k\right),$$

where $m \sim M$, $n \in \mathcal{I}(m, pp'd_1d_2)$ and p, p', d_1, d_2 are subject to

$$p' < p < z, \quad d_1 \mid \mathcal{P}(z), \quad d_2 \mid \mathcal{P}(p'), \quad d_1 p \le z_0 P^{2\rho} < d_1 pp'.$$

Hence, using Lemma 2.7 to remove the summation conditions

$$p' < p, \quad d_1 pp' > z_0 P^{2\rho}, \quad npp'd_1d_2 \sim N,$$

we get

$$g_{2,2}(\alpha) \ll L^3 \left| \sum_{P^{2\rho} \le u \le z_0 P^{2\rho}} \sum_{uv \sim P} \tilde{\xi}_u \tilde{\eta}_v e\left(\alpha(uv)^k\right) \right| + L^2,$$

with coefficients $|\tilde{\xi}_u| \ll u^{\varepsilon}$, $|\tilde{\eta}_v| \le 1$ (the new variables being $u = mnp'd_2$ and $v = pd_1$). Applying Lemma 3.1 with $(m, n) = (u, v)$, we deduce

$$g_{2,2}(\alpha) \ll P^{1-\rho+\varepsilon} + \frac{w_k(q)^{1/2} P^{1+\varepsilon}}{\left(1 + P^k|\alpha - a/q|\right)^{1/2}}. \tag{3.46}$$

Combining (3.44)–(3.46), we complete the proof of (3.43) and establish the lemma. □

4. Proof of Theorem 1

Let z_0 denote the right side of (3.41) and let $z_1 = 2P^{1/3}$. We apply Buchstab's combinatorial identity in the form

$$\psi(n, z_1) = \psi(n, z_2) - \sum_{\substack{z_2 \le p < z_1 \\ n = pj}} \psi(j, p) \qquad (2 \le z_2 < z_1). \tag{4.1}$$

Applying (4.1), we obtain

$$\begin{aligned} f(\alpha) &= \sum_{n \sim P} \psi(n, \sqrt{2P}) e\left(\alpha n^k\right) \\ &= \sum_{n \sim P} \psi(n, z_0) e\left(\alpha n^k\right) - \sum_{z_0 \le p < \sqrt{2P}} \sum_{j \sim Pp^{-1}} \psi(j, p) e\left(\alpha (jp)^k\right). \end{aligned} \tag{4.2}$$

Lemma 3.3 applies to the first sum on the right side of this identity. On the other hand, the second sum on the right side of (4.2) is equal to

$$\sum_{z_0 \le p \le z_1} \sum_{j \sim Pp^{-1}} \psi(j, p) e\left(\alpha(jp)^k\right) + \sum_{z_1 < p < \sqrt{2P}} \sum_{j \sim Pp^{-1}} \psi(j, p) e\left(\alpha(jp)^k\right).$$

The first of these sums can be estimated by Lemma 3.1 and the second can be rewritten as

$$g(\alpha) = \sum_{z_1 < p < \sqrt{2P}} \sum_{j \in \mathcal{I}(p)} \psi(j, z_1) e\left(\alpha(jp)^k\right),$$

where $\mathcal{I}(p)$ is the interval $\max(p, P/p) \le j < 2P/p$. Another appeal to (4.1) yields

$$g(\alpha) = \sum_{z_1 < p < \sqrt{2P}} \sum_{j \in \mathcal{I}(p)} \psi(j, z_0) e\left(\alpha (jp)^k\right) - \sum_{\substack{z_1 < p_1 < \sqrt{2P} \\ z_0 \le p_2 < z_1}} \sum_{jp_2 \in \mathcal{I}(p_1)} \psi(j, p_2) e\left(\alpha (jp_1p_2)^k\right).$$

Since these two sums can be estimated by Lemmas 3.3 and 3.1, respectively, this completes the proof. □

5. Multilinear Weyl sums, II

In this section, we derive bounds for exponential sums from the large sieve. As we mentioned in the Introduction, this idea goes back to Linnik [15], who used the large sieve to prove zero-density estimates for Dirichlet L-functions and then applied the latter to deduce bounds for exponential sums. We use a variant of Linnik's method that was introduced by Vaughan [23] and has also been used by Harman [5]. It derives exponential sum estimates directly from large sieve inequalities for Dirichlet polynomials. We start with two lemmas that relate upper bounds for exponential sums to mean-value estimates for Dirichlet polynomials.

Lemma 5.1. *Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, and suppose that $\alpha = a/q + \beta$, where $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $(a, q) = 1$. Define*

$$g(\alpha) = \sum_{n \sim P} \xi_n e\left(\alpha n^k\right),$$

with coefficients ξ_n subject to $|\xi_n| \le \tau(n)^c$, and suppose that there exists $z \ge 2$ such that $\xi_n = 0$ unless $(n, \mathcal{P}(z)) = 1$. Then

$$g(\alpha) \ll q^{-1/2+\varepsilon}(|g(\beta)| + \Sigma(\beta)) + q^{\varepsilon} L^c P z^{-1}, \tag{5.1}$$

where

$$\Sigma(\beta) = \sideset{}{'}\sum_{\chi \bmod q} \left| \sum_{n \sim P} \xi_n \chi(n) e\left(\beta n^k\right) \right|. \tag{5.2}$$

Here, $\sum'$ denotes summation over the non-principal characters mod q.

Proof. We have

$$g(\alpha) = g_1(\alpha) + O(g_2), \tag{5.3}$$

where

$$g_1(\alpha) = \sum_{(n,q)=1} \xi_n e\left(\alpha n^k\right), \quad g_2 = \sum_{(n,q)>1} |\xi_n|.$$

Using the properties of the coefficients ξ_n, we obtain

$$g_2 \le \sum_{\substack{d | q \\ d > z}} \sum_{\substack{n \sim P \\ n \equiv 0 \ (\mathrm{mod}\ d)}} \tau(n)^c \ll q^{\varepsilon} L^c P z^{-1}. \tag{5.4}$$

On the other hand, by the orthogonality of the Dirichlet characters mod q,

$$g_1(\alpha) = \frac{1}{\phi(q)} \sum_{\chi \bmod q} S_\chi(q,a) \sum_{n\sim P} \xi_n \chi(n) e\left(\beta n^k\right), \tag{5.5}$$

where

$$S_\chi(q,a) = \sum_{x=1}^{q} \bar{\chi}(x)\, e\left(\frac{ax^k}{q}\right).$$

By [26, Problem VI.14], we have

$$S_\chi(q,a) \ll q^{1/2+\varepsilon}, \tag{5.6}$$

so separating the contribution from the principal character, we deduce from (5.5) that

$$g_1(\alpha) \ll q^{-1/2+\varepsilon}\left(|g(\beta)| + g_2 + \Sigma(\beta)\right). \tag{5.7}$$

Clearly, (5.1) follows from (5.3), (5.4) and (5.7). □

Lemma 5.2. *Let $k \in \mathbb{N}$, $\beta \in \mathbb{R}$, $q \in \mathbb{N}$, and suppose that $q \le P^c$. Define*

$$g(\beta,\chi) = \sum_{\substack{m\sim M \\ mn\sim P}} \sum_{n} \xi_m \eta_n \chi(mn) e\left(\beta (mn)^k\right)$$

and

$$G(s,\chi) = \sum_{m\asymp M} \sum_{n\asymp N} \xi_m \eta_n \chi(mn)(mn)^{-s},$$

where ξ_m, η_n are complex numbers subject to $|\xi_m| \le \tau(m)^c$, $|\eta_n| \le \tau(n)^c$, and $N = PM^{-1}$. Then

$$\sum_{\chi} g(\beta,\chi) \ll L \max_{2\le T\le P^5} \frac{\sqrt{PT_0}}{T_0+T} \sum_{\chi} \int_{-T}^{T} \left|G\left(\tfrac12 + it,\chi\right)\right| dt + qT_0P^{-1+\varepsilon}, \tag{5.8}$$

where $T_0 = P^k|\beta| + 1$ and $\sum_\chi$ denotes summation over a set of characters mod q*.*

Proof. Applying (2.10) with $b = \frac12$ and $T_1 = P^5$, we get

$$\sum_{y_1 \le mn < y_2} \xi_m\eta_n\chi(mn) = \frac{1}{2\pi i}\int_{\frac12 - iT_1}^{\frac12 + iT_1} G(s,\chi)\frac{y_2^s - y_1^s}{s} ds + O\left(P^{-2+\varepsilon}\right),$$

whenever $P \le y_1 < y_2 \le 2P$ and $\min_{n\in\mathbb{Z}} |n - y_j| \ge P^{-2}$. Hence, by partial summation,

$$g(\beta,\chi) = \frac{1}{2\pi i}\int_{\frac12 - iT_1}^{\frac12 + iT_1} G(s,\chi)h(s) ds + O\left(T_0P^{-1+\varepsilon}\right), \tag{5.9}$$

where

$$h(s) = h(s;\beta) = \int_P^{2P} y^{s-1} e\left(\beta y^k\right) dy.$$

We now observe that

$$h(\sigma+it) \ll P^\sigma \min\left(1, |t|^{-1/2}\right),$$

and that, unless $k\pi|\beta|P^k \le |t| \le 2^{k+2}k\pi|\beta|P^k$, we also have

$$h(\sigma+it) \ll P^\sigma \min\left(T_0^{-1}, |t|^{-1}\right).$$

Hence,

$$h(1/2+it) \ll \frac{\sqrt{PT_0}}{T_0+|t|},$$

and (5.8) follows from (5.9) by a standard splitting argument. □

The next lemma provides preliminary estimates for sums of the form appearing on the right side of (5.1) by combining Lemma 5.2 with Lemmas 2.4 and 2.6. Define

$$g(\beta,\chi) = \sum_{\substack{m\sim M \\ mnr\sim P}} \sum_{n\sim N} \sum_{r} \xi_m \eta_n \delta_r \chi(mnr) e\left(\beta(mnr)^k\right), \tag{5.10}$$

where the coefficients ξ_m, η_n, δ_r are complex numbers with

$$|\xi_m| \le \tau(m)^c, \quad |\eta_n| \le \tau(n)^c, \quad |\delta_r| \le \tau(r)^c.$$

Also, through the remainder of this section, $\sum'$ has the same meaning as in Lemma 2.6: it represents summation over the non-principal characters mod q or a single term with $\chi = \chi^0$ according as $q > 1$ or $q = 1$.

Lemma 5.3. *Let $k \in \mathbb{N}$, $\beta \in \mathbb{R}$, $q \in \mathbb{N}$, and suppose that $q \le P$ and $q|\beta| \le P^{1-k}$. Let $g(\beta,\chi)$ be defined by* (5.10)*, and suppose that* $\max(M,N) \le P^{11/20}$ *and either $MN \ge P^{27/35}$, or $\delta_r = 1$ for all r. Then*

$$\sideset{}{'}\sum_{\chi \bmod q} |g(\beta,\chi)| \ll L^c \left(P\Psi(\beta)^{-1/2} + qP^{11/20}\Psi(\beta)^{1/2}\right), \tag{5.11}$$

where $\Psi(\beta) = P^k|\beta| + 1$.

Proof. By Lemma 5.2,

$$\sideset{}{'}\sum_{\chi \bmod q} |g(\beta,\chi)| \ll \frac{L\sqrt{P\Psi(\beta)}}{\Psi(\beta)+T} \sum_{\chi} \int_{-T}^{T} \left|G\left(\tfrac12+it,\chi\right)\right| dt + P^{\varepsilon}, \tag{5.12}$$

where $2 \le T \le P^5$ and

$$G(s,\chi) = \sum_{m\asymp M} \sum_{n\asymp N} \sum_{r\asymp P(MN)^{-1}} \xi_m\eta_n\delta_r\chi(mnr)(mnr)^{-s}.$$

When $MN \ge P^{27/35}$, we can bound the right side of (5.12) by Lemma 2.4; when $\delta_r = 1$, we can apply Lemma 2.6. □

Lemma 5.4. *Let $k \in \mathbb{N}$, $\beta \in \mathbb{R}$, $q \in \mathbb{N}$, and suppose that $q \le P$ and $q|\beta| \le P^{1-k}$. Also, let $g(\beta,\chi)$ be defined by* (5.10) *with $\delta_r = \psi(r,z)$, and suppose that*

$$z \le P^{23/140} \quad \textit{and} \quad \max(M,N) \le P^{11/20}.$$

Then (5.11) *holds.*

Proof. We consider two cases depending on the sizes of M and N. By symmetry, we may assume that $N \le M$.

Case 1. Suppose that $MN \ge P^{27/35}$ or $M \ge P^{9/20}$. If the former condition holds, we apply Lemma 5.3 with $\delta_r = \psi(r,z)$. Otherwise, we write $n' = nr$ and apply Lemma 5.3 with $(m,n,r) = (m,n',1)$.

Case 2. Suppose that $M \le P^{9/20}$ and $MN \le P^{27/35}$. We have

$$\sum_{\chi \bmod q}{}' |g(\beta,\chi)| \ll L \sum_{\chi \bmod q}{}' \left| \sum_{m,n,r,d} \xi_m \eta_n \mu(d) \chi(mnrd) e\left(\beta (mnrd)^k\right) \right|, \tag{5.13}$$

where m, n, r, d are subject to

$$d \mid \mathcal{P}(z), \quad d \sim D, \quad m \sim M, \quad n \sim N, \quad mnrd \sim P.$$

We now consider two subcases depending on the size of D.

Case 2.1. Suppose that $ND \le P^{11/20}$. Then, we estimate the right side of (5.13) by Lemma 5.3 with $(m,n,r) = (m, nd, r)$.

Case 2.2. Suppose that $ND \ge P^{11/20}$. Our argument is similar to that used in the proof of Lemma 3.3. Suppose that d occurs on the right side of (5.13). We observe d has at least two prime divisors, as otherwise we would have

$$DN < zN \le z(MN)^{1/2} \le zP^{27/70} \le P^{11/20}.$$

Decomposing d into prime factors, we have $d = p_1 \cdots p_j$, where

$$p_j < \cdots < p_1 < z, \quad p_1 N \le P^{11/20} < p_1 \cdots p_j N.$$

Hence, there is a unique i, $1 \le i < j$, such that

$$p_1 \cdots p_i N \le P^{11/20} < p_1 \cdots p_{i+1} N,$$

and consequently, d has a unique factorization $d = pp'd_1d_2$ in which

$$p < p', \quad (d_1, \mathcal{P}(p')) = 1, \quad d_2 \mid \mathcal{P}(p), \quad p'd_1 \le P^{11/20} < pp'd_1.$$

Thus, the sum over d on the right side of (5.13) can be rearranged in the form

$$\sum_{p<p'} \sum_{d_1 \mid \mathcal{P}(z)} \sum_{d_2 \mid \mathcal{P}(p)} \mu(d_1)\mu(d_2)\psi(d_1,p')\chi(pp'd_1d_2) e\left(\alpha(mnrpp'd_1d_2)^k\right),$$

where p, p', d_1, d_2 are subject to

$$p'd_1 \le N^{11/20} < pp'd_1, \quad pp'd_1d_2 \sim D, \quad rpp'd_1d_2 \sim R, \quad mnrpp'd_1d_2 \sim P.$$

Using Lemma 2.7 to simplify the summation conditions, we obtain

$$L^{-c} \sum_{\chi \bmod q}{}' |g(\beta,\chi)| \ll \sum_{\chi \bmod q}{}' \left| \sum_{u \sim M'} \sum_{\substack{v \sim N' \\ uvp \sim P}} \sum_{p} \tilde{\xi}_u \tilde{\eta}_v \theta_p \chi(uvp) e\left(\alpha(uvp)^k\right) \right| + 1,$$

where the new summation variables are $u = mrd_2$ and $v = p'd_1$, the coefficients satisfy $|\tilde{\xi}_u| \le \tau(u)^c$, $|\tilde{\eta}_v| \le \tau(v)^c$, $|\theta_p| \le 1$, and M' and N' are subject to

$$N' \ll P^{11/20}, \quad M' \ll P^{9/20}, \quad M'N' \gg Pz^{-1}.$$

The desired estimate then follows from Lemma 5.3 with $(m,n,r) = (u,v,p)$. □

Lemma 5.5. *Let $k \in \mathbb{N}$, $\beta \in \mathbb{R}$, $q \in \mathbb{N}$, and suppose that $q \le P$ and $q|\beta| \le P^{1-k}$. Let ξ_m be complex numbers with $|\xi_m| \le \tau(m)^c$, and define*

$$g(\beta,\chi) = \sum_{\substack{m\sim M \\ mn\sim P}} \sum_{n} \xi_m \psi(n,z)\chi(mn) e\left(\beta(mn)^k\right),$$

where $\psi(n,z)$ is given by (1.15). *Then* (5.11) *holds, provided that*

$$M \le P^{11/20}, \quad z \le \sqrt{2P/M}. \tag{5.14}$$

Proof. We use Buchstab's identity (4.1) to write $g(\beta,\chi)$ as a linear combination of exponential sums for which (5.11) can be established by means of Lemmas 5.3 or 5.4. We may assume that $z > P^{23/140}$, for otherwise the result is an immediate corollary to Lemma 5.4.

Set $z_0 = P^{23/140}$. Applying (4.1) twice, we get

$$g(\beta,\chi) = g_1(\beta,\chi) - g_2(\beta,\chi) + g_3(\beta,\chi),$$

where

$$g_i(\beta,\chi) = \sum_{\substack{m\sim M \\ mn\sim P}} \sum_{n} \xi_m \eta_{n,i} \chi(mn) e\left(\beta(mn)^k\right) \quad (i = 1,2,3),$$

with

$$\eta_{n,1} = \psi(n,z_0), \quad \eta_{n,2} = \sum_{\substack{n=pj \\ z_0\le p<z}} \psi(j,z_0) \quad \text{and} \quad \eta_{n,3} = \sum_{n=p_1p_2j} \psi(j,p_2),$$

the primes p_1, p_2 in $\eta_{n,3}$ being subject to

$$z_0 \le p_2 < p_1 < z, \quad p_1p_2^2 \le 2PM^{-1}. \tag{5.15}$$

The desired estimates for $g_1(\beta,\chi)$ and $g_2(\beta,\chi)$ follow from Lemma 5.4. We decompose $g_3(\beta,\chi)$ further. We write

$$g_3(\beta,\chi) = \left(\sum_{p_1p_2\le P^{11/20}} + \sum_{p_1p_2>P^{11/20}} \right) \sum_{m,j} \cdots = g_4(\beta,\chi) + g_5(\beta,\chi), \quad \text{say.}$$

Consider $g_4(\beta,\chi)$. Using (4.1) once more, we obtain

$$g_4(\beta,\chi) = g_6(\beta,\chi) - g_7(\beta,\chi),$$

where $g_6(\beta,\chi)$ and $g_7(\beta,\chi)$ are obtained from $g_4(\beta,\chi)$ by replacing $\eta_{n,3}$ with

$$\eta_{n,6} = \sum_{n=p_1p_2j} \psi(j,z_0) \quad \text{and} \quad \eta_{n,7} = \sum_{n=p_1p_2p_3j} \psi(j,p_3),$$

the prime p_3 in $\eta_{n,7}$ being subject to

$$z_0 \le p_3 < p_2, \quad p_1p_2p_3^2 \le 2PM^{-1}.$$

The sum $g_6(\beta,\chi)$ is covered by Lemma 5.4 and we will show that $g_7(\beta,\chi)$ can be dealt with by Lemma 5.3. Indeed, either

$$P^{9/20} \le p_1p_2 \le P^{11/20}, \quad jp_3M \le 2P^{11/20},$$

or

$$p_1p_2 < P^{9/20}, \quad jM \le 2Pz_0^{-3} \le P^{11/20}, \quad p_3 < \sqrt{p_1p_2} < P^{9/40}.$$

In the former case we can apply Lemma 5.3 with $(m,n,r) = (mjp_3, p_1p_2, 1)$ and in the latter with $(m,n,r) = (mj, p_1p_2, p_3)$. (Also, we need to appeal to Lemma 2.7 to remove the 'unwanted' summation conditions.)

We now turn to $g_5(\beta,\chi)$. By (4.1),

$$g_5(\beta,\chi) = g_8(\beta,\chi) - g_9(\beta,\chi),$$

where $g_8(\beta,\chi)$ and $g_9(\beta,\chi)$ are defined similarly to $g_6(\beta,\chi)$ and $g_7(\beta,\chi)$. We can estimate $g_8(\beta,\chi)$ by Lemma 5.4 (note that $p_1p_2 \geq P^{11/20}$ and the second inequality in (5.15) yield $p_2M \leq 2P^{9/20}$). On the other hand, the summation variables in $g_9(\beta,\chi)$ satisfy

$$j < 2P/(Mp_1p_2p_3) < 2P^{9/20}z_0^{-1} < z_0^2,$$

so $j = p_4 \geq p_3$ and we can replace the coefficient $\psi(j,p_3)$ by $\psi(j,z_0)$ whenever $j \geq p_3$. Furthermore,

$$p_1M < 2P/(p_2p_3j) < 2Pz_0^{-3} < P^{11/20} \quad \text{and} \quad (p_2p_3)^2 \leq p_1p_2p_3^2 \leq 2PM^{-1},$$

so any subsum of $g_9(\beta,\chi)$ in which the constraints on m, p_1, p_2, p_3 make the summation condition $j \geq p_3$ superfluous can be dealt with via Lemma 5.4.[1] In particular, Lemma 5.4 applies to the subsum of $g_9(\beta,\chi)$ with $p_1p_2p_3M \leq P^{27/35}$, as in this case,

$$p_3 \leq \sqrt{2P/p_1p_2} \leq P^{9/40} < P^{8/35} \leq j.$$

Finally, in the remainder of $g_9(\beta,\chi)$, we have

$$p_1M \leq 2Pz_0^{-3} \leq P^{11/20}, \quad p_2p_3 \leq \sqrt{2P/M} \leq P^{11/20}, \quad j \leq 2P^{8/35},$$

and we can refer to Lemma 5.3 with $(m,n,r) = (mp_1, p_2p_3, j)$. □

We are finally in position to state the main result of this section. Combining Lemmas 5.1 and 5.5, we obtain the following estimate for bilinear Weyl sums over almost primes.

Lemma 5.6. *Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, and suppose that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying* (1.10) *with $Q \leq P$. Let ξ_m be complex numbers with $|\xi_m| \leq \tau(m)^c$, and define*

$$g(\alpha) = \sum_{\substack{m\sim M \\ mn\sim P}} \sum_{n} \xi_m \psi(mn,z) e\left(\alpha(mn)^k\right),$$

with $\psi(n,z)$ given by (1.15)*. Suppose that conditions* (5.14) *hold. Then*

$$g(\alpha) \ll q^{\varepsilon}L^c \left(P\Psi(\alpha)^{-1/2} + \Psi(\alpha)^{1/2}P^{11/20} + Pz^{-1}\right), \tag{5.16}$$

where $\Psi(\alpha) = q + P^k|q\alpha - a|$.

Remark 5.1. Sometimes, one needs a slight variation of Lemma 5.6 in which z, instead of being fixed, depends on m. Let ξ_m and η_n be complex numbers as above, and let $z(m)$ be defined by $z(m) = m$ or $z(m) = Zm^{-1}$ with $Z \in \mathbb{R}$. Suppose that the sequences (ξ_m) and (η_n) are supported on integers free of prime divisors $< z$ and that $z(m) \geq z$ for all $m \sim M$. We claim that the exponential sum

$$g(\alpha) = \sum_{\substack{m\sim M \\ mnr\sim P}} \sum_{n\sim N} \sum_{r} \xi_m \eta_n \psi(r, z(m)) e\left(\alpha(mnr)^k\right)$$

[1]We can use Lemma 2.7 to remove troublesome summation conditions involving m, p_1, p_2, p_3, but using that lemma to remove $j \geq p_3$ would alter the coefficients $\psi(j,z_0)$, which we need in order to apply Lemma 5.4.

satisfies (5.16), provided that

$$MN \le P^{11/20} \quad \text{and} \quad z(m) \le \sqrt{2P/MN}.$$

The proof of this estimate is similar to the proof of Lemma 5.6. Only a few 'cosmetic' changes are needed because of the interdependence between m and r. For example, in the proof of the respective variant of Lemma 5.4, in place of (5.13) we have a bound of the form

$$\sideset{}{'}\sum_{\chi \bmod q} |g(\beta, \chi)| \ll L \sideset{}{'}\sum_{\chi \bmod q} \bigg| \sum_{m \sim M} \sum_{d | \mathcal{P}(z(m))} \xi_m \mu(d) \Sigma(m, d) \bigg|,$$

where $\Sigma(m, d)$ represents the double sum over n and r. Since

$$\sum_{d | \mathcal{P}(z(m))} \mu(d) \Sigma(m, d) = - \sum_{p < z(m)} \sum_{d | \mathcal{P}(p)} \mu(d) \Sigma(m, pd),$$

an appeal to Lemma 2.7 gives

$$\sum_{m,d} \xi_m \mu(d) \Sigma(m, d) \ll L \bigg| \sum_{m \sim M} \sum_{p < z_0} \sum_{d | \mathcal{P}(p)} \xi_m' \theta_p \mu(d) \Sigma(m, pd) \bigg|,$$

where $z_0 = \max_m z(m)$. The sum on the right side of this inequality can then be dealt with in the same fashion as that on the right side of (5.13). □

6. Proofs of Theorems 2, 3 and 4

We are finally in position to complete the proofs of Theorems 2–4.

Proof of Theorem 2. Apply Lemma 5.6 with $M = 1$ and $z = \sqrt{2P}$. □

Proof of Theorem 3. If a and q satisfy (1.10) with $Q = P^{2k\rho(k)}$, Theorem 2 yields the bound

$$f(\alpha) \ll P^{4/5+\varepsilon} + \frac{q^{\varepsilon} P (\log P)^c}{(q + P^k |q\alpha - a|)^{1/2}},$$

which is even stronger than (1.12). On the other hand, if a and q satisfy (1.7) but not (1.10) with $Q = P^{2k\rho(k)}$, the estimate

$$f(\alpha) \ll P^{1-\rho(k)+\varepsilon}$$

follows from Theorem 1 or (1.4) according as $k \ge 3$ or $k = 2$. □

Proof of Theorem 4. We fix $k = 2$. Let $\mathfrak{M}$ be the set of $\alpha \in [0, 1]$ for which there exist integers a and q satisfying (1.10) with $Q = P^{1/3-\varepsilon}$ and let $\mathfrak{m} = [0, 1] \setminus \mathfrak{M}$. By the argument in [18], the desired bound will follow, if we show that

$$\max_{\alpha \in \mathfrak{m}} |f(\alpha)| \ll P^{7/8+\varepsilon}. \tag{6.1}$$

By Dirichlet's theorem on diophantine approximation, for every $\alpha \in \mathbb{R}$ there exist integers a and q satisfying (1.7) with $Q = P^{3/2}$. Since for $\alpha \in \mathfrak{m}$ we also have

$$q + P^2 |q\alpha - a| > P^{1/3-\varepsilon},$$

the desired bound (6.1) follows from Theorem 3. □

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Department of Mathematics, Towson University, 8000 York Road, Towson, MD 21252
E-mail address: akumchev@towson.edu