AN ADDITIVE PROBLEM WITH PRIME NUMBERS FROM A THIN SET

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In 1937 I. M. Vinogradov [10] found an asymptotic formula for the number of the solutions in prime numbers of the equation

$$p_1 + p_2 + p_3 = N \ . \tag{1}$$

This formula implies that every sufficiently large odd integer may be represented as the sum of three primes. In [11] he proved a similar theorem for the equation

$$p_1^n + p_2^n + \cdots + p_k^n = N \ .$$

Later, Hua Lo-Keng using the method of I. M. Vinogradov obtained an asymptotic formula for the number of the solutions $J = J(N_1, \ldots, N_n)$ of the system

$$\begin{cases} p_1 + p_2 + \cdots + p_k = N_1 \\ p_1^2 + p_2^2 + \cdots + p_k^2 = N_2 \\ \cdots\cdots\cdots \\ p_1^n + p_2^n + \cdots + p_k^n = N_n \end{cases} \tag{2}$$

with prime unknowns $p_1, \ldots, p_k$. He showed [7, Chapter 10] that if $k_0(n)$ is defined by the table

n	2	3	4	5	6	7	8	9	10
$k_0(n)$	7	19	49	113	243	413	675	1083	1773

Table 1.

in the case of $2 \le n \le 10$, and by the formula

$$k_0(n) = 2[n^2(3\log n + \log\log n + 4)] - 21 \tag{3}$$

in the case of $n \ge 11$, and if $k \ge k_0(n)$, then

$$J = \frac{P^{k-0.5n(n+1)}}{(\log P)^k} \cdot \left(\gamma\sigma + \mathcal{O}\left(\frac{\log\log P}{\log P} \right) \right) \ . \tag{4}$$

Here $P = N_n^{1/n}$ and γ and σ are the singular integral and the singular series, which values are given in [7, pp.139–140]. Hua Lo-Keng also proved [7, Chapter 11] that if $N_1, \ldots, N_n$ satisfy some arithmetical conditions (conditions of congruent solvability) then $\sigma \ge \sigma_0 > 0$ where σ_0 does not depend on $N_1, \ldots, N_n$, and that if the orders of magnitude of $N_1, \ldots, N_n$ satisfy some other conditions (conditions of positive solvability) then $\gamma \ge \gamma_0 > 0$ where γ_0 does not depend on $N_1, \ldots, N_n$.

In 1986 Wirsing [12] considered (1) for prime numbers from a thin set (the set of prime numbers S is called to be *thin* if the number of the primes $p \le x$ such that $p \in S$ equals $o(\pi(x))$, $x \to \infty$).

He showed that there exists a thin set of prime numbers S such that for every sufficiently large odd integer N the equation (1) has solutions in prime numbers from S. However, Wirsing did not give an example for such a set because his method does not allow this.

In 1988 Gritsenko [3] (see also [4] and [5]) considered some additive problems with prime numbers from the set

$$\{p \text{ — prime} \mid (2n)^\alpha \le p < (2n+1)^\alpha \text{ for some } n \in \mathbb{N}\}$$

where $1 < \alpha \le 2$ is a fixed number. This set is not thin but the method of Gritsenko may be used for studying of additive problems with primes from thin sets.

In 1990 Balog and Friedlander [1] proved that if $1 < c < 21/20$ then every sufficiently large odd integer N can be represented in the form (1) where p_j belong to the set of the Piatetski-Shapiro prime numbers

$$\{p - \text{prime} \mid p = [n^c] \text{ for some } n \in \mathbb{N}\}\ .$$

In this paper we study the system (2) in prime numbers p "close to squares", i.e. such that $\sqrt{p}$ is close to an integer.

We use the standard notation: $A \ll B$ means that $A = \mathcal{O}(B)$; $A \asymp B$ means that $A \ll B \ll A$; $p,\ p_1, \dots, p_k$ are prime numbers; ε is an arbitrary small positive number, not necessary the same in all appearances; $e(x) = e^{2\pi i x}$; $\|\alpha\| = \min(\{\alpha\}, 1 - \{\alpha\})$, $\Lambda(n)$ is von Mangoldt's function. The constants are absolute or depend only on n, k, λ.

Let $J_0 = J_0(N_1, \dots, N_n)$ be the number of the solutions of (2) in prime numbers $p_1, \dots, p_k$ such that $\|\sqrt{p_1}\|, \dots, \|\sqrt{p_k}\| < P^{-\lambda}$, where $\lambda > 0$ is a fixed number. One can expect that if λ is sufficienly small and $N_1, \dots, N_n$ satisfy the conditions of congruent and positive solvability, then J_0 is close to $2^k J P^{-\lambda k}$.

We prove the following theorem:

Theorem. *Let $n \ge 2$ and $k \ge k_0(n)$, where $k_0(n)$ is defined by* Table 1 *and* (3). *There exists an absolute constant $c > 0$ such that if*

$$0 < \lambda < \lambda_n = \begin{cases} \dfrac{c}{n^3 \log^2 n} & ,\text{if } n \ge 3 \\ \dfrac{1}{64} & ,\text{if } n = 2 \end{cases} \tag{5}$$

then the asymptotic formula

$$J_0 = \frac{P^{k(1-\lambda) - 0.5n(n+1)}}{(\log P)^k} \cdot \left(2^k \gamma \sigma + \mathcal{O}\left(\frac{\log\log P}{\log P} \right) \right)\ . \tag{6}$$

holds. Here P, σ, γ are the same as in (4).

The upper bound for λ depends on the estimate for an exponential sum. To obtain this estimate we use in the case $n = 2$ the method of van der Corput. In the case $n \ge 3$ we can not use this method and so we use the method of I. M. Vinogradov. For the constant c we may get for example $c = 10^{-6}$ but for small values of n we may obtain a better result (for instance in the case $n = 3$ we may get $c = 1/13000$).

For the proof of the theorem we need some lemmas:

Lemma 1. *Let r be an integer, α and β be real, $0 < \Delta < 0.25$, $\Delta < \beta - \alpha < 1 - \Delta$. There exists a periodic with period 1 function $\psi(t)$ such that:*

1° $\psi(t) = 1$, *if $\alpha + 0.5\Delta \le t \le \beta - 0.5\Delta$;*

2° $\psi(t) = 0$, *if* $\beta + 0.5\Delta \le t \le 1 + \alpha - 0.5\Delta$*;*
3° $0 < \psi(t) < 1$, *if* $\alpha - 0.5\Delta < t < \alpha + 0.5\Delta$ *or* $\beta - 0.5\Delta < t < \beta + 0.5\Delta$*;*
4° $\psi(t)$ *has the Fourier expansion*

$$\psi(t) = \sum_{h=-\infty}^{\infty} g(h)\, e(ht)$$

where $g(0) = \beta - \alpha$ *and*

$$|g(h)| \le \min\left(\beta - \alpha, \frac{1}{\pi|h|}, \frac{1}{\pi|h|}\left(\frac{r}{\pi|h|\Delta}\right)^r\right)$$

for $h \neq 0$.

Proof. See [9], p. 23.

We denote as usual

$$J_{r,n}(P) = \int_0^1 \cdots \int_0^1 \left|\sum_{x \le P} e(\alpha_1 x + \cdots + \alpha_n x^n)\right|^r d\alpha_1 \ldots d\alpha_n \ . \tag{7}$$

Lemma 2. *If* $n > 10$ *and* $r \ge r_0(n) = 2[n^2(2\log n + \log\log n + 4)]$*, then we have*

$$J_{r,n}(P) \ll P^{r-0.5n(n+1)}.$$

Proof. See [9, p. 70].

Lemma 3. *Let* $2 \le n \le 10$ *and* $r \ge r_0(n)$ *where* $r_0(n)$ *is defined by the table*

n	2	3	4	5	6	7	8	9	10
$r_0(n)$	6	16	46	110	240	410	672	1080	1770

Table 2.

Then we have

$$J_{r,n}(P) \ll P^{r-0.5n(n+1)+\varepsilon} \ .$$

Proof. See [7, Chapter 5].

Lemma 4. *Let* $k \ge 2$*,* $K = 2^{k-2}$ *and* $F(x)$ *be a real-valued function with* k *contionuous derivatives in* $[a, b]$ *such that*

$$|F^{(k)}(x)| \asymp h \quad \text{, uniformly in } x \in [a, b] \ .$$

We have

$$\left|\sum_{a<n\le b} e(F(n))\right| \ll (b-a)h^{1/(4K-2)} + (b-a)^{1-2/K} + (b-a)^{1-2/K+1/K^2} h^{-1/2K}.$$

Proof. This is Theorem 2.8 on p. 14 of [2].

Let $0 < M < M_1 \le 2M$, $0 < D < D_1 \le 2D$, $MD \asymp X$. Suppose that a_m, $M < m \le M_1$, and b_d, $D < d \le D_1$ are complex numbers such that $|a_m| \ll X^\varepsilon$, $|b_d| \ll X^\varepsilon$. We consider sums of two types:

Type I sums:

$$W_1 = \sum_{M<m\le M_1} a_m \sum_{\substack{D<d\le D_1 \\ X<md\le X_1}} e(f(md)) \, ,$$

$$W_1' = \sum_{M<m\le M_1} a_m \sum_{\substack{D<d\le D_1 \\ X<md\le X_1}} (\log d)\, e(f(md)) \, ;$$

Type II sums:

$$W_2 = \sum_{M<m\le M_1} a_m \sum_{\substack{D<d\le D_1 \\ X<md\le X_1}} b_d\, e(f(md)) \, .$$

Lemma 5. *Let $f(n)$ be a real-valued function. Let $X > 2$, $X_1 \le 2X$ and u, v, z be positive numbers ($z - 1/2 \in \mathbb{N}$) satisfying*

$$2 \le u < v \le z < X \quad , \quad 4u^2 \le z \quad , \quad 64uz^2 \le X_1 \quad , \quad v^3 \le 32X_1 \, .$$

Then the sum

$$\sum_{X<n\le X_1} \Lambda(n)\, e(f(n))$$

may be decomposed into $\mathcal{O}(\log^{10} x)$ sums, each either of Type I *with $D > z$, or of Type* II *with $u < D < v$.*

Proof. See Lemma 3 of [6].

Lemma 6. *Suppose that $U > 0$, $P \ge 1$ and $\alpha = a/q + \theta/q^2$, $(a, q) = 1$, $q \ge 1$, $|\theta| \le 1$. Then for every β we have*

$$\sum_{x=1}^{P} \min\left(U, \frac{1}{\|\alpha x + \beta\|}\right) \ll \left(\frac{P}{q} + 1\right)(U + q\log q) \, .$$

Proof. This is Lemma 2 on p. 111 of [8].

Lemma 7. *Suppose that $n \ge 3$, $x_1, \dots, x_n$ are arbitrary real numbers and $h \le P^{1/n^3} \log^2 P$. For the sum*

$$T = \sum_{p\le P} e(h\sqrt{p} + x_1 p + \cdots + x_n p^n)$$

we have

$$|T| \ll P^{1-c/n^3 \log^2 n}.$$

The constant $c > 0$ is absolute.

Proof. In order to prove the lemma we are following the approach of I. M. Vinogradov (see [8, pp. 112–113]). It is easy to see that

$$|T| \le |T_1| + P^{1-1/n^3} \tag{8}$$

where

$$T_1 = \sum_{X<m\le X_1} \Lambda(m)\, e(f(m))$$

and $P^{1-1/n^3} \le X < X_1 \le \min(2X, P)$, $f(t) = h\sqrt{t} + x_1 t + \cdots + x_n t^n$.

Applying Lemma 5 with $u = X^{1/n^3}$, $v = 2^{10}X^{1/3}$, $z = [X^{2/5}] + 1/2$, we decompose T_1 into $\mathcal{O}(\log^{10} X)$ sums of Type I with $D > z$ and sums of Type II with $u < D < v$.

Let us consider a Type II sum W_2. Using the Cauchy inequality and changing the order of summation we obtain

$$|W_2|^2 \ll X^{2-1/n^3+\varepsilon} + MX^{\varepsilon} \sum_{DX^{-1/n^3}<q\le D}\ \sum_{D<d\le D_1-q} \left| \sum_{M'<m\le M_1'} e(f_1(m)) \right| \tag{9}$$

where $M \le M' < M_1' \le M_1$ and $f_1(m) = f(m(d+q)) - f(md)$.

Let $A = [M^{1/2}X^{-1/6n}]$. Then

$$\left| \sum_{M'<m\le M_1'} e(f_1(m)) \right| \ll \frac{1}{A^2} \sum_{M<m\le 2M} |U(m)| + A^2 \tag{10}$$

where

$$U(m) = \sum_{x,y=1}^{A} e(f_1(m+xy))\ .$$

We choose $r = [3n/2]$, $r_1 = n+1$, $r_2 = [(3n-1)/2]$ and define $k_1(n)$ by the table

n	3	4	5	6	7	>7
$k_1(n)$	46	240	410	1080	1770	$2[r^2(2\log r + \log\log r + 4)]$

Table 3.

Using Taylor's formula we get

$$f_1(m+xy) = f_1(m) + \sum_{j=1}^{r} \alpha_j (xy)^j + \mathcal{O}(X^{-1/5n^3}) \qquad ;\ \alpha_j = \frac{f_1^{(j)}(m)}{j!}\ .$$

Hence

$$|U(m)| \ll |U_1(m)| + A^2 X^{-1/5n^3} \tag{11}$$

where

$$U_1(m) = \sum_{x,y=1}^{A} e(\alpha_1 xy + \cdots + \alpha_r x^r y^r)\ .$$

After some calculations we obtain

$$|U_1(m)|^{k_1^2} \ll A^{2(k_1^2-k_1)} J_{k_1,r}^2(A) \prod_{j=1}^{r} T_j \tag{12}$$

where $J_{k_1,r}(A)$ is given by (7) and

$$T_j = \sum_{|d|<k_1 A^j} \min\left(k_1 A^j, \frac{1}{\|\alpha_j d\|}\right) .$$

The trivial estimate for the quantity T_j is

$$T_j \ll A^{2j} . \tag{13}$$

If $r_1 \le j \le r_2$ we can find a better estimate. It is easy to see that in this case we have

$$\alpha_j = \frac{a_j}{q_j} + \frac{\theta_j}{q_j^2} \qquad , \ |\theta_j| \le 1$$

where

$$a_j = \frac{\alpha_j}{|\alpha_j|} \quad , \quad q_j = \left[\frac{1}{|\alpha_j|}\right] \ge 1 \quad , \quad (a_j, q_j) = 1 .$$

Hence Lemma 6 gives

$$T_j \ll A^2 X^{-1/2+j/3n+1/n^3} \tag{14}$$

for $r_1 \le j \le r_2$.

Using Lemmas 2 and 3 and the inequalities (12)–(14) we get

$$|U_1(m)| \ll A^{2-c/n^3 \log^2 n} \tag{15}$$

for some absolute constant $c > 0$, and combining (9)–(11) and (15):

$$|W_2| \ll X^{1-c/n^3 \log^2 n} \tag{16}$$

where $c > 0$ is another absolute constant.

For the Type I sums we have

$$|W_1| \ , \ |W_1'| \ll M X^{\varepsilon} \left| \sum_{D' < d \le D_1'} e(f(md)) \right|$$

where $D \le D' < D_1' \le D_1$. Choosing $A = [D^{1/2} X^{-1/6n}]$, $r = [3n/2]$, $r_1 = n+1$, $r_2 = [(3n-1)/2]$ and arguing as above we obtain

$$|W_1| \ , \ |W_1'| \ll X^{1-c/n^3 \log^2 n}. \tag{17}$$

The inequalities (8), (16) and (17) prove the lemma.

Lemma 8. *Suppose that $h \le P^{1/16} \log^2 P$ and x_1, x_2 are real numbers. Then we have*

$$\left| \sum_{p \le P} e(h\sqrt{p} + x_1 p + x_2 p^2) \right| \ll P^{15/16+\varepsilon}.$$

Proof. Using Lemma 5 with $u = 2^{-10}X^{1/5}$, $v = 2^{10}X^{1/3}$ and $z = [X^{2/5}] - 1/2$ we reduce the estimation of the above sum to the estimation of Type I and Type II sums. Let us consider a Type II sum W_2. Using the Cauchy inequality we obtain

$$|W_2|^2 \ll MX^{1+\varepsilon} + MX^{\varepsilon} \sum_{q\le D} \sum_{D<d\le D_1-q} \left| \sum_{M'<m\le M_1'} e(\varphi(m)) \right| \tag{18}$$

where $M \le M' < M_1' \le M_1$ and

$$\varphi(m) = h(\sqrt{d+q} - \sqrt{d})\sqrt{m} + x_1 mq + x_2((d+q)^2 - d^2)m^2 \ .$$

Since $|\varphi'''(m)| \asymp hqD^{-1/2}M^{-5/2}$ Lemma 4 with $k = 3$ gives

$$\left| \sum_{M'<m\le M_1'} e(\varphi(m)) \right| \ll h^{1/6}q^{1/6}D^{-1/12}M^{7/12} + M^{3/4} + h^{-1/4}q^{-1/4}D^{1/8}M^{7/8} \ . \tag{19}$$

Using (18) and (19) we get

$$|W_2| \ll X^{15/16+\varepsilon}. \tag{20}$$

For the Type I sums we use again Lemma 4 with $k = 3$ and we obtain

$$|W_1| \ , \ |W_1'| \ll X^{9/10+\varepsilon}.$$

The last inequality and (20) prove the lemma.

Proof of the Theorem. We define the function

$$\chi(t) = \begin{cases} 1 & , \text{ if } \|t\| \le P^{-\lambda} \\ 0 & , \text{ if } \|t\| > P^{-\lambda} \end{cases} .$$

Obviously

$$J_0 = \sum \chi(\sqrt{p_1}) \cdots \chi(\sqrt{p_k})$$

where the summation is over all primes $p_1, \ldots, p_k$ satisfying (2). Using Lemma 1 we construct two functions $\chi_1(t)$ and $\chi_2(t)$ corresponding to the next values of α, β, Δ, r:

$$\begin{aligned} &\chi_1(t): \quad \alpha = -P^{-\lambda} + 0.5\Delta\,, \quad \beta = P^{-\lambda} - 0.5\Delta\,, \quad \Delta = P^{-\lambda}/\log P\,, \quad r = [\log^2 P]\ ; \\ &\chi_2(t): \quad \alpha = -P^{-\lambda} - 0.5\Delta\,, \quad \beta = P^{-\lambda} + 0.5\Delta\,, \quad \Delta = P^{-\lambda}/\log P\,, \quad r = [\log^2 P]\ . \end{aligned}$$

Then $\chi_1(t) \le \chi(t) \le \chi_2(t)$ and so

$$J_1(P) \le J_0(P) \le J_2(P) \tag{21}$$

where

$$J_j = \sum \chi_j(\sqrt{p_1}) \cdots \chi_j(\sqrt{p_k}) \qquad , \ j = 1, 2\ .$$

Let us consider for example J_1. We have

$$J_1 = \int_0^1 \cdots \int_0^1 \left(S(x_1, \ldots, x_n)\right)^k e(-x_1N_1 - \cdots - x_nN_n)\, dx_1 \ldots dx_n \tag{22}$$

where

$$S = S(x_1, \dots, x_n) = \sum_{p \le P} \chi_1(\sqrt{p})\, e(x_1 p + \dots + x_n p^n) \ .$$

According to Lemma 1 we get

$$S = (2P^{-\lambda} - \Delta)G + V \tag{23}$$

where

$$G = G(x_1, \dots, x_n) = \sum_{p \le P} e(x_1 p + \dots + x_n p^n) \ , \tag{24}$$

$$V = V(x_1, \dots, x_n) = \sum_{|h| \ne 0} g(h) \sum_{p \le P} e(h\sqrt{p} + x_1 p + \dots + x_n p^n) \ . \tag{25}$$

The equalities (22) and (23) imply

$$J_1 = (2P^{-\lambda} - \Delta)^k J + \mathcal{O}(R)$$

where J is the number of solutions of (2), and

$$R = \sum_{s=1}^{k} P^{-\lambda(k-s)} R_s \tag{26}$$

$$R_s = \int_0^1 \cdots \int_0^1 |G(x_1, \dots, x_n)|^{(k-s)} |V(x_1, \dots, x_n)|^s \, dx_1 \dots dx_n \ .$$

Using the asymptotic formula (4) we have

$$J_1 = \frac{P^{k(1-\lambda)-0.5n(n+1)}}{(\log P)^k} \cdot \left(2^k \gamma\sigma + \mathcal{O}\left(\frac{\log\log P}{\log P} \right) \right) + \mathcal{O}(R) \ . \tag{27}$$

From the properties of the coeficients $g(h)$ and Lemmas 7 and 8 we obtain

$$V_0 = \max_{x_1, \dots, x_n} |V(x_1, \dots, x_n)| \ll P^{1-\delta} \tag{28}$$

where

$$\delta = \begin{cases} \dfrac{c}{n^3 \log^2 n} & , \text{ if } n \ge 3, \\ 1/16 & , \text{ if n=2.} \end{cases}$$

If $n \ge 3$ and $r_0(n)$ is the function defined in Lemmas 2 and 3, then following the proof of Theorem 2 of [4] we get

$$R_s \ll \begin{cases} P^{k-0.5n(n+1)-s\delta} & , \text{ if } 1 \le s \le (k - r_0), \\ P^{k-0.5n(n+1)-\delta(k-r_0-1)+\varepsilon} & , \text{ if } (k - r_0) < s \le k. \end{cases} \tag{29}$$

If $0 < \lambda < \delta/2000$, then (26) and (29) imply

$$R \ll P^{k(1-\lambda)-0.5n(n+1)-\rho_1} \tag{30}$$

for some $\rho_1 = \rho_1(n, k, \lambda) > 0$.

In the case $n = 2$ we have

$$R \ll V_0 \left(P^{-(k-1)\lambda} \int_0^1 \int_0^1 |G(x_1, x_2)|^{k-1} \, dx_1 dx_2 + \int_0^1 \int_0^1 |S(x_1, x_2)|^{k-1} \, dx_1 dx_2 \right) . \tag{31}$$

Lemma 3 implies

$$\int_0^1 \int_0^1 |G(x_1, x_2)|^{k-1} \, dx_1 dx_2 \ll P^{(k-1)-3+\varepsilon}. \tag{32}$$

Using Lemmas 1 and 4 it is easy to prove that

$$\sum_{\substack{p \le P \\ \|\sqrt{p}\| \le P^{-\lambda}}} 1 \ll P^{1-\lambda}, \tag{33}$$

and therefore

$$|S(x_1, x_2)| \ll P^{1-\lambda}.$$

The last inequality implies

$$\int_0^1 \int_0^1 |S(x_1, x_2)|^{k-1} \, dx_1 dx_2 \ll P^{(k-7)\lambda} R^* \tag{34}$$

where R^* denotes the number of the solutions of the system

$$\begin{aligned} x_1 + x_2 + x_3 &= y_1 + y_2 + y_3 \\ x_1^2 + x_2^2 + x_3^2 &= y_1^2 + y_2^2 + y_3^2 \end{aligned}$$

in integers $1 \le x_1, \ldots, y_3 \le P$ such that $\|\sqrt{x_1}\|, \ldots, \|\sqrt{y_3}\| \le P^{-\lambda}$. Arguing as in the proof of Lemma 5.4 of [7] and using (33) we get

$$R^* \ll P^{3-3\lambda+\varepsilon}. \tag{35}$$

From (34) and (35) we obtain

$$\int_0^1 \int_0^1 |S(x_1, x_2)|^{k-1} \, dx_1 dx_2 \ll P^{(k-4)(1-\lambda)+\varepsilon}. \tag{36}$$

Inequalities (28), (31), (32) and (36) imply

$$R \ll P^{k(1-\lambda)-3-\rho_2} \tag{37}$$

for some $\rho_2 = \rho_2(k, \lambda) > 0$.

From (27) and (30) in the case $n \ge 3$, and from (27) and (37) in the case $n = 2$ we derive

$$J_1 = \frac{P^{k(1-\lambda)-0.5n(n+1)}}{(\log P)^k} \cdot \left(2^k \gamma \sigma + \mathcal{O}\left(\frac{\log \log P}{\log P} \right) \right) .$$

Since similar conclusions hold for J_2 the assertion of the theorem follows from (21).

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