Abstract. Let \( r_3(n) \) be the number of representations of a positive integer \( n \) as a sum of three squares of integers. We give two alternative proofs of a conjecture of Wagon concerning the asymptotic value of the mean square of \( r_3(n) \).

1. Introduction

Problems concerning sums of three squares have a rich history. It is a classical result of Gauss that

\[
n = x_1^2 + x_2^2 + x_3^2
\]

has a solution in integers if and only if \( n \) is not of the form \( 4^a(8k + 7) \) with \( a, k \in \mathbb{Z} \). Let \( r_3(n) \) be the number of representations of \( n \) as a sum of three squares (counting signs and order). It was conjectured by Hardy and proved by Bateman [1] that

\[
(1) \quad r_3(n) = 4\pi n^{1/2} \Xi_3(n),
\]

where the singular series \( \Xi_3(n) \) is given by (16) with \( Q = \infty \).

While in principle this exact formula can be used to answer almost any question concerning \( r_3(n) \), the ensuing calculations can be tricky because of the slow convergence of the singular series \( \Xi_3(n) \). Thus, one often sidesteps (1) and attacks problems involving \( r_3(n) \) directly. For example, concerning the mean value of \( r_3(n) \), one can adapt the method of solution of the circle problem to obtain the following

\[
\sum_{n \leq x} r_3(n) \sim \frac{4}{3} \pi x^{3/2}.
\]

Moreover, such a direct approach enables us to bound the error term in this asymptotic formula. An application of a result of Landau [9, pp. 200–218] yields

\[
\sum_{n \leq x} r_3(n) = \frac{4}{3} \pi x^{3/2} + O(x^{5/4 + \epsilon})
\]

for all \( \epsilon > 0 \), and subsequent improvements on the error term have been obtained by Vinogradov [19], Chamizo and Iwaniec [3], and Heath-Brown [6].

In this note we consider the mean square of \( r_3(n) \). The following asymptotic formula was conjectured by Wagon and proved by Crandall (see [4] or [2]).
Theorem. Let \( r_3(n) \) be the number of representations of a positive integer \( n \) as a sum of three squares of integers. Then

\[
\sum_{n \leq x} r_3(n)^2 \sim \frac{8\pi^4}{21\zeta(3)} x^2.
\]

Apparently, at the time they discussed this conjecture Crandall and Wagon were unaware of the earlier work of Müller [11, 12]. He obtained a more general result which, in a special case, gives

\[
\sum_{n \leq x} r_3(n)^2 = B x^2 + O(x^{14/9}),
\]

where \( B \) is a constant. However, since in Müller’s work \( B \) arises as a specialization of a more general (and more complicated) quantity, it is not immediately clear that \( B = \frac{8\pi^4}{21\zeta(3)} \). The purpose of this paper is to give two distinct proofs of this fact: one that evaluates \( B \) in the form given by Müller and a direct proof using the Hardy–Littlewood circle method.

2. A direct proof: the circle method

Our first proof exploits the observation that the left side of (2) counts solutions of the equation

\[
m_1^2 + m_2^2 + m_3^2 = m_4^2 + m_5^2 + m_6^2
\]

in integers \( m_1, \ldots, m_6 \) with \( |m_j| \leq x \). This is exactly the kind of problem that the circle method was designed for. The additional constraint \( m_1^2 + m_2^2 + m_3^2 \leq x \) causes some technical difficulties, but those are minor.

Set \( N = \sqrt{x} \) and define

\[
f(\alpha) = \sum_{m \leq N} e(\alpha m^2),
\]

where \( e(z) = e^{2\pi iz} \). Then for an integer \( n \leq x \), the number \( r^*(n) \) of representations of \( n \) as a sum of three squares of positive integers is

\[
r^*(n) = \int_0^1 f(\alpha)^3 e(-an) d\alpha.
\]

Since \( r_3(n) = 8r^*(n) + O(r_2(n)) \), where \( r_2(n) \) is the number of representations of \( n \) as a sum of two squares, we have

\[
\sum_{n \leq x} r_3(n)^2 = 64 \sum_{n \leq x} r^*(n)^2 + O(x^{1/2+\epsilon}).
\]

Therefore, it suffices to evaluate the mean square of \( r^*(n) \). Let

\[
P = N/4 \quad \text{and} \quad Q = N^{1/2}.
\]

We introduce the sets

\[
\mathcal{M}(q, a) = \{ \alpha \in [Q^{-1}, 1 + Q^{-1}] : |q\alpha - a| \leq PN^{-2} \}
\]

and

\[
\mathcal{M} = \bigcup_{q \leq Q} \bigcup_{1 \leq a \leq q} \mathcal{M}(q, a), \quad m = [Q^{-1}, 1 + Q^{-1}] \setminus \mathcal{M}.
\]
We have

\[ r^*(n) = \left( \int_{\mathbb{R}} + \int_{\mathbb{M}} \right) f(\alpha)^3 e(-\alpha n) \, d\alpha = r^*(n, \mathbb{M}) + r^*(n, m), \quad \text{say.} \]

(4)

We now proceed to approximate the mean square of \( r^*(n) \) by that of \( r^*(n, \mathbb{M}) \). By (4) and Cauchy’s inequality,

\[ \sum_{n \leq x} r^*(n)^2 = \sum_{n \leq x} r^*(n, \mathbb{M})^2 + O((\Sigma_1 \Sigma_2)^{1/2} + \Sigma_2), \]

where

\[ \Sigma_1 = \sum_{n \leq x} |r^*(n, \mathbb{M})|^2, \quad \Sigma_2 = \sum_{n \leq x} |r^*(n, m)|^2. \]

By Bessel’s inequality,

\[ |\Sigma_2| = \sum_{n \leq x} \left| \int_{m} f(\alpha)^3 e(-\alpha n) \, d\alpha \right|^2 \leq \int_{m} |f(\alpha)|^6 \, d\alpha. \]

(6)

By Dirichlet’s theorem of diophantine approximation, we can write any real \( \alpha \) as \( \alpha = a/q + \beta \), where

\[ 1 \leq q \leq N^2 P^{-1}, \quad (a, q) = 1, \quad |\beta| \leq P/(qN^2). \]

When \( \alpha \in \mathbb{M} \), we have \( q \geq Q \), and hence Weyl’s inequality (see Vaughan [18, Lemma 2.4]) yields

\[ |f(\alpha)| \ll N^{1+\epsilon} (q^{-1} + N^{-1} + qN^{-2})^{1/2} \ll N^{1+\epsilon} Q^{-1}. \]

(7)

Furthermore, we have

\[ \int_{0}^{1} |f(\alpha)|^4 \, d\alpha \ll N^{2+\epsilon}, \]

(8)

because the integral on the left equals the number of solutions of

\[ m_1^2 + m_2^2 = m_3^2 + m_4^2 \]

in integers \( m_1, \ldots, m_4 \leq N \). For each choice of \( m_1 \) and \( m_2 \), this equation has \( \ll N^\epsilon \) solutions. Combining (6)–(8) and replacing \( \epsilon \) by \( \epsilon/3 \), we obtain

\[ \Sigma_2 \ll N^{4+\epsilon} Q^{-1}. \]

Furthermore, another appeal to Bessel’s inequality and appeals to (8) and to the trivial estimate \( |f(\alpha)| \leq N \) yield

\[ \Sigma_1 \leq \int_{\mathbb{R}} |f(\alpha)|^6 \, d\alpha \leq \int_{0}^{1} |f(\alpha)|^6 \, d\alpha \ll N^{4+\epsilon}. \]

(10)

We now define a function \( f^* \) on \( \mathbb{M} \) by setting

\[ f^*(\alpha) = q^{-1} S(q, a)v(\alpha - a/q) \quad \text{for } \alpha \in \mathbb{M}(q, a) \subseteq \mathbb{M}; \]

where

\[ S(q, a) = \sum_{1 \leq h \leq q} e(ah^2/q), \quad v(\beta) = \frac{1}{2} \sum_{m \leq x} m^{-1/2} e(\beta m). \]
Similarly to (5),

\[ \sum_{n \leq x} r^*(n, \mathfrak{M})^2 = \sum_{n \leq x} R^*(n)^2 + O(\Sigma_3 + (\Sigma_1 \Sigma_3)^{1/2}), \]

where

\[ \Sigma_3 = \sum_{n \leq x} \left| \int_{\mathfrak{M}} \left[ f(\alpha)^3 - f^*(\alpha)^3 \right] e(-\alpha n) \, d\alpha \right|^2 \leq \int_{\mathfrak{M}} \left| f(\alpha)^3 - f^*(\alpha)^3 \right|^2 \, d\alpha, \]

after yet another appeal to Bessel’s inequality. By [18, Theorem 4.1], when \( \alpha \in \mathfrak{M}(q,a) \),

\[ f(\alpha) = f^*(\alpha) + O(q^{1/2+\epsilon}). \]

Thus,

\[ \int_{\mathfrak{M}(q,a)} \left| f(\alpha)^3 - f^*(\alpha)^3 \right|^2 \, d\alpha \ll q^{1+2\epsilon} \int_{\mathfrak{M}(q,a)} (|f(\alpha)|^4 + q^{2+4\epsilon}) \, d\alpha, \]

whence

\[ \int_{\mathfrak{M}} \left| f(\alpha)^3 - f^*(\alpha)^3 \right|^2 \, d\alpha \ll Q^{1+2\epsilon} \int_0^1 |f(\alpha)|^4 \, d\alpha + PQ^4N^{-2}. \]

Bounding the last integral using (8) and substituting the ensuing estimate into (12), we obtain

\[ \Sigma_3 \ll QN^{2+2\epsilon} + PQ^4N^{-2+3\epsilon} \ll QN^{2+2\epsilon}. \]

Combining (5), (9)–(11), and (13), we deduce that

\[ \sum_{n \leq x} r^*(n)^2 = \sum_{n \leq x} R^*(n)^2 + O(\Sigma_4 + Q^{1/2} + N^{3+\epsilon} Q^{1/2}). \]

We now proceed to evaluate the main term in (14). We have

\[ \int_{\mathfrak{M}(q,a)} f^*(\alpha)^3 e(-\alpha n) \, d\alpha = q^{-3} S(q,a)^3 e(-an/q) \int_{\mathfrak{M}(q,0)} \nu(\beta)^3 e(-\beta n) \, d\beta, \]

so

\[ R^*(n) = \sum_{q \leq Q} A(q,n) I(q,n), \]

where

\[ A(q,n) = \sum_{\substack{1 \leq a \leq q \\ (a,q) = 1}} q^{-3} S(q,a)^3 e(-an/q), \quad I(q,n) = \int_{\mathfrak{M}(q,0)} \nu(\beta)^3 e(-\beta n) \, d\beta. \]

Hence,

\[ \sum_{n \leq x} R^*(n)^2 = \sum_{n \leq x} I(n)^2 \Xi_3(n, Q)^2 + O(\Sigma_4 \Xi_3)^{1/2} + \Sigma_5), \]
where

\begin{align}
\Xi_3(n, Q) &= \sum_{q \leq Q} A(q, n), \\
I(n) &= \int_{-1/2}^{1/2} v(\beta)^3 e(-\beta n) \, d\beta,
\end{align}

\begin{align*}
\Sigma_4 &= \sum_{n \leq x} I(n)^2 \left( \sum_{q \leq Q} |A(q, n)| \right)^2, \\
\Sigma_5 &= \sum_{n \leq x} \left( \sum_{q \leq Q} |A(q, n)(I(n) - I(q, n))| \right)^2.
\end{align*}

By [18, Theorem 2.3] and [18, Theorem 4.2],

\begin{equation}
I(n) = \Gamma(3/2)^2 \sqrt{n} + O(1) = \frac{\pi}{4} \sqrt{n} + O(1), \quad A(q, n) \ll q^{-1/2}.
\end{equation}

Furthermore, since \( A(q, n) \) is multiplicative in \( q \), [18, Lemma 4.7] yields

\begin{equation}
\sum_{q \leq Q} |A(q, n)| \leq \prod_{p \leq Q} \left( 1 + |A(p, n)| + |A(p^2, n)| + \cdots \right) \ll \prod_{p \leq Q} \left( 1 + c_1(p, n)p^{-3/2} + 3c_1p^{-1} \right) \ll (nQ)^c,
\end{equation}

where \( c_1 > 0 \) is an absolute constant. In particular, we have

\begin{equation}
\Sigma_4 \ll N^{4+\epsilon}.
\end{equation}

We now turn to the estimation of \( \Sigma_5 \). By Cauchy’s inequality and the second bound in (17),

\begin{equation}
\Sigma_5 \ll (\log Q) \sum_{n \leq x} \sum_{q \leq Q} |I(n) - I(n, q)|^2
\end{equation}

Another application of Bessel’s inequality gives

\begin{equation}
\sum_{n \leq x} |I(n) - I(n, q)|^2 \leq 2 \int_{P/qN^2}^{1/2} |v(\beta)|^6 \, d\beta.
\end{equation}

Using [18, Lemma 2.8] to estimate the last integral, we deduce that

\begin{equation}
\Sigma_5 \ll \log Q \sum_{q \leq Q} \left( q^2 N^4 p^{-2} + 1 \right) \ll N^2 Q^{3+\epsilon}.
\end{equation}

Substituting this inequality and (19) into (15), we conclude that

\begin{equation}
\sum_{n \leq x} R^*(n)^2 = \sum_{n \leq x} I(n)^2 \Xi_3(n, Q)^2 + O(N^{3+\epsilon} Q^{3/2}).
\end{equation}

We then use (17) and (18) to replace \( I(n) \) on the right side of (20) by \( \frac{\pi}{4} \sqrt{n} \). We get

\begin{equation}
\sum_{n \leq x} I(n)^2 \Xi_3(n, Q)^2 = \frac{\pi^2}{16} \sum_{n \leq x} n\Xi_3(n, Q)^2 + O(N^{3+\epsilon}).
\end{equation}

Together with (14) and (20), this leads to the asymptotic formula

\begin{equation}
\sum_{n \leq x} R^*(n)^2 = \frac{\pi^2}{16} \sum_{n \leq x} n\Xi_3(n, Q)^2 + O(N^{4+\epsilon} Q^{-1/2} + N^{3+\epsilon} Q^{3/2}).
\end{equation}
Finally, we evaluate the sum on the right side of (21). On observing that \( \Xi_3(n, Q) \) is in fact a real number, we have

\[
\sum_{n \leq t} \Xi_3(n, Q)^2 = \sum_{q_1, q_2 \leq Q} \sum_{1 \leq a_1 \leq q_1} \sum_{1 \leq a_2 \leq q_2} ( q_1 q_2 )^{-3} S(q_1, a_1)^3 S(q_2, -a_2)^3 \sum_{n \leq t} e( (a_1/q_1 - a_2/q_2) n ).
\]

As the sum over \( n \) equals \( t + O(1) \) when \( a_1 = a_2 \) and \( q_1 = q_2 \) and \( O(q_1 q_2) \) otherwise, we get

\[
\sum_{n \leq t} \Xi_3(n, Q)^2 = t \sum_{q \leq Q} \sum_{1 \leq a \leq q} q^{-6} |S(q, a)|^6 + O(\Sigma_6),
\]

where

\[
\Sigma_6 = \sum_{q \leq Q} \sum_{1 \leq a \leq q} q^{-2} |S(q, a)|^3 \ll Q^{3/2}.
\]

We find that

\[
\sum_{n \leq t} \Xi_3(n, Q)^2 = B_1 t + O(tQ^{-1} + Q^3),
\]

with

\[
B_1 = \sum_{q=1}^{\infty} \sum_{1 \leq a \leq q} q^{-6} |S(q, a)|^6.
\]

Thus, by partial summation,

\[
\sum_{n \leq x} n \Xi_3(n, Q)^2 = (B_1/2) x^2 + O(x^2 Q^{-1} + x Q^3).
\]

Combining this asymptotic formula with (21), we deduce that

\[
\sum_{n \leq x} r^6(n)^2 = \frac{\pi^2}{32} B_1 x^2 + O(x^{15/8+\epsilon}).
\]

Recalling (3), we see that (2) will follow if we show that

\[
B_1 = \frac{8 \zeta(2)}{7 \zeta(3)}.
\]

This, however, follows easily from the well-known formula (see [7, §7.5])

\[
|S(q, a)| = \begin{cases} \sqrt{q} & \text{if } q \equiv 1 \pmod{2}, \\ \sqrt{2q} & \text{if } q \equiv 0 \pmod{4}, \\ 0 & \text{if } q \equiv 2 \pmod{4}. \end{cases}
\]

Indeed, (22) yields

\[
B_1 = \frac{4}{3} \sum_{q \text{ odd}} q^{-3} \phi(q) = \frac{8 \zeta(2)}{7 \zeta(3)},
\]

where the last step uses the Euler product of \( \zeta(s) \). This completes the proof of our theorem.
3. Second Proof of Theorem

Rankin [13] and Selberg [17] independently introduced an important method which allows one to study the analytic behavior of the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

where \(a(n)\) are Fourier coefficients of a holomorphic cusp form for some congruence subgroup of \(\Gamma = SL_2(\mathbb{Z})\). Originally the method was for holomorphic cusp forms. Zagier [20] extended the method to cover forms that are not cuspidal and may not decay rapidly at infinity. Mülter [11, 12] considered the case where \(a(n)\) is the Fourier coefficient of non-holomorphic cusp or non-cusp form of real weight with respect to a Fuchsian group of the first kind. It is this last approach we wish to discuss. Note that if we apply a Tauberian theorem to the above Dirichlet series, we then gain information on the asymptotic behavior of the partial sum

$$\sum_{n \leq x} a(n).$$

We now discuss Mülter’s elegant work. For details regarding discontinuous groups and automorphic forms, see [8, 10, 11, 14, 15, 16]. Let \(\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}\) denote the upper half plane and \(G = SL(2, \mathbb{R})\) the special linear group of all \(2 \times 2\) matrices with determinant 1. \(G\) acts on \(\mathbb{H}\) by

$$z \mapsto gz = \frac{az + b}{cz + d}$$

for \(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G\). We write \(y = y(z) = \Im(z)\). Thus we have

$$y(gz) = \frac{y}{|cz + d|^2}.$$  

Let \(dx\,dy\) denote the Lebesgue measure in the plane. Then the measure

$$d\mu = \frac{dx\,dy}{y^2}$$

is invariant under the action of \(G\) on \(\mathbb{H}\). A discrete subgroup \(\Gamma\) of \(G\) is called a **Fuchsian group of the first kind** if its fundamental domain \(\Gamma \setminus \mathbb{H}\) has finite volume. Let \(\Gamma\) be a Fuchsian group of the first kind containing \(\pm I\) where \(I\) is the identity matrix. Let \(\mathcal{F}(\Gamma, \chi, k, \lambda)\) denote the space of (non-holomorphic) automorphic forms of real weight \(k\), eigenvalue \(\lambda = \frac{1}{4} - \rho^2, \Re(\rho) \geq 0\), and multiplier system \(\chi\). For \(k \in \mathbb{R}, g \in SL(2, \mathbb{R})\) and \(f : \mathbb{H} \to \mathbb{C}\), we define the stroke operator \(|_k\) by

$$(f|_k g)(z) := \left(\frac{cz + d}{|cz + d|}\right)^{-k} f(gz)$$

where \(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\). The transformation law for \(f \in \mathcal{F}(\Gamma, \chi, k, \lambda)\) is then

$$(f|_k g)(z) = \chi(g)f(z)$$

for all \(g \in \Gamma\). Automorphic forms \(f \in \mathcal{F}(\Gamma, \chi, k, \lambda)\) have a Fourier expansion at every cusp \(\kappa\) of \(\Gamma\), namely

$$A_{\kappa,0}(y) + \sum_{n \neq 0} a_{\kappa,n} W_{(\text{sgn} n)\chi}(4\pi|n + \mu_n| y) e((n + \mu_{\kappa})x),$$

where \(\mu_{\kappa}\) is the cusp parameter and \(a_{\kappa,n}\) are the Fourier coefficients of \(f\) at \(\kappa\). The functions \(W_{a,\rho}\) are Whittaker functions (see [11, §3]), \(A_{\kappa,0}(y) = 0\) if \(\mu_k \neq 0\) and
\[ A_{\kappa,0}(y) = \begin{cases} 
  a_{\kappa,0}y^{1/2+\rho} + b_{\kappa,0}y^{1/2-\rho} & \text{if } \mu_\kappa = 0, \rho \neq 0, \\
  a_{\kappa,0}y^{1/2} + b_{\kappa,0}y^{1/2} \log y & \text{if } \mu_\kappa = 0, \rho = 0. 
\end{cases} \]

An automorphic form \( f \) is called a cusp form if \( a_{\kappa,0} = b_{\kappa,0} = 0 \) for all cusps \( \kappa \) of \( \Gamma \). Now consider the Dirichlet series

\[ S_\kappa(f, s) = \sum_{n>0} \frac{|a_{\kappa,n}|^2}{(n + \mu_\kappa)^s}. \]

This series is absolutely convergent for \( \Re(s) > 2\Re(\rho) \) and has been shown [12] to have meromorphic continuation in the entire complex plane. In what follows, we will only be interested in the case \( f \) is not a cusp form. If \( f \) is not a cusp form and \( \Re(\rho) > 0 \), then \( S_\kappa(f, s) \) has a simple pole at \( s = 2\Re(\rho) \) with residue

\[ \beta_\kappa(f) = \res_{s=2\Re(\rho)} S_\kappa(f, s) = (4\pi)^{2\Re(\rho)} b^+(k/2, \rho) \sum_{\substack{\varphi \in \Gamma \backslash \kappa,\iota \\gamma \in \Gamma \\varphi \kappa,\iota \\gamma}} \varphi_{\kappa,\iota}(1 + 2\Re(\rho))|a_{\kappa,0}|^2, \]

where \( K \) denotes a complete set of \( \Gamma \)-inequivalent cusps, \( \varphi_{\kappa,\iota}(1 + 2\Re(\rho)) > 0 \) and \( b^+(k/2, \rho) > 0 \) if \( \rho + 1/2 \pm k/2 \) is a non-negative integer. For the definition of the functions \( \varphi_{\kappa,\iota} \) and \( b^+ \), see Lemma 3.6 and (69) in [12]. This result (23) and a Tauberian argument then provide the asymptotic behaviour of the summatory function

\[ \sum_{n \leq x} |a_{\kappa,n}|^2 n + \mu_\kappa'|. \]

Precisely, we have (see [11, Theorem 2.1] or [12, Theorem 5.2]) that

\[ \sum_{n \leq x} |a_{\kappa,n}|^2 n + \mu_\kappa'| = \sum_{\varepsilon = \pm 1} \res_{s=\Re(\rho)} S_\kappa(f, s) \frac{x^{s+\rho}}{s+r} + O(x^{s+2\Re(\rho) - \gamma}(\log x)^\varepsilon), \]

where \( 2\Re(\rho) + r \geq 0, R = \pm 2\Re(\rho), \pm 2i\Im(\rho), 0, -r \), \( \gamma = (2 + 8\Re(\rho))(5 + 16\Re(\rho))^{-1} \), and \( g = \max(0, b - 1) \); \( b \) denotes the order of the pole of \( S_\kappa(f, s)(r + s)^{-1}x^{s+\rho} \) at \( s = 2\Re(\rho) \) \((0 \leq b \leq 5)\).

We now consider an application of (24). Let \( Q \in \mathbb{Z}^{m\times m} \) be a non-singular symmetric matrix with even diagonal entries and \( g(x) = \frac{1}{2} Qg(x) = \frac{1}{2} x^T Qx, x \in \mathbb{Z}^m \), the associated quadratic form in \( m \geq 3 \) variables. Here we assume that \( g(x) \) is positive definite. Let \( r(Q, n) \) denote the number of presentations of \( n \) by the quadratic form \( Q \). Now consider the theta function

\[ \theta_Q(z) = \sum_{x \in \mathbb{Z}^m} e^{\pi i z Q(x)}. \]

By [11, Lemma 6.1], the Dirichlet series associated with the automorphic form \( \theta_Q \) is

\[ (4\pi)^{-m/4} \zeta_Q(m/4 + s) \]

where

\[ \zeta_Q(s) = \sum_{n=1}^{\infty} \frac{r(Q, n)}{n^s} = \sum_{x \in \mathbb{Z}^m \setminus \{0\}} g(x)^{-s} \]

for \( \Re(s) > m/2 \). Using (24), Müller proved the following (see [11, Theorem 6.1])

**Theorem** (Müller). Let \( g(x) = \frac{1}{2} Qg(x) = \frac{1}{2} x^T Qx, x \in \mathbb{Z}^m \) be a primitive positive definite quadratic form in \( m \geq 3 \) variables with integral coefficients. Then

\[ \sum_{n \leq x} r(Q, n)^2 = Bx^{m-1} + O(x^{(m-1)4m-15/11m+5}) \]
Thus for

\[
B = (4\pi)^{m/2} \frac{\beta_{\infty}(\theta_Q)}{m-1}
\]

and \(\beta_{\infty}(\theta_Q)\) is given by (23).

We are now in a position to prove our theorem in page 2.

**Proof.** We are interested in the case \(q(x) = x_1^2 + x_2^2 + x_3^2\) and so \(r(Q, n) = r_3(n)\) counts the number of representations of \(n\) as a sum of three squares. By Müller’s Theorem above,

\[
\sum_{n \leq x} r_3(n)^2 = B x^2 + O\left(x^{14/9}\right)
\]

where \(B\) is a computable constant. Specifically, we have by (23) (with \(k = 3/2\) and \(\rho = 1/4\))

\[
B = \frac{4\pi^2}{3-1} b^+ (3/4, 1/4) \sum_{i \in K} \varphi_{\infty,i}(3/2) |a_{i,0}|^2.
\]

where \(K\) denotes a complete set of \(\Gamma_0(4)\)-inequivalent cusps and \(a_{i,0}\) is the 0-th Fourier coefficient of \(\theta_Q(z)\) at a rational cusp \(i\). Choose \(K = \{1, \frac{1}{2}, \frac{1}{4}\}\). Then by p. 145 and (67) in [11], we have

\[
|a_{i,0}|^2 = W_i^3 |G(S_i)|^2
\]

where \(i = u/w, (u, w) = 1, w \geq 1, W_i\) is width of the cusp \(i\), and

\[
|G(S_i)|^2 = 2^{-3} w^{-3} \left| \sum_{x=1}^w \frac{\varphi_i(w)}{w^2} \right|^6.
\]

As \(W_{1/4} = W_{1/2} = 1, W_1 = 4\), we have \(|a_{1,0}|^2 = 1, |a_{1/2,0}|^2 = 0,\) and \(|a_{1/4,0}|^2 = 1\). An explicit description of the functions \(\varphi_{\infty,i}(s)\) in the case \(\Gamma_0(4)\) is given by (see (1.17) and p. 247 in [5])

\[
\varphi_{\infty,1/4}(s) = 2^{1-4s} (1 - 2^{-2s})^{-1} \pi^{1/2} \frac{\Gamma(s-1/2) \zeta(2s-1)}{\Gamma(s) \zeta(2s)},
\]

\[
\varphi_{\infty,1/2}(s) = \varphi_{\infty,1}(s) = 2^{-2s} (1 - 2^{-2s})^{-1} \left(1 - 2^{-2s}\right) \pi^{1/2} \frac{\Gamma(s-1/2) \zeta(2s-1)}{\Gamma(s) \zeta(2s)}.
\]

Thus for \(s = 3/2\), we have

\[
\varphi_{\infty,1/4}(3/2) = 2^{-5} (1 - 2^{-3})^{-1} \pi^2 \frac{\zeta(2)}{\Gamma(3/2) \zeta(3)},
\]

\[
\varphi_{\infty,1/2}(3/2) = \varphi_{\infty,1}(3/2) = 2^{-3} (1 - 2^{-3})^{-1} \left(1 - 2^{-3}\right) \pi^2 \frac{\zeta(2)}{\Gamma(3/2) \zeta(3)}.
\]

Now, from p. 65 in [12], we have

\[
b^+(3/4, 1/4) = G_{1/4,1/4}^*(3/2).
\]

By Lemma 3.3 and (16) in [12],

\[
G_{1/4,1/4}^*(s) = \frac{\Gamma(s+1/2)^{-1}}{s}
\]
and so $b^{+}(3/4, 1/4) = \Gamma(2)^{-1}$. In total,

\[
B = \frac{(4\pi)^2}{(3-1)} \frac{1}{\Gamma(2)} \left( 2^{-3}(1 - 2^{-3})^{-1}(1 - 2^{-2})\pi^{1/2} \frac{\zeta(2)}{\Gamma(3/2)\zeta(3)} \\
+ 2^{-5}(1 - 2^{-3})^{-1}\pi^{1/2} \frac{\zeta(2)}{\Gamma(3/2)\zeta(3)} \right)
\]

\[
= \frac{8\pi^4}{21\zeta(3)},
\]

Thus

\[
\sum_{n \leq x} r_3(n)^2 \sim \frac{8\pi^4}{21\zeta(3)} x^2.
\]

**Remark.** Müller’s Theorem can also be used to obtain the mean square value of sums of $N > 3$ squares. Precisely, if $r_N(n)$ is the number of representations of $n$ by $N > 3$ squares, then a calculation similar to the second proof of our theorem yields (compare with Theorem 3.3 in [2])

\[
\sum_{n \leq x} r_N(n)^2 = W_N x^{N-1} + O\left( x^{(N-1)\frac{4N-5}{4N-3}} \right)
\]

where

\[
W_N = \frac{1}{(N-1)(1-2^{-N}) \Gamma(N/2)^2} \frac{\pi^N \zeta(N-1)}{\zeta(N)}.
\]

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