ON THE WARING–GOLDBACH PROBLEM: EXCEPTIONAL SETS FOR SUMS OF CUBES AND HIGHER POWERS

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1. Introduction

The Waring–Goldbach problem is concerned with the solvability of the equation

$$p_1^k + \cdots + p_s^k = n, \tag{1.1}$$

where $p_1, \dots, p_s$ are prime unknowns. It is conjectured that for any pair of integers $k, s \in \mathbb{N}$ with $s \geq k+1$ there exist a fixed modulus $q_{k,s}$ and a collection $\mathcal{N}_{k,s}$ of congruence classes mod $q_{k,s}$ such that (1.1) is solvable for all sufficiently large $n \in \mathcal{N}_{k,s}$. While a proof of this conjecture appears to be beyond the reach of present methods, some significant progress has been made. Let $H(k)$ denote the least s for which a set of integers $\mathcal{N}_{k,s}$ as above exists. The first breakthrough came in 1937, when I. M. Vinogradov [23] developed a new method for estimating sums over primes and used it to solve the ternary Goldbach problem for sufficiently large n, that is, he proved that $H(1) \leq 3$. Shortly thereafter, Hua [8] showed that

$$H(k) \leq 2^k + 1 \quad \text{for all } k \geq 1, \tag{1.2}$$

which is the best result to date for $k \leq 3$. On the other hand, when $k \geq 4$, (1.2) has been improved on. If k is large, an approach based on I. M. Vinogradov's mean-value theorem gives the best results. In particular, using this approach, Hua [9, Theorem 14] showed that

$$H(k) \leq 2k(2\log k + \log\log k + O(1)).$$

For smaller $k \geq 4$, the sharpest bounds for $H(k)$ have been obtained by variants of Davenport's iterative method. Thanigasalam [19] obtained

$$H(6) \leq 33, \quad H(7) \leq 47 \quad H(8) \leq 63, \quad H(9) \leq 83, \quad \text{and} \quad H(10) \leq 107,$$

and Kawada and Wooley [10] proved recently that

$$H(4) \leq 14 \quad \text{and} \quad H(5) \leq 21.$$

One can reduce further the number of variables needed to solve (1.1) by trying to represent *almost all* $n \in \mathcal{N}_{k,s}$ instead of all but finitely many such n. Let $E_{k,s}(x)$ be the number of $n \in \mathcal{N}_{k,s} \cap (1, x]$ for which

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(1.1) cannot be solved in primes $p_1, \dots, p_s$. Exploiting the nature of the proofs of the above bounds for $H(k)$, one can show that along with each estimate of the form $H(k) \le s_0(k)$ one also has

$$E_{k,s}(x) \ll x(\log x)^{-A} \tag{1.3}$$

for any fixed $A > 0$ and for any $s \ge \frac{1}{2}s_0(k)$. In this paper, we pursue improvements on the right side of (1.3) for $s < s_0(k)$. The first such improvement was obtained by Vaughan [21], who showed that

$$E_{1,2}(x) \ll x \exp\left(-c\sqrt{\log x}\right)$$

for some constant $c > 0$. Shortly afterward, Montgomery and Vaughan [17] proved that there exists an absolute constant $\theta < 1$ such that

$$E_{1,2}(x) \ll x^{\theta},$$

and several authors used their method to give such estimates with explicit values of θ, the most recent result being $\theta = 0.914$ due to Li [13]. The first to obtain a similar estimate for an exceptional set for sums of squares or higher powers of primes were Leung and Liu [12], who showed that

$$E_{2,3}(x) \ll x^{\theta}, \tag{1.4}$$

with an absolute constant $\theta < 1$. Later, Bauer, Liu and Zhan [2, 15, 16] obtained a series of refinements of this estimate, the most recent being given by Liu and Zhan [16]. They establish (1.4) for every fixed θ in the interval $11/12 < \theta < 1$ and, in fact, their work and Lemma 2.3 below suffice to extend the range for θ to $7/8 < \theta < 1$ (see Kumchev [11, Theorem 6]). Furthermore, Liu and Liu [14] and Ren [18] proved that for any fixed $\varepsilon > 0$ one has

$$E_{2,4}(x) \ll x^{13/15+\varepsilon} \quad \text{and} \quad E_{3,5}(x) \ll x^{152/153+\varepsilon}, \tag{1.5}$$

respectively, and Ren's method can be easily adjusted to produce the bounds

$$E_{3,s}(x) \ll x^{1-(s-4)/153+\varepsilon} \qquad (s = 6, 7, 8). \tag{1.6}$$

Using a recent refinement of the treatment of exceptional sets for additive representations (see Brüdern, Kawada and Wooley [4]), Wooley [25, 26] improved further on (1.5) and (1.6). He proved that for any fixed $\varepsilon > 0$ one has

$$\begin{gathered} E_{2,4}(x) \ll x^{13/30+\varepsilon}, \\ E_{3,5}(x) \ll x^{35/36+\varepsilon}, \quad E_{3,6}(x) \ll x^{17/18+\varepsilon}, \\ E_{3,7}(x) \ll x^{23/36+\varepsilon}, \quad E_{3,8}(x) \ll x^{11/36+\varepsilon}. \end{gathered} \tag{1.7}$$

Our first theorem improves on (1.7). Let us define the sets $\mathcal{N}_{3,s}$, $5 \le s \le 8$, by

$$\mathcal{N}_{3,5} = \{n \in \mathbb{N} : \ n \equiv 1 \ (\mathrm{mod}\ 2), \ n \not\equiv 0, \pm 2 \ (\mathrm{mod}\ 9), \ n \not\equiv 0 \ (\mathrm{mod}\ 7)\},$$

$$\mathcal{N}_{3,6} = \{n \in \mathbb{N} : \ n \equiv 0 \ (\mathrm{mod}\ 2), \ n \not\equiv \pm 1 \ (\mathrm{mod}\ 9)\},$$

$$\mathcal{N}_{3,7} = \{n \in \mathbb{N} : \ n \equiv 1 \ (\mathrm{mod}\ 2), \ n \not\equiv 0 \ (\mathrm{mod}\ 9)\},$$

$$\mathcal{N}_{3,8} = \{n \in \mathbb{N} : \ n \equiv 0 \ (\mathrm{mod}\ 2)\}.$$

We can state our result as follows.

Theorem 1. *Let $5 \le s \le 8$ be an integer, let $E_{3,s}(x)$ be defined as above, and define θ_s by*

$$\theta_5 = 79/84, \quad \theta_6 = 31/35, \quad \theta_7 = 17/28, \quad \theta_8 = 23/84.$$

Then

$$E_{3,s}(x) \ll x^{\theta_s}.$$

Remark 1.1. The method of proof produces a bound of the form

$$E_{3,s}(x) \ll x^{\theta'_s + \varepsilon},$$

where θ'_s depends on the lowest positive zero of a function defined in terms of certain multiple integrals (see (7.28)). Since finding the exact value of θ'_s seems to be an impossible task, we have replaced θ'_s by a reasonably close upper bound. Thus, one can easily improve the value of θ_5 in Theorem 1 to, say, $\theta_5 = 79/84 - 10^{-7}$, but one needs a new idea in order to obtain $\theta_5 = 79/84 - 10^{-4}$.

We establish Theorem 1 in §§6 and 7, using the following tools.

(1) New exponential sum estimates from [11].

(2) A larger than usual set of major arcs. Using the standard treatment of the major arcs in the Waring–Goldbach problem, one can only obtain estimates of the form (1.3) for exceptional sets. Earlier estimates for $E_{3,s}(x)$ by Ren [18] and Wooley [25] rely on the more refined treatment in [18, Theorem 2], but that result is insufficient for our purposes. In §3, we obtain new results concerning the major arcs in the Waring–Goldbach problem, which let us take the set of major arcs in the proof of Theorem 1 larger than allowed by Ren's theorem. Our approach is somewhat different from that used in [18] or in related work on sums of squares [14, 15] and also enables us to treat fourth and higher powers as well as quasi-Waring–Goldbach problems (i.e., problems in which some of the unknowns are almost prime instead of prime).

(3) Sieve ideas. Using only (1) and (2), we can already obtain a considerable improvement on (1.7), but the result would be somewhat weaker than Theorem 1. For example, we would only have

$$E_{3,5}(x) \ll x^{20/21+\varepsilon}.$$

In order to establish Theorem 1, we employ sieve techniques. We use the sieve method in Harman [6, 7], and for the result on sums of six cubes we also need a variant of the vector sieve of Brüdern and Fouvry [3]. These matters are discussed in §7.

The methods we develop for the proof of Theorem 1 can be generalized quite naturally to yield estimates for exceptional sets for sums of fourth and higher powers of primes. Next we state a few such estimates. Theorem 2, which we establish in §§4 and 5, contains the results on biquadrates of primes that one can deduce from the estimates in [11] and §3. For the sake of brevity, we have avoided the use of sieve methods, although such use would undoubtedly lead to a slightly stronger result. We have also excluded $E_{4,7}(x)$, $E_{4,8}(x)$ and $E_{4,9}(x)$ from consideration, since the treatment of those cases—while possible—would complicate further the treatment of the major arcs. Theorem 3, whose proof we omit, lists the results we can prove for fifth and higher powers as well as the results on seven, eight and nine biquadrates (and also some results on ten and more biquadrates that are superseded by Theorem 2).

Theorem 2. *Let $10 \leq s \leq 13$ be an integer, let*

$$\mathcal{N}_{4,s} = \{n \in \mathbb{N} : \ n \equiv s \ (\mathrm{mod}\ 240)\},$$

and let $E_{4,s}(x)$ be defined as above. Also, define θ_s by

$$\theta_{10} = 15/16, \quad \theta_{11} = 3/4, \quad \theta_{12} = 35/48, \quad \theta_{13} = 1/2.$$

Then for any fixed $\varepsilon > 0$ one has

$$E_{4,s}(x) \ll x^{\theta_s - \delta + \varepsilon},$$

where $\delta = 335/56832$ and the implied constant depends at most on ε.

Theorem 3. *Let k and s be integers with $4 \leq k \leq 10$ and $\frac{1}{2}s_0(k) \leq s < s_0(k)$, where $s_0(k)$ is given by the following table:*

k	4	5	6	7	8	9	10
$s_0(k)$	14	21	33	47	63	83	107

Also, let $\rho(k) = \frac{2}{3} \times (k2^k)^{-1}$ and define

$$\theta_{k,s} = \begin{cases} 1 - (s-7)/48, & \text{if } k = 4, \\ 1 - (2s-21)/240, & \text{if } k = 5, \ 11 \leq s \leq 18, \\ 4/5 - (2s-37)/240, & \text{if } k = 5, \ s = 19, 20, \\ 1 - (2s - s_0(k)) \times \rho(k), & \text{if } 6 \leq k \leq 10. \end{cases}$$

Then there exists an absolute constant $\delta > 0$ such that

$$E_{k,s}(x) \ll x^{\theta_{k,s} - \delta}.$$

Notation. Throughout the paper, the letter ε denotes a sufficiently small positive real number. Any statement in which ε occurs holds for each positive ε, and any implied constant in such a statement is allowed to depend on ε. Implicit constants are also allowed to depend on k (when it appears in a statement). Any additional dependence will be mentioned explicitly. The letter p, with or without indices, is reserved for prime numbers; c denotes an absolute constant, not necessarily the same in all occurrences. Also, we often use P to denote the "main parameter"; in such situations, we write $L = \log P$.

As usual in number theory, $\mu(n)$, $\phi(n)$ and $\tau(n)$ denote, respectively, the Möbius function, the Euler totient function and the number of divisors function. Also, if $z \geq 2$, we define

$$\psi(n,z) = \begin{cases} 1, & \text{if } (n, \mathcal{P}(z)) = 1, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{where} \quad \mathcal{P}(z) = \prod_{p<z} p. \tag{1.8}$$

We write $e(x) = \exp(2\pi i x)$ and $(a,b) = \gcd(a,b)$ and use $m \sim M$ as an abbreviation for the condition $M \leq m < 2M$.

Throughout the paper, we use decompositions of the unit interval into major and minor arcs. If $1 \leq Y \leq X$, we define the set of major arcs $\mathfrak{M}(Y,X)$ as the union of the intervals

$$\mathfrak{M}(q,a;Y,X) = \left\{ \alpha \in [0,1] : \; |q\alpha - a| \leq YX^{-1} \right\}$$

with $0 \leq a \leq q \leq Y$ and $(a,q) = 1$. The corresponding set of minor arcs is denoted by $\mathfrak{m}(Y,X) = [0,1] \setminus \mathfrak{M}(Y,X)$.

2. Exponential sum estimates

In this section, we record several exponential sum estimates from [11].

Lemma 2.1. *Let $0 < \rho < 1/10$. Suppose that $\alpha \in \mathbb{R}$ and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that*

$$1 \leq q \leq Q, \quad (a,q) = 1, \quad |q\alpha - a| \leq Q^{-1}, \tag{2.1}$$

with

$$Q = P^{(9-6\rho)/5}. \tag{2.2}$$

Let $M \geq N \geq 2$, $|\xi_m| \leq 1$, $|\eta_n| \leq 1$. Then

$$\sum_{\substack{m \sim M \\ mn \sim P}} \sum_{n \sim N} \xi_m \eta_n e\left(\alpha (mn)^3\right) \ll P^{1-\rho+\varepsilon} + \frac{q^{-1/6} P^{1+\varepsilon}}{\left(1 + P^3 |\alpha - a/q|\right)^{1/2}}, \tag{2.3}$$

provided that

$$\max\left(P^{8\rho}, P^{(2+12\rho)/5}\right) \leq M \leq P^{1-2\rho}. \tag{2.4}$$

Furthermore, if $\psi(n,z)$ is defined by (1.8),

$$\sum_{\substack{m\sim M \\ mn\sim P}}\sum_{n\sim N}\xi_m\psi(n,z)e\left(\alpha(mn)^3\right)\ll P^{1-\rho+\varepsilon}+\frac{q^{-1/6}P^{1+\varepsilon}}{\left(1+P^3|\alpha-a/q|\right)^{1/2}}, \tag{2.5}$$

provided that

$$z\le\min\left(P^{(3-22\rho)/5},P^{1-10\rho}\right) \tag{2.6}$$

and

$$M\le\min\left(P^{(3-7\rho)/5},P^{1-6\rho}\right). \tag{2.7}$$

Proof. These estimates are [11, Lemmas 3.1 and 3.3] with $k=3$. ☐

Lemma 2.2. *Let $k\ge 4$ and define $\rho(k)=\frac{2}{3}\times 2^{-k}$. Suppose that $\alpha\in\mathbb{R}$ and that there exist $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfying (2.1) with*

$$Q=P^{(k^2-2k\rho(k))/(2k-1)}.$$

Then for any fixed $\varepsilon>0$ one has

$$\sum_{p\sim P}e\left(\alpha p^k\right)\ll P^{1-\rho(k)+\varepsilon}+\frac{P^{1+\varepsilon}}{\left(q+P^k|q\alpha-a|\right)^{1/2}}. \tag{2.8}$$

Proof. This is the case $k\ge 4$ of [11, Theorem 3]. ☐

Lemma 2.3. *Let $k\in\mathbb{N}$ and $\alpha\in\mathbb{R}$, and suppose that there exist $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfying*

$$1\le q\le Q,\quad (a,q)=1,\quad |q\alpha-a|<QP^{-k} \tag{2.9}$$

with $Q\le P$. Then for any fixed $\varepsilon>0$ one has

$$\sum_{p\sim P}e\left(\alpha p^k\right)\ll Q^{1/2}P^{11/20+\varepsilon}+\frac{q^{\varepsilon}PL^c}{\left(q+P^k|q\alpha-a|\right)^{1/2}}. \tag{2.10}$$

Proof. This is [11, Theorem 2]. ☐

Lemma 2.4. *Let $k\in\mathbb{N}$ and $\alpha\in\mathbb{R}$, and suppose that there exist $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfying (2.9) with $Q\le P$. Let ξ_m, η_n be complex numbers with $|\xi_m|\le\tau(m)^c$, $|\eta_n|\le\tau(n)^c$, and let z_m be defined as $z_m=m$ or $z_m=Zm^{-1}$ with $Z\in\mathbb{R}$. Let $z\ge 2$ be such that $z_m\ge z$ for all $m\sim M$, and define*

$$g(\alpha)=\sum_{\substack{m\sim M \\ mn\sim P}}\sum_{n\sim N}\sum_{r}\xi_m\eta_n\psi(mn,z)\psi(r,z_m)e\left(\alpha(mnr)^k\right),$$

where $\psi(n,z)$ is given by (1.8). Suppose that

$$MN\le P^{11/20}\quad\text{and}\quad\max_{m\sim M}z_m\le\sqrt{2P/M}.$$

Then

$$g(\alpha)\ll q^{\varepsilon}L^c\left(P\Psi(\alpha)^{-1/2}+\Psi(\alpha)^{1/2}P^{11/20}+Pz^{-1}\right), \tag{2.11}$$

where $\Psi(\alpha) = q + P^k|q\alpha - a|$.

Proof. This result combines [11, Lemma 5.6] with the remark following its statement. □

Occasionally, we also need the following simple tool that reduces the estimation of a bilinear sum to the estimation of a similar sum subject to "nicer" summation conditions. The proof can be found in [11, Lemma 2.7].

Lemma 2.5. *Let* $M, N, X \geq 2$, *let* $\Phi : \mathbb{N} \to \mathbb{C}$ *satisfy* $|\Phi(x)| \leq X$, *and define the bilinear form*

$$\mathcal{B}(M, N) = \sum_{m\sim M} \sum_{n\sim N} \xi_m \eta_n \Phi(mn),$$

where $|\xi_m| \leq 1$, $|\eta_n| \leq 1$, *and* m *and* n *are subject to a joint condition of one of the forms*

$$m < n \quad \text{or} \quad U \leq mn < U'.$$

Then

$$\mathcal{B}(M, N) \ll L \left| \sum_{m\sim M} \sum_{n\sim N} \xi'_m \eta'_n \Phi(mn) \right| + 1, \tag{2.12}$$

where $|\xi'_m| \leq |\xi_m|$, $|\eta'_n| \leq |\eta_n|$ *and* $L = \log(2MNX)$.

3. The major arcs in the Waring–Goldbach problem

In this section, we establish two general results concerning estimates for mean values of Weyl sums over major arcs.

Let $P_1, \dots, P_s$ be real numbers with

$$P^\varepsilon \leq P_s \leq \cdots \leq P_1 \leq P, \tag{3.1}$$

and define

$$S^*(q, a) = \sum_{\substack{x=1 \\ (x,q)=1}}^{q} e\left(\frac{ax^k}{q}\right), \quad B(n, q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{S^*(q, a)^s}{\phi(q)^s} e\left(-\frac{an}{q}\right),$$

$$f_i(\alpha) = \sum_{p\sim P_i} e\left(\alpha p^k\right), \quad v_i(\beta) = \int_{P_i}^{2P_i} \frac{e\left(\beta y^k\right)}{\log y} dy.$$

We obtain the following proposition.

Proposition 1. *Let* $k \geq 2$, $s \geq 5$, *and* $P^k \ll N \ll P^k$. *Let* $N \leq n < 2N$ *and define*

$$\mathfrak{S}_{k,s}(n) = \sum_{q=1}^{\infty} B(n, q), \quad J_{k,s}(n) = \int_{\mathbb{R}} v_1(\beta) \cdots v_s(\beta) e(-n\beta) d\beta. \tag{3.2}$$

Suppose that $P_1, \dots, P_s$ *satisfy* (3.1), *that* $P_5 \gg P$, *and that* $Q \leq P^{1/2-\varepsilon}$. *Then for any* $A > 0$, *we have*

$$\int_{\mathfrak{M}(Q,N)} f_1(\alpha) \cdots f_s(\alpha) e(-n\alpha) d\alpha = \mathfrak{S}_{k,s}(n) J_{k,s}(n) + O\left(P_1 \cdots P_s N^{-1} L^{-A}\right), \tag{3.3}$$

where the implied constant depends at most on A, k, s, *and* ε.

We should point out that (3.3) is not necessarily an asymptotic formula. The hypothesis $s \geq 5$ implies the absolute convergence of the singular series $\mathfrak{S}_{k,s}(n)$ and the singular integral $J_{k,s}(n)$, but we need to make additional assumptions to ensure that the "main term" in (3.3) dominates the error term. We have left the related analysis of $\mathfrak{S}_{k,s}(n)$ and $J_{k,s}(n)$ out of Proposition 1, since it is quite standard and can be carried quickly (and often more efficiently) in any particular case, in which one may want to refer to the proposition.

We derive Proposition 1 from a more general result, Proposition 2 below, in which $f_1(\alpha), \dots, f_s(\alpha)$ are replaced by the exponential sums

$$g_i(\alpha) = \sum_{m \sim P_i} \lambda_i(m) e\left(\alpha m^k\right) \qquad (1 \leq i \leq s), \tag{3.4}$$

where $\lambda_1(m), \dots, \lambda_s(m)$ are arithmetic functions having the following properties:

- (A_1) $\lambda_i(m) \ll 1$;
- (A_2) $\lambda_i(m) = 0$ unless $(m, \mathcal{P}(P^\varepsilon)) = 1$;
- (A_3) $\lambda_i(m)$ is well distributed in arithmetic progressions to small moduli, that is, for any $y \sim P_i$, $q \geq 1$, $(a, q) = 1$ and $A > 0$, we have

$$\sum_{\substack{m \leq y \\ m \equiv a \ (\mathrm{mod}\ q)}} \lambda_i(m) - \frac{1}{\phi(q)} \sum_{\substack{m \leq y \\ (m,q)=1}} \lambda_i(m) \ll y L^{-A},$$

where the implied constant depends at most on A.

Using (A_1)–(A_3) and partial summation, we find that if $\alpha \in \mathfrak{M}(q, a; L^A, N)$,

$$g_1(\alpha) \cdots g_s(\alpha) = \frac{S^*(q,a)^s}{\phi(q)^s} g_1(\beta) \cdots g_s(\beta) + O\left(P_1 \cdots P_s L^{-3A}\right), \tag{3.5}$$

where $\beta = \alpha - a/q$. Integrating both sides of (3.5) over $\mathfrak{M}(L^A, N)$, we get

$$\begin{aligned}\int_{\mathfrak{M}(L^A,N)} g_1(\alpha) \cdots g_s(\alpha) e(-n\alpha) d\alpha = \sum_{q \leq L^A} & B(n,q) J(n, L^A/(qN)) \\ & + O\left(P_1 \cdots P_s N^{-1} L^{-A}\right),\end{aligned}$$

where

$$J(n, Z) = \int_{-Z}^{Z} g_1(\beta) \cdots g_s(\beta) e(-n\beta) d\beta.$$

We now record the bounds

$$B(n,q) \ll q^{1-s/2+\varepsilon} \quad \text{and} \quad J(n,Z) \ll Z P_1 \cdots P_s. \tag{3.6}$$

The latter is trivial and the former is a direct consequence from the estimate (see [24, Problem VI.14])

$$S^*(q,a) \ll q^{1/2+\varepsilon}. \tag{3.7}$$

Using (3.6), we conclude that if $s \geq 5$,

$$\sum_{q \leq L^A} B(n,q) J(n, L^A/(qN)) = \mathfrak{R}(n, L^A) + O\left(P_1 \cdots P_s N^{-1} L^{-A/3}\right),$$

where

$$\mathfrak{R}(n,X) = \sum_{q=1}^{\infty} B(n,q)J(n,X/(qN)). \tag{3.8}$$

Therefore, if $s \geq 5$ and $A > 0$, we have

$$\int_{\mathfrak{M}(L^{3A},N)} g_1(\alpha)\cdots g_s(\alpha)e(-n\alpha)d\alpha = \mathfrak{R}(n, L^{3A}) + O\left(P_1\cdots P_s N^{-1}L^{-A}\right), \tag{3.9}$$

where the implied constant in the O-term depends at most on A and s.

Suppose now that $L^{3A} \leq Q \leq P$, that $P \ll P_t \ll P$ $(t \leq s)$, and that $\lambda_1(m), \dots, \lambda_t(m)$ satisfy the following additional hypothesis:

(A_4) if $\alpha \in \mathfrak{M}(q,a;Q,N)$, with $(a,q)=1$ and $q \leq Q$, we have

$$g_i(\alpha) \ll q^{\varepsilon}L^{c}\left(P\Psi(\alpha)^{-1/2} + P^{1-\rho_1}Q^{1/2} + P^{1-\rho_2}\right),$$

where $\rho_1, \rho_2 > 0$ and $\Psi(\alpha) = q + N|q\alpha - a|$.

Then for $\alpha \in \mathfrak{M}(q,a;Q,N) \subset \mathfrak{M}(Q,N)$, we have

$$g_1(\alpha)\cdots g_s(\alpha) \ll q^{\varepsilon}L^{c}P_1\cdots P_s\left(\Psi(\alpha)^{-t/2} + P^{-t\rho_1}Q^{t/2} + P^{-t\rho_2}\right).$$

If $t \geq 5$, we deduce that

$$\int_{\mathfrak{N}} g_1(\alpha)\cdots g_s(\alpha)e(-n\alpha)d\alpha \ll L^{c}P_1\cdots P_s N^{-1}E, \tag{3.10}$$

where $\mathfrak{N} = \mathfrak{M}(Q,N)\setminus\mathfrak{M}(L^{3A},N)$ and

$$E = L^{-A} + Q^{(t/2)+2+\varepsilon}P^{-t\rho_1} + Q^{2+\varepsilon}P^{-t\rho_2}.$$

Since the condition

$$Q \leq \min\left(P^{2t\rho_1/(t+4)-\varepsilon}, P^{t\rho_2/2-\varepsilon}\right)$$

implies that $E \ll L^{-A}$, combining (3.9) and (3.10), we obtain the following result.

Proposition 2. *Let $k \geq 2$, $s \geq 5$, and $P^k \ll N \ll P^k$. Let $g_i(\alpha)$, $1 \leq i \leq s$, be defined by* (3.4) *with $\lambda_i(m)$ satisfying* (A_1)–(A_3) *and P_i satisfying* (3.1)*. Suppose that $P_5 \gg P$, that $\lambda_1(m), \dots, \lambda_5(m)$ satisfy* (A_4)*, and that*

$$Q \leq \min\left(P^{10\rho_1/9-\varepsilon}, P^{5\rho_2/2-\varepsilon}\right), \tag{3.11}$$

where ρ_1 and ρ_2 are the numbers appearing in (A_4)*. Then for any $A > 0$ we have*

$$\int_{\mathfrak{M}(Q,N)} g_1(\alpha)\cdots g_s(\alpha)e(-n\alpha)d\alpha = \mathfrak{R}(n, L^{3A}) + O\left(P_1\cdots P_s N^{-1}L^{-A}\right),$$

where $\mathfrak{R}(n,X)$ is defined by (3.8) *and the implied constant depends at most on A, k, s, and ε.*

Proof of Proposition 1. We apply Proposition 2, choosing each function $\lambda_i(m)$ equal to the characteristic function of the set of primes. Axioms (A_1) and (A_2) are then immediate, and (A_3) is the Siegel–Walfisz theorem. Furthermore, Lemma 2.3 yields (A_4) with $\rho_1 = 9/20$ and $\rho_2 = \infty$, so (3.11) follows from the assumption $Q \le P^{1/2-\varepsilon}$ of Proposition 1. Thus, it remains to show that

$$\mathfrak{R}(n, L^{3A}) = \mathfrak{S}(n)J(n) + O\left(P_1 \cdots P_s N^{-1} L^{-A}\right).$$

This asymptotic formula follows easily from (3.6) and the estimates

$$g_i(\beta) = v_i(\beta) + O\left(P_i L^{-4A}\right) \quad \text{for} \quad |\beta| \le L^{3A} N^{-1}, \tag{3.12}$$

and

$$v_i(\beta) \ll P_i L^{-1}\left(1 + P_i^k|\beta|\right)^{-1} \quad \text{for all} \quad \beta \in \mathbb{R}. \tag{3.13}$$

The approximation (3.12) follows from the Prime Number Theorem by partial summation, while (3.13) can be obtained by partial integration (see [22, Lemma 6.2]). □

4. Minor arc estimates, I: biquadrates

The results of this section will be used in the proof of Theorem 2 to estimate the contribution from the minor arcs. We start by fixing some notation. We put

$$\nu_1 = 1, \quad \nu_2 = \tfrac{13}{16}, \quad \nu_3 = \left(\tfrac{13}{16}\right)^2, \quad \nu_4 = \left(\tfrac{13}{16}\right)^2 \tfrac{91}{111}, \quad \nu_5 = \left(\tfrac{13}{16}\right)^2 \tfrac{78}{111},$$
$$P_0 = \tfrac{1}{4}P, \quad P_j = P^{\nu_j} \quad (1 \le j \le 5), \quad f_j(\alpha) = \sum_{p \sim P_j} e\left(\alpha p^4\right) \quad (0 \le j \le 5).$$

Also, we write

$$F(\alpha) = f_1(\alpha) \cdots f_4(\alpha) f_5(\alpha)^2.$$

The exponents ν_j were suggested by Thanigasalam [20, Theorem 3], which we state in the following form.

Lemma 4.1. *Define*

$$g_j(\alpha) = \sum_{x \sim P_j} e\left(\alpha x^4\right) \quad \textit{and} \quad G(\alpha) = g_1(\alpha) \cdots g_4(\alpha) g_5(\alpha)^2.$$

Then

$$\int_0^1 |G(\alpha)|^2 d\alpha \ll P^{\varepsilon} G(0).$$

Lemma 4.2. *Let $Q \ge P^{1/12}$, let $P^4 \ll N \ll P^4$, and write $\mathfrak{m} = \mathfrak{m}(Q, N)$. We have*

$$\int_{\mathfrak{m}} |f_0(\alpha) F(\alpha)|^2 \, d\alpha \ll F(0)^2 P^{-2-\delta+\varepsilon}, \tag{4.1}$$

where $\delta = 355/14208 = 0.0249\ldots$.

Proof. We have

$$\int_{\mathfrak{m}} |f_0(\alpha)F(\alpha)|^2 \, d\alpha \le \sup_{\alpha\in\mathfrak{m}} |f_0(\alpha)|^2 \int_0^1 |F(\alpha)|^2 d\alpha. \tag{4.2}$$

Comparing the underlying diophantine equations, we deduce

$$\int_0^1 |F(\alpha)|^2 d\alpha \le \int_0^1 |G(\alpha)|^2 d\alpha. \tag{4.3}$$

Since $G(0) \ll F(0)L^6$, (4.1) follows from (4.2), (4.3), Lemma 4.1 and the estimate

$$\sup_{\alpha\in\mathfrak{m}} |f_0(\alpha)| \ll P^{23/24+\varepsilon}. \tag{4.4}$$

By Dirichlet's theorem on diophantine approximation, every real α has a rational approximation a/q subject to

$$1 \le q \le P^{47/21}, \quad (a,q)=1, \quad |q\alpha - a| \le P^{-47/21}.$$

Since for $\alpha \in \mathfrak{m}$ we also have

$$q + P^4|q\alpha - a| > Q \ge P^{1/12},$$

(4.4) follows from Lemma 2.2 with $k=4$. □

The next lemma is the main result of this section.

Lemma 4.3. *Let s be an integer with $10 \le s \le 13$. Let $Q \ge P^{1/12}$, $P^4 \ll N \ll P^4$, and write $\mathfrak{m} = \mathfrak{m}(Q,N)$. Also, let the set $\mathcal{Z} \subset \mathbb{N}$ have cardinality Z, and write*

$$K(\alpha) = \sum_{n\in\mathcal{Z}} e(\alpha n).$$

Then

$$\int_{\mathfrak{m}} \left| f_0(\alpha)^{s-6}F(\alpha)K(\alpha)\right| d\alpha \ll F(0)Z^{1/2}P^{\sigma_s-\delta/2+\varepsilon} + F(0)ZP^{s-10-\delta/3}, \tag{4.5}$$

where $\delta = 335/14208$ and σ_s is defined as follows:

$$\sigma_{10} = 15/8, \quad \sigma_{11} = 5/2, \quad \sigma_{12} = 83/24, \quad \sigma_{13} = 4.$$

Proof. We write

$$I_s(P) = \int_{\mathfrak{m}} \left| f_0(\alpha)^{s-7}K(\alpha)\right|^2 d\alpha.$$

By Cauchy's inequality and Lemma 4.2, the integral on the left side of (4.5) is bounded above by

$$I_s(P)^{1/2} \left(\int_{\mathfrak{m}} |f_0(\alpha)F(\alpha)|^2 \, d\alpha \right)^{1/2} \ll F(0)P^{-1-\delta/2+\varepsilon} I_s(P)^{1/2}.$$

Therefore, it suffices to show that

$$I_s(P) \ll P^{2\sigma_s+2+\varepsilon}Z + P^{2s-18+\varepsilon}Z^2. \tag{4.6}$$

Case 1: $s = 10$. By orthogonality,

$$\int_0^1 |K(\alpha)|^2 d\alpha = Z. \tag{4.7}$$

Hence, recalling (4.4), we get

$$I_{10}(P) \leq \sup_{\alpha \in \mathfrak{m}} |f_0(\alpha)|^6 \int_0^1 |K(\alpha)|^2 d\alpha \ll P^{23/4+\varepsilon} Z,$$

which establishes (4.6) for $s = 10$.

Case 2: $s = 11$. We have

$$I_{11}(P) \leq \int_0^1 \left| g(\alpha)^8 K(\alpha)^2 \right| d\alpha, \tag{4.8}$$

where

$$g(\alpha) = \sum_{x \sim P_0} e\left(\alpha x^4\right),$$

By Weyl's differencing lemma (see [22, Lemma 2.3]),

$$|g(\alpha)|^8 \ll P^4 \sum_{|h_1| < P_0} \sum_{|h_2| < P_0} \sum_{|h_3| < P_0} \sum_{x \in \mathcal{I}} e\left(\alpha \Delta(x^4; \mathbf{h})\right), \tag{4.9}$$

where $\mathcal{I} = \mathcal{I}(\mathbf{h})$ is a subinterval of $[P_0, 2P_0)$ and $\Delta(x^4; \mathbf{h})$ is the third-order forward difference of the function $x \mapsto x^4$ with steps h_1, h_2, h_3, that is,

$$\Delta(x^4; \mathbf{h}) = 12 h_1 h_2 h_3 (2x + h_1 + h_2 + h_3).$$

Thus, we deduce from (4.8) that

$$I_{11}(P) \ll P^4 J(P), \tag{4.10}$$

where $J(P)$ is the number of solutions of the diophantine equation

$$\Delta(x^4; \mathbf{h}) = n_1 - n_2 \tag{4.11}$$

subject to

$$P_0 \leq x \leq 2P_0, \quad |h_i| < P_0, \quad n_j \in \mathcal{Z}. \tag{4.12}$$

The number of solutions of (4.11), (4.12) with $n_1 = n_2$ is bounded by $P^3 Z$. Also, for each pair (n_1, n_2) with $n_1 \neq n_2$ there are at most P^ε choices for $x, \mathbf{h}$. Hence,

$$J(P) \ll P^3 Z + P^\varepsilon Z^2.$$

In conjunction with (4.10), this estimate establishes (4.6) for $s = 11$.

Case 3: $s = 12$. We have

$$I_{12}(P) \leq \sup_{\alpha \in \mathfrak{m}} |f_0(\alpha)|^2 I_{11}(P) \ll P^{107/12+\varepsilon} Z + P^{6+\varepsilon} Z^2,$$

on combining (4.4) and Case 2.

Case 4: $s = 13$. We have

$$I_{13}(P) \le \int_0^1 \left| g(\alpha)^{12} K(\alpha)^2 \right| d\alpha.$$

Hence, by (4.9) and Parseval's identity,

$$I_{13}(P) \ll P^4 J(P), \tag{4.13}$$

where $J(P)$ is the number of solutions of the diophantine equation

$$\Delta(x^4; \mathbf{h}) = n_1 - n_2 + x_1^4 + x_2^4 - x_3^4 - x_4^4, \tag{4.14}$$

subject to

$$P_0 \le x, x_1, \dots, x_4 \le 2P_0, \quad |h_i| < P_0, \quad n_j \in \mathcal{Z}. \tag{4.15}$$

Let $J_0(P)$ denote the number of solutions of (4.14), (4.15) subject to

$$n_2 - n_1 = x_1^4 + x_2^4 - x_3^4 - x_4^4, \tag{4.16}$$

and let $J_1(P)$ be the number of solutions of (4.14), (4.15) for which (4.16) fails. A standard divisor function argument reveals that for each choice of $n_1, n_2, x_1, \dots, x_4$, $J_1(P)$ counts at most P^ε solutions of (4.14). On the other hand, every solution counted by $J_0(P)$ must have $h_1 h_2 h_3 (2x + h_1 + h_2 + h_3) = 0$, and therefore, each solution of (4.16) gives rise of $O(P^3)$ solutions of (4.14), (4.15). We conclude that

$$J(P) = J_0(P) + J_1(P) \ll P^3 J_2(P) + P^{4+\varepsilon} Z^2, \tag{4.17}$$

where $J_2(P)$ denotes the number of solutions of (4.16) in n_i, x_j subject to (4.15).

We now estimate $J_2(P)$. We have

$$J_2(P) = \int_0^1 \left| g(\alpha)^4 K(\alpha)^2 \right| d\alpha.$$

By Weyl's differencing lemma,

$$|g(\alpha)|^4 \ll P \sum_{|h_1| < P_0} \sum_{|h_2| < P_0} \sum_{x \in \mathcal{I}} e\left(\alpha \Delta(x^4; \mathbf{h}) \right),$$

where $\mathcal{I} = \mathcal{I}(\mathbf{h})$ is a subinterval of $[P_0, 2P_0)$ and $\Delta(x^4; \mathbf{h})$ is the second-order forward difference of the function $x \mapsto x^4$ with steps h_1, h_2,

$$\Delta(x^4; \mathbf{h}) = 2h_1 h_2 \left(6x^2 + 6(h_1 + h_2)x + 2h_1^2 + 3h_1 h_2 + 2h_2^2 \right).$$

Thus,

$$J_2(P) \ll P J_3(P),$$

where $J_3(P)$ is the number of solutions of

$$\Delta(x^4; \mathbf{h}) = n_1 - n_2$$

subject to

$$P_0 \le x \le 2P_0, \quad |h_i| < P_0, \quad n_j \in \mathcal{Z}.$$

Estimating $J_3(P)$ similarly to the quantity $J(P)$ from Case 2, we obtain

$$J_2(P) \ll P\left(P^2 Z + P^{\varepsilon} Z^2\right). \tag{4.18}$$

Combining (4.13), (4.17) and (4.18), we get (4.6) with $s = 13$ and complete the proof of the lemma. □

5. Proof of Theorem 2

Let $10 \le s \le 13$ and let $\mathcal{E}_s = \mathcal{E}_{4,s}(N)$ denote the set of those $n \in \mathcal{N}_{4,s} \cap [N, 2N)$ for which (1.1) with $k = 4$ has no solution in prime numbers $p_1, \dots, p_s$. It suffices to show that

$$|\mathcal{E}_s(N)| \ll N^{\theta_s + \varepsilon} \tag{5.1}$$

for all $N \ge N_0(\varepsilon)$.

We define P by $2P^4 = N$ and adopt the notation set in §4. Also, we write $R_s(n)$ for the number of solutions of (1.1) with $k = 4$ in primes $p_1, \dots, p_s$ subject to

$$P_j \le p_j < 2P_j \quad (1 \le j \le 5), \quad P_5 \le p_6 < 2P_5, \quad P_0 \le p_7, \dots, p_s < 2P_0,$$

and if $\mathfrak{B}$ is a measurable subset of $[0, 1]$, we define

$$R_s(n; \mathfrak{B}) = \int_{\mathfrak{B}} f_0(\alpha)^{s-6} F(\alpha) e(-n\alpha) d\alpha.$$

Let us partition the unit interval into sets of major and minor arcs, $\mathfrak{M} = \mathfrak{M}(Q, N)$ and $\mathfrak{m} = \mathfrak{m}(Q, N)$, where $Q = P^{1/2-\varepsilon}$. By orthogonality,

$$R_s(n) = R_s(n; [0, 1]) = R_s(n; \mathfrak{M}) + R_s(n; \mathfrak{m}). \tag{5.2}$$

By Proposition 1,

$$R_s(n; \mathfrak{M}) = \mathfrak{S}_{4,s}(n) J_{4,s}(n) + O\left(F(0) P^{s-10} L^{-s-1}\right). \tag{5.3}$$

Here, $\mathfrak{S}_{4,s}(n)$ is defined by (3.2) and

$$J_{4,s}(n) = \int_{\mathbb{R}} v_1(\beta) \cdots v_4(\beta) v_5(\beta)^2 v_0(\beta)^{s-6} e(-n\beta) d\beta,$$

where

$$v_j(\beta) = \int_{P_j}^{2P_j} \frac{e\left(\beta t^4\right)}{\log t} dt = \int_{P_j^4}^{16P_j^4} \frac{u^{-3/4} e(\beta u)}{\log u} du.$$

A simple argument using Fourier transforms reveals that

$$F(0) P^{s-10} L^{-s} \ll J_{4,s}(n) \ll F(0) P^{s-10} L^{-s} \tag{5.4}$$

for all $n \in [N, 2N)$ (the reader can find detailed expositions of the treatments of similar singular integrals in [10, §5] or in §7.2 of the present paper). Furthermore, if $n \in \mathcal{N}_{4,s}$, we have

$$1 \ll \mathfrak{S}_{4,s}(n) \ll 1. \tag{5.5}$$

The upper bound follows immediately from [9, Lemma 8.10], and the argument of [9, Lemma 8.12] gives

$$\mathfrak{S}_{4,s}(n) \gg \left(1 + \sum_{j=1}^{4} B\left(n, 2^j\right) \right) \prod_{p \in \{3,5\}} (1 + B(n,p)) \tag{5.6}$$

whenever $s \geq 8$. A direct computation then reveals that the right side of (5.6) is non-zero if and only if $n \in \mathcal{N}_{4,s}$. Combining (5.3)–(5.5), we conclude that

$$R_s(n; \mathfrak{M}) \gg F(0) P^{s-10} L^{-s} \tag{5.7}$$

for all $n \in \mathcal{N}_{4,s} \cap [N, 2N)$.

We now turn to $R_s(n; \mathfrak{m})$. Using (5.2) and the definition of $\mathcal{E}_s$, we obtain

$$\sum_{n \in \mathcal{E}_s} R_s(n; \mathfrak{M}) = - \sum_{n \in \mathcal{E}_s} R_s(n; \mathfrak{m}),$$

so (5.7) implies that

$$F(0) P^{s-10-\varepsilon} |\mathcal{E}_s| \ll \left| \sum_{n \in \mathcal{E}_s} R_s(n; \mathfrak{m}) \right|. \tag{5.8}$$

Furthermore, on writing

$$K_s(\alpha) = \sum_{n \in \mathcal{E}_s} e(\alpha n),$$

Lemma 4.3 yields

$$\begin{aligned} \left| \sum_{n \in \mathcal{E}_s} R_s(n; \mathfrak{m}) \right| &= \left| \int_{\mathfrak{m}} f_0(\alpha)^{s-6} F(\alpha) K_s(-\alpha) d\alpha \right| \\ &\ll F(0) |\mathcal{E}_s|^{1/2} P^{\sigma_s - \delta/2 + \varepsilon} + F(0) |\mathcal{E}_s| P^{s-10-\delta/3}. \end{aligned} \tag{5.9}$$

The desired estimate (5.1) follows immediately from (5.8) and (5.9). □

6. Minor arc estimates, II: cubes

In this section, we prove minor arc estimates to be used in the proof of Theorem 1. Throughout the section, $\mathcal{Z}$ denotes a set of integers having cardinality Z and

$$K(\alpha) = \sum_{n \in \mathcal{Z}} e(\alpha n).$$

We also define the exponential sums

$$f_j(\alpha) = \sum_{p \sim P_j} e\left(\alpha p^3\right), \quad g_j(\alpha) = \sum_{m \sim P_2} \lambda_j(m) e\left(\alpha m^3\right) \qquad (j = 1, 2), \tag{6.1}$$

where $\lambda_1(m)$ and $\lambda_2(m)$ are bounded arithmetic functions and $P \ll P_1, P_2 \ll P$.

Lemma 6.1. *Let $0 < \rho < 1/10$ and suppose that $g_1(\alpha)$ satisfies*

$$g_1(\alpha) \ll P^{1-\rho+\varepsilon} + \frac{q^{1/6}P^{1+\varepsilon}}{(1+P^3|\alpha - a/q|)^{1/2}}, \tag{6.2}$$

whenever there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying (2.1) with $Q = P^{(9-6\rho)/5}$. Also, let

$$X \geq P^{2\rho} \quad \textit{and} \quad P^3 \ll N \ll P^3, \tag{6.3}$$

and write $\mathfrak{m} = \mathfrak{m}(X, N)$. Then

$$\int_{\mathfrak{m}} \left| f_1(\alpha) f_2(\alpha)^2 g_1(\alpha) g_2(\alpha) K(\alpha) \right| d\alpha \ll Z^{1/2} P^{7/2-\rho+\varepsilon}. \tag{6.4}$$

Proof. By Cauchy's inequality, the integral on the left side of (6.4) does not exceed

$$\left(\int_0^1 |K(\alpha)|^2 d\alpha \right)^{1/2} \left(\int_{\mathfrak{m}} \left| f_1(\alpha) f_2(\alpha)^2 g_1(\alpha) g_2(\alpha) \right|^2 d\alpha \right)^{1/2}.$$

By orthogonality,

$$\int_0^1 |K(\alpha)|^2 d\alpha = Z,$$

so the lemma will follow if we show that

$$\int_{\mathfrak{m}} \left| f_1(\alpha) f_2(\alpha)^2 g_1(\alpha) g_2(\alpha) \right|^2 d\alpha \ll P^{7-2\rho+\varepsilon}. \tag{6.5}$$

We start the proof of (6.5) by defining the sets

$$\mathfrak{N} = \mathfrak{M}(P^{6\rho}, N) \quad \text{and} \quad \mathfrak{n} = \mathfrak{m}(P^{6\rho}, N).$$

By Dirichlet's theorem on diophantine approximation, every real α has a rational approximation a/q satisfying (2.1) with $Q = P^{(9-6\rho)/5}$. Thus, by hypothesis (6.2),

$$\sup_{\alpha \in \mathfrak{n}} |g_1(\alpha)| \ll P^{1-\rho+\varepsilon}. \tag{6.6}$$

On the other hand, if $X < P^{6\rho}$, any $\alpha \in \mathfrak{m} \cap \mathfrak{N}$ has a rational approximation a/q subject to

$$X < q + N|q\alpha - a| \leq P^{6\rho},$$

and Lemma 2.3 with $k = 3$ yields

$$\sup_{\alpha \in \mathfrak{m} \cap \mathfrak{N}} |f_1(\alpha)| \ll P^{1+\varepsilon} X^{-1/2} + P^{11/20+3\rho+\varepsilon} \ll P^{1-\rho+\varepsilon}. \tag{6.7}$$

Furthermore, using Hölder's inequality, we deduce from Hua's lemma [22, Lemma 2.5] that

$$\int_0^1 |G(\alpha)|^2 d\alpha \ll P^{5+\varepsilon}, \tag{6.8}$$

whenever $G(\alpha)$ is a product of the form

$$G(\alpha) = f_1(\alpha)^{u_1} f_2(\alpha)^{u_2} g_1(\alpha)^{v_1} g_2(\alpha)^{v_2},$$

with $u_1, u_2, v_1, v_2 \geq 0$ and $u_1 + u_2 + v_1 + v_2 = 4$. The desired estimate (6.5) follows readily from (6.6)–(6.8). □

Lemma 6.2. *Let $0 < \rho < 1/10$ and suppose that $g_1(\alpha)$ satisfies* (6.2)*, whenever there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying* (2.1) *with $Q = P^{(9-6\rho)/5}$. Also, assume* (6.3) *and write $\mathfrak{m} = \mathfrak{m}(X, N)$. Then*

$$\int_{\mathfrak{m}} \left| f_1(\alpha) f_2(\alpha) g_1(\alpha)^2 g_2(\alpha)^2 K(\alpha) \right| d\alpha \ll Z^{1/2} P^{9/2-2\rho+\varepsilon}.$$

Proof. The proof is similar to the proof of Lemma 6.1, using the bounds

$$\sup_{\alpha\in\mathfrak{n}} |g_1(\alpha)|^2 \ll P^{2-2\rho+\varepsilon} \quad \text{and} \quad \sup_{\alpha\in\mathfrak{m}\cap\mathfrak{N}} |f_1(\alpha) f_2(\alpha)| \ll P^{2-2\rho+\varepsilon}$$

instead of (6.6) and (6.7). □

Lemma 6.3. *Let $0 < \rho < 1/10$ and suppose that $g_1(\alpha)$ satisfies* (6.2)*, whenever there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying* (2.1) *with $Q = P^{(9-6\rho)/5}$. Also, assume* (6.3) *and write $\mathfrak{m} = \mathfrak{m}(X, N)$. Then for $j = 4, 5$ we have*

$$\int_{\mathfrak{m}} \left| f_1(\alpha) f_2(\alpha)^j g_1(\alpha) g_2(\alpha) K(\alpha) \right| d\alpha \ll Z^{1/2} P^{3+j/2-\rho+\varepsilon} + Z P^{j-\rho+\varepsilon}. \tag{6.9}$$

Proof. By Cauchy's inequality, the left side of (6.9) does not exceed

$$\left(\int_0^1 \left| f_2(\alpha)^{2j-4} K(\alpha)^2 \right| d\alpha \right)^{1/2} \left(\int_{\mathfrak{m}} \left| f_1(\alpha) f_2(\alpha)^2 g_1(\alpha) g_2(\alpha) \right|^2 d\alpha \right)^{1/2}.$$

Thus, (6.9) follows from (6.5) and the estimates

$$\int_0^1 \left| f_2(\alpha)^{2j-4} K(\alpha)^2 \right| d\alpha \ll P^{\varepsilon} \left(P^{2j-7} Z^2 + P^{j-1} Z \right),$$

for which we refer to [25, Lemmas 5.1 and 6.2]. □

7. Proof of Theorem 1

Let $\mathcal{E}_s = \mathcal{E}_{3,s}(N)$ denote the set of those $n \in \mathcal{N}_{3,s} \cap [N, 2N)$ for which (1.1) with $k = 3$ has no solution in prime numbers $p_1, \dots, p_s$. We proceed to show that

$$|\mathcal{E}_s(N)| \ll N^{\theta_s} \tag{7.1}$$

for all $N \geq N_0(\varepsilon)$, our approach being similar to that used in the proof of Theorem 2. We choose $P = N^{1/3}$ and set

$$P_1 = \tfrac{3}{4} P, \quad P_2 = \tfrac{1}{6} P. \tag{7.2}$$

Let $R_s(n)$ denote the number of solutions of (1.1) with $k = 3$ in primes $p_1, \dots, p_s$ subject to

$$P_1 \leq p_1 < 2P_1, \quad P_2 \leq p_2, \dots, p_s < 2P_2. \tag{7.3}$$

Also, let ρ be a parameter with $0 < \rho < 1/11$ and define the sets of major and minor arcs,

$$\mathfrak{M} = \mathfrak{M}(Q, N) \quad \text{and} \quad \mathfrak{m} = \mathfrak{m}(Q, N),$$

with

$$Q = \min\left(P^{1/2-\varepsilon}, P^{5/2-25\rho-\varepsilon} \right). \tag{7.4}$$

We shall construct a generating function $F_s(\alpha) = F_s(\rho;\alpha)$ with the following properties:

(F_1) $F_s(\alpha)$ is a trigonometric polynomial with real coefficients;

(F_2) if $N \le n < 2N$,

$$R_s(n) \ge \int_0^1 F_s(\alpha)e(-n\alpha)d\alpha;$$

(F_3) if $\mathcal{Z} \subset [N, 2N)$,

$$\sum_{n\in\mathcal{Z}} \int_{\mathfrak{m}} F_s(\alpha)e(-n\alpha)d\alpha \ll |\mathcal{Z}|^{1/2} P^{\sigma_s-\rho+\varepsilon} + |\mathcal{Z}|P^{s-3-\rho+\varepsilon},$$

where $\sigma_5 = 7/2$, $\sigma_6 = 9/2 - \rho$, $\sigma_7 = 5$ and $\sigma_8 = 11/2$;

(F_4) if $n \in \mathcal{N}_{3,s} \cap [N, 2N)$,

$$\int_{\mathfrak{M}} F_s(\alpha)e(-n\alpha)d\alpha = \big(C_s(\rho) + O\big(L^{-1}\big)\big)\, P^{s-3}L^{-s},$$

where $C_s(\rho)$ is a non-increasing function of ρ satisfying $C_s\left(\rho_s + 10^{-6}\right) > 0$ with $\rho_6 = 3/35$ and $\rho_s = 5/56$ for $s = 5, 7, 8$.

By (F_1) and the symmetry of $\mathfrak{M}$ mod 1, the integral in (F_4) is a real number. Thus, on choosing $\rho = \rho_s + 10^{-6}$, the estimate

$$|\mathcal{E}_s(N)| \ll N^{\theta_s-(2/3)\times 10^{-6}+\varepsilon}$$

follows from (F_2)–(F_4) in the same way as (5.1) follows from (5.2), (5.7) and (5.9). This establishes (7.1).

7.1. **Construction of $F_5(\alpha)$.** We now build a generating function $F_5(\alpha)$ having properties (F_1)–(F_4) above. Given an arithmetic function λ, define

$$g(\alpha;\lambda) = \sum_{m\sim P_2} \lambda(m)e\left(\alpha m^3\right).$$

We start by expressing $R_5(n)$ as a linear combination of three types of terms:

- sums of the form

$$\sum_{m\sim P_2} \gamma(m)R_4\left(n - m^3\right),$$

  where $g(\alpha;\gamma)$ satisfies (6.2);
- sums of the form

$$\sum_{m_1,m_2\sim P_2} \delta(m_1)\gamma(m_2)R_3\left(n - m_1^3 - m_2^3\right),$$

  where $g(\alpha;\gamma)$ satisfies (6.2);
- other positive terms.

We produce such a representation for $R_5(n)$ by successive decompositions using Buchstab's identity

$$\psi(n,z_1) = \psi(n,z_2) - \sum_{\substack{z_2\le p<z_1\\ n=pj}} \psi(j,p) \qquad (2 \le z_2 < z_1). \tag{7.5}$$

We navigate the decompositions as to lead to exponential sums $g(\alpha;\gamma)$ that can be estimated by Lemma 2.1 (possibly in conjunction with Lemma 2.5). Also, we try to keep the terms of the last kind—“other”—to a minimum, since the larger those terms the sooner the function $C_5(\rho)$ in (F_4) vanishes as ρ increases.

We now proceed with the decomposition of $R_5(n)$. For the sake of brevity, we define $z = P^{1-10\rho}$, $X = \sqrt{2P_2}$, $U = P^{2\rho}$, $V = P^{1-8\rho}$ and

$$W = \min\left(P^{1-6\rho}, P^{(3-7\rho)/5}\right) = \begin{cases} P^{1-6\rho}, & \text{if } \rho > 2/23, \\ P^{(3-7\rho)/5}, & \text{if } \rho \le 2/23; \end{cases}$$

notice that, with the exception of X, these quantities have been chosen in accordance with Lemma 2.1. We have

$$R_5(n) = \sum_{p\sim P_2} R_4\left(n - p^3\right) = \sum_{m\sim P_2} \psi(m,X) R_4\left(n - m^3\right). \tag{7.6}$$

We now decompose the function $\psi(m, X)$.

Applying twice Buchstab’s identity, we get

$$\begin{aligned} \psi(m,X) &= \psi(m,z) - \sum_{\substack{m=pj \\ z\le p\le W}} \psi(j,z) \\ &\quad - \sum_{\substack{m=pj \\ W<p<X}} \psi(j,p) + \sum_{\substack{m=p_1p_2j \\ z\le p_2<p_1\le W}} \psi(j,p_2) \\ &= \gamma_1(m) - \gamma_2(m) - \gamma_3(m) + \gamma_4(m), \quad \text{say.} \end{aligned} \tag{7.7}$$

We remark that (2.5) applies to $g(\alpha;\gamma_1)$ and $g(\alpha;\gamma_2)$. Furthermore, we can apply (2.3) to the exponential sums corresponding to some subsums of $\gamma_4(m)$, while other subsums of $\gamma_4(m)$ we can decompose further. Thus, we now partition $\gamma_4(m)$ into five subsums. We write

$$\gamma_4(m) = \gamma_5(m) + \cdots + \gamma_9(m), \tag{7.8}$$

where $\gamma_5(m), \dots, \gamma_9(m)$ are the subsums of $\gamma_4(m)$ subject to the following constraints:

$$\begin{aligned} \gamma_5(m): &\quad p_1, p_2 \text{ or } p_1p_2 \text{ lies in } [U,V]; \\ \gamma_6(m): &\quad z \le p_2 < p_1 < U < V < p_1p_2; \\ \gamma_7(m): &\quad z \le p_2 < U < V < p_1 < p_1p_2 \le W; \\ \gamma_8(m): &\quad z \le p_2 < U < V < p_1 < W < p_1p_2; \\ \gamma_9(m): &\quad V < p_2 < p_1 \le W. \end{aligned}$$

We remark that $\gamma_5(m)$ is the portion of $\gamma_4(m)$ that induces an exponential sum which can be estimated by (2.3), while $\gamma_6(m)$ and $\gamma_7(m)$ are the portions for which we can give further decompositions.

Next, we decompose $\gamma_6(m)$. Another appeal to (7.5) gives

$$\gamma_6(m) = \sum_{\substack{p_1,p_2\\ m=p_1p_2j}} \psi(j,z) - \sum_{\substack{p_1,p_2\\ m=p_1p_2p_3j}} \sum_{z\le p_3<p_2} \psi(j,p_3) \tag{7.9}$$
$$= \gamma_{10}(m) - \gamma_{11}(m), \quad \text{say.}$$

We can apply (2.5) to $g(\alpha;\gamma_{10})$, and we deal with $\gamma_{11}(m)$ similarly to $\gamma_4(m)$. We have

$$\gamma_{11}(m) = \gamma_{12}(m) + \gamma_{13}(m) + \gamma_{14}(m), \tag{7.10}$$

where $\gamma_{12}(m), \gamma_{13}(m)$ and $\gamma_{14}(m)$ are the subsums of $\gamma_{11}(m)$ subject to

$$p_2p_3 \le V, \quad V < p_2p_3 < p_1p_2p_3 \le W \quad \text{and} \quad V < p_2p_3 < W < p_1p_2p_3,$$

respectively. We now apply (7.5) to $\gamma_{13}(m)$ and obtain

$$\gamma_{13}(m) = \sum_{\substack{p_1,p_2,p_3\\ m=p_1p_2p_3j}} \psi(j,z) - \sum_{\substack{p_1,p_2,p_3\\ m=p_1\cdots p_4j}} \sum_{z\le p_4<p_3} \psi(j,p_4) \tag{7.11}$$
$$= \sum_{m=p_1p_2p_3j} \psi(j,z) - \sum_{\substack{m=p_1\cdots p_4j\\ p_3p_4\le V}} \psi(j,p_4) - \sum_{\substack{m=p_1\cdots p_4j\\ p_3p_4>V}} \psi(j,p_4)$$
$$= \gamma_{15}(m) - \gamma_{16}(m) - \gamma_{17}(m), \quad \text{say.}$$

Combining (7.9)–(7.11), we conclude that

$$\gamma_6(m) = \gamma_{10}(m) - \gamma_{12}(m) - \gamma_{14}(m) - \gamma_{15}(m) + \gamma_{16}(m) + \gamma_{17}(m), \tag{7.12}$$

where (2.5) applies to $g(\alpha;\gamma_{10})$ and $g(\alpha;\gamma_{15})$ and, since $z^2 > U$, (2.3) applies to $g(\alpha;\gamma_{12})$ and $g(\alpha;\gamma_{16})$.

We now turn to $\gamma_7(m)$. Applying (7.5) once more, we get

$$\gamma_7(m) = \sum_{\substack{p_1,p_2\\ m=p_1p_2j}} \psi(j,z) - \sum_{\substack{p_1,p_2\\ m=p_1p_2p_3j}} \sum_{z\le p_3<p_2} \psi(j,p_3) \tag{7.13}$$
$$= \sum_{m=p_1p_2j} \psi(j,z) - \sum_{\substack{m=p_1p_2p_3j\\ p_2p_3\le V}} \psi(j,p_3) - \sum_{\substack{m=p_1p_2p_3j\\ p_2p_3>V}} \psi(j,p_3)$$
$$= \gamma_{18}(m) - \gamma_{19}(m) - \gamma_{20}(m), \quad \text{say,}$$

where $g(\alpha;\gamma_{18})$ and $g(\alpha;\gamma_{19})$ can be estimated by (2.5) and (2.3), respectively.

Finally, combining (7.7), (7.8), (7.12) and (7.13), we deduce that

$$\psi(m,X) = \gamma_1(m) - \gamma_2(m) - \gamma_3(m) + \gamma_5(m) + \gamma_8(m) + \gamma_9(m) \tag{7.14}$$
$$+ \gamma_{10}(m) - \gamma_{12}(m) - \gamma_{14}(m) - \gamma_{15}(m)$$
$$+ \gamma_{16}(m) + \gamma_{17}(m) + \gamma_{18}(m) - \gamma_{19}(m) - \gamma_{20}(m).$$

Substituting (7.14) into the right side of (7.6), we obtain the identity

$$R_5(n) = R_5(n;\lambda_1) - R_5(n;\lambda_2) + R_5(n;\lambda_3), \tag{7.15}$$

where

$$\begin{aligned} R_5(n;\lambda) &= \sum_{m\sim P_2} \lambda(m) R_4\left(n-m^3\right), \\ \lambda_1(m) &= \gamma_1(m) - \gamma_2(m) + \gamma_5(m) + \gamma_{10}(m) - \gamma_{12}(m) \\ &\qquad - \gamma_{15}(m) + \gamma_{16}(m) + \gamma_{18}(m) - \gamma_{19}(m), \\ \lambda_2(m) &= \gamma_3(m) + \gamma_{14}(m) + \gamma_{20}(m), \end{aligned}$$

and

$$\lambda_3(m) = \gamma_8(m) + \gamma_9(m) + \gamma_{17}(m).$$

We remark that $\lambda_1(m)$ is the sum of those $\gamma_j(m)$'s for which we can estimate $g(\alpha;\gamma_j)$ using one of the bounds in Lemma 2.1, while $\lambda_2(m)$ and $\lambda_3(m)$ collect the terms in the decomposition for which we cannot estimate the induced exponential sums and which contribute, respectively, negative and positive quantities to $\psi(m,X)$. Accordingly, $R_5(n;\lambda_1)$ represents the terms in the decomposition of $R_5(n)$ of the first kind mentioned above, $R_5(n;\lambda_2)$ will give rise of the terms of the second kind and some of the third kind, and $R_5(n;\lambda_3)$ contributes terms of the third kind.

Next, we decompose $R_5(n;\lambda_2)$. Similarly to (7.6), we have

$$R_5(n;\lambda_2) = \sum_{m_1,m_2\sim P_2} \lambda_2(m_1)\psi(m_2,X) R_3\left(n-m_1^3-m_2^3\right).$$

We use the following identity for $\psi(m,X)$:

$$\begin{aligned} \psi(m,X) &= \psi(m,z) - \sum_{\substack{m=pj \\ U\le p\le V}} \psi(j,p) - \sum_{\substack{m=pj \\ V<p<X}} \psi(j,p) - \sum_{\substack{m=pj \\ z\le p<U}} \psi(j,z) \\ &\quad + \sum_{\substack{m=p_1p_2j \\ z\le p_2<p_1<U}} \psi(j,z) - \sum_{\substack{m=p_1p_2p_3j \\ z\le p_3<p_2<p_1<U}} \Bigg\{ \sum_{p_2p_3\le V} + \sum_{p_2p_3>V} \Bigg\} \psi(j,p_3) \\ &= \gamma_1^*(m) - \gamma_2^*(m) - \gamma_3^*(m) - \gamma_4^*(m) + \gamma_5^*(m) - \gamma_6^*(m) - \gamma_7^*(m), \quad \text{say.} \end{aligned} \tag{7.16}$$

The proof of this identity is similar to (and simpler than) that of (7.14), using only three Buchstab decompositions; thus, we omit it. The main point is that we only have terms for which we can estimate $g(\alpha;\gamma_i^*)$—those with $i\in\{1,2,4,5,6\}$—and negative terms—$\gamma_3^*(m)$ and $\gamma_7^*(m)$. We conclude that

$$R_5(n;\lambda_2) = R_5(n;\lambda_2,\lambda_1^*) - R_5(n;\lambda_2,\lambda_3^*), \tag{7.17}$$

where

$$R_5(n;\lambda,\nu) = \sum_{m_1,m_2\sim P_2} \lambda(m_1)\nu(m_2)R_3\left(n-m_1^3-m_2^3\right),$$

$$\lambda_1^*(m) = \gamma_1^*(m)-\gamma_2^*(m)-\gamma_4^*(m)+\gamma_5^*(m)-\gamma_6^*(m),$$

$$\lambda_3^*(m) = \gamma_3^*(m)+\gamma_7^*(m).$$

Combining (7.15) and (7.17), we obtain

$$R_5(n) = R_5(n;\lambda_1) - R_5(n;\lambda_2,\lambda_1^*) + R_5(n;\lambda_3) + R_5(n;\lambda_2,\lambda_3^*),$$

which is the decomposition for $R_5(n)$ we have been pursuing in this section. Let $f_1(\alpha)$ and $f_2(\alpha)$ be the exponential sums defined in (6.1). We set

$$F_5(\alpha) = f_1(\alpha)f_2(\alpha)^2\big(g(\alpha;\lambda_1)f_2(\alpha) - g(\alpha;\lambda_2)g(\alpha;\lambda_1^*)\big), \tag{7.18}$$

so that

$$\int_0^1 F_5(\alpha)e(-n\alpha)d\alpha = R_5(n;\lambda_1) - R_5(n;\lambda_2,\lambda_1^*) \le R_5(n). \tag{7.19}$$

Then, $F_5(\alpha)$ satisfies (F_1) and (F_2). Also, $g(\alpha;\lambda_1)$ and $g(\alpha;\lambda_1^*)$ satisfy (6.2) by construction, and (7.4) implies $Q \ge P^{2\rho}$, so Lemma 6.1 yields (F_3). Therefore, in order to complete the proof of (7.1) for $s=5$, it remains to verify that $F_5(\alpha)$ satisfies (F_4).

7.2. **The case $s = 5$ completed.** We intend to verify that the generating function (7.18) satisfies (F_4) by referring to Proposition 2. Let $\lambda(m)$ denote any of the functions $\gamma_j(m)$ or $\gamma_j^*(m)$ appearing in the definitions of λ_1, λ_2 or λ_1^*. We note that λ satisfies axioms (A_1) and (A_2) from §3 by construction and proceed to check that it satisfies axioms (A_3) and (A_4) as well.

We start our discussion with (A_3). It suffices to show that if $P \ll y \ll P$, $A > 0$, and χ is a non-principal character mod q, $q \le L^{2A}$, then

$$\sum_{m\le y} \lambda(m)\chi(m) \ll yL^{-A}, \tag{7.20}$$

with an implied constant depending at most on A. To this end, we observe that the left side of (7.20) is always bounded above by an expression of the form

$$LK\left|\sum_{J\le j<J_1} \psi(j,w)\chi(j)\right|,$$

where

$$w\ge z, \quad z\le J<J_1\le 2J \quad \text{and} \quad JK\ll y.$$

Thus, (7.20) follows from the estimate

$$\sum_{J\le j<J_1} \psi(j,w)\chi(j) \ll JL^{-A},$$

which can be derived by partial summation from [1, Lemma 5].

We now turn toward axiom (A_4), which we verify by means of Lemma 2.4. When λ is one of γ_1, γ_2, γ_{10}, γ_{12}, γ_{14}, γ_{15}, γ_{18}, γ_1^*, γ_2^*, γ_4^*, γ_5^*, γ_6^* or the portion of γ_5 subject to $U \le p_1p_2 \le V$, Lemma 2.4 yields (A_4) with $\rho_1 = 9/20$ and $\rho_2 = 1 - 10\rho$. In order to deal with $\gamma_{16}, \gamma_{19}, \gamma_{20}$ and the remainder of γ_5, we first apply Buchstab's identity in the reverse. Consider, for example, γ_{20}. We have

$$\begin{aligned}\gamma_{20}(m) &= \sum_{p_1>V} \sum_{\substack{V^{1/2}<p_2<U \\ p_1p_2\le W}} \sum_{m=p_1p_2j} \left\{\psi(j, Vp_2^{-1}) - \psi(j, p_2)\right\} \\ &= \gamma'_{20}(m) - \gamma''_{20}(m), \quad \text{say,}\end{aligned}$$

and we can apply Lemma 2.4 to both γ'_{20} and γ''_{20}. Finally, since γ_3 is the characteristic function of the products p_1p_2 with $p_1 \ge p_2 > W$, we can estimate $g(\alpha; \gamma_3)$ by combining Lemmas 5.1 and 5.3 in [11]. In all the cases, we obtain (at least) that λ satisfies axiom (A_4) with $\rho_1 = 9/20$ and $\rho_2 = 1 - 10\rho$.

We conclude that λ_1, λ_2 and λ_1^* satisfy axioms (A_1)–(A_4) from §3. We also note that when $\rho_1 = 9/20$ and $\rho_2 = 1 - 10\rho$, (3.11) follows from (7.4). Hence, we can apply Proposition 2 to the integral appearing in (F_4). We obtain

$$\int_{\mathfrak{M}} F_5(\alpha)e(-n\alpha)d\alpha = \mathfrak{R}(n, L^{50}) + O\left(P^2L^{-6}\right), \tag{7.21}$$

where, in accordance with the notation in §3,

$$\mathfrak{R}(n, X) = \sum_{q=1}^{\infty} B(n, q) \int_{-X/(qN)}^{X/(qN)} F_5(\beta)e(-n\beta)d\beta.$$

We now define the exponential integrals

$$v_i(\beta) = \int_{P_i}^{2P_i} \frac{e\left(\beta y^3\right)}{\log y} dy \qquad (i = 1, 2)$$

and the singular integral

$$\tilde{J}(n) = \int_{\mathbb{R}} v_1(\beta)v_2(\beta)^2 \left(g(\beta; \lambda_1)v_2(\beta) - g(\beta; \lambda_2)g(\beta; \lambda_1^*)\right) e(-n\beta)d\beta.$$

Recalling the bounds (3.12) and (3.13), we obtain

$$\mathfrak{R}(n, L^{50}) = \mathfrak{S}_{3,5}(n)\tilde{J}(n) + O\left(P^2L^{-6}\right), \tag{7.22}$$

where $\mathfrak{S}_{3,5}(n)$ is defined in (3.2).

In order to analyze the singular integral $\tilde{J}(n)$, we require approximations for the mean values of λ_1, λ_2 and λ_1^*. We use that there exist functions $\ell_1(\rho), \ell_2(\rho)$ and $\ell_1^*(\rho)$ such that if $P_2 < y \le 2P_2$,

$$\sum_{P_2\le m<y} \lambda_i(m) = \ell_i \cdot (y - P_2)L^{-1} + O(PL^{-2}) \qquad (i = 1, 2) \tag{7.23}$$

and

$$\sum_{P_2\le m<y} \lambda_1^*(m) = \ell_1^* \cdot (y - P_2)L^{-1} + O(PL^{-2}). \tag{7.24}$$

These asymptotic formulas follow by partial summation from the Prime Number Theorem and the following lemma, whose proof can be found in [5, Lemma 1].

Lemma 7.1. *Let $y^\varepsilon \le z \le y$, and let $\omega(u)$ be the continuous solution of the differential delay equation*

$$\begin{cases} (u\omega(u))' = \omega(u-1), & \text{if } u > 2, \\ \omega(u) = u^{-1}, & \text{if } 1 < u \le 2. \end{cases}$$

Then

$$\sum_{m<y} \psi(m,z) = \omega\left(\frac{\log y}{\log z}\right)\frac{y}{\log z} + O\left(y(\log y)^{-2}\right),$$

with an implied constant depending at most on ε.

The resulting expressions for the ℓ_i's are sums of multiple integrals involving the function $\omega(\cdot)$, each of them accounting for the contribution of one of the γ_i's or γ_i^*'s. For example, if $P_2 \le y \le 2P_2$, we have

$$\sum_{m \le y} \gamma_{15}(m) = yL^{-1}\left(I_{15} + O(L^{-1})\right),$$

where

$$I_{15} = \frac{1}{1-10\rho}\iiint_{\mathfrak{D}_{15}} \omega\left(\frac{1-u_1-u_2-u_3}{1-10\rho}\right)\frac{du_1du_2du_3}{u_1u_2u_3},$$

the integration being over u_1, u_2, u_3 subject to

$$\begin{cases} 1-10\rho < u_3 < u_2 < u_1 < 2\rho, \quad u_2+u_3 > 1-8\rho, \\ u_1+u_2+u_3 < \min(1-6\rho, (3-7\rho)/5). \end{cases}$$

Now, we consider

$$\tilde{J}_1(n) = \int_{\mathbb{R}} v_1(\beta)v_2(\beta)^3 g(\beta;\lambda_1)e(-n\beta)d\beta.$$

Changing the order of integration, we obtain

$$\tilde{J}_1(n) = \iiint_{P_2 \le y_1,y_2,y_3 \le 2P_2} \sum_{m \sim P_2} \lambda_1(m)\Phi(m,\mathbf{y})\frac{dy_1dy_2dy_3}{(\log y_1)(\log y_2)(\log y_3)}, \tag{7.25}$$

where

$$\Phi(m,\mathbf{y}) = \int_{\mathbb{R}}\left\{\int_{P_1^3}^{8P_1^3}\frac{y^{-2/3}e(\beta y)}{\log y}dy\right\}e\left(-\beta\left(n-y_1^3-y_2^3-y_3^3-m^3\right)\right)d\beta.$$

If $N \le n < 2N$ and $P_2 \le m, y_1, y_2, y_3 \le 2P_2$, (7.2) implies

$$P_1^3 < n - y_1^3 - y_2^3 - y_3^3 - m^3 < 8P_1^3,$$

so an appeal to Fourier's inversion formula gives

$$\Phi(m,\mathbf{y}) = \frac{\left(n-y_1^3-y_2^3-y_3^3-m^3\right)^{-2/3}}{\log\left(n-y_1^3-y_2^3-y_3^3-m^3\right)}.$$

Hence, (7.23) and partial summation lead to the approximate formula

$$\sum_{m\sim P_2} \lambda_1(m)\Phi(m,\mathbf{y}) = \ell_1 L^{-1}\int_{P_2}^{2P_2} \Phi(y_4,\mathbf{y})dy_4 + O(P^{-1}L^{-3}). \tag{7.26}$$

Combining (7.25) and (7.26), we conclude that

$$\tilde{J}_1(n) = \tfrac{1}{3}\ell_1 L^{-5} J(n) + O(P^2L^{-6}), \tag{7.27}$$

where

$$J(n) = \iiiint\limits_{P_2\le y_1,\dots,y_4\le 2P_2} \left(n - y_1^3 - \cdots - y_4^3\right)^{-2/3} dy_1\cdots dy_4.$$

Similarly, we obtain

$$\int_{\mathbb{R}} v_1(\beta)v_2(\beta)^2 g(\beta;\lambda_2)g(\beta;\lambda_1^*)e(-n\beta)d\beta = \tfrac{1}{3}\ell_2\ell_1^* L^{-5}J(n) + O(P^2L^{-6}),$$

which in conjunction with (7.27) gives

$$\tilde{J}(n) = \tfrac{1}{3}(\ell_1 - \ell_2\ell_1^*)L^{-5}J(n) + O\left(P^2L^{-6}\right). \tag{7.28}$$

Finally, we observe that

$$P^2 \ll J(n) \ll P^2$$

for $N \le n < 2N$, and that an argument similar to that leading to (5.5) yields

$$1 \ll \mathfrak{S}_{3,5}(n) \ll 1$$

for $n \in \mathfrak{N}_{3,5}$. Therefore, (F_4) follows from (7.21), (7.22), (7.28) and the inequality

$$(\ell_1 - \ell_2\ell_1^*)|_{\rho=5/56+10^{-6}} > 0,$$

which can be verified by numerical integration. This completes the proof of Theorem 1 in the case $s = 5$.

7.3. **The cases** $s = 6, 7$ **and** 8**.** When $s = 7$ or 8, the proof is essentially the same as in the case $s = 5$, using the generating functions

$$F_s(\alpha) = f_1(\alpha)f_2(\alpha)^{s-3}\big(g(\alpha;\lambda_1)f_2(\alpha) - g(\alpha;\lambda_2)g(\alpha;\lambda_1^*)\big)$$

instead of $F_5(\alpha)$ and Lemma 6.3 instead of Lemma 6.1. The case $s = 6$ is also similar, with Lemma 6.2 in place of Lemma 6.1. However, since Lemma 6.2 requires upper bounds for two exponential sums instead for one, we need to sift two pairs of variables instead of one. This forces some changes, which we now describe.

We intend to combine the sieve construction used in §7.1 with a variant of the vector sieve of Brüdern and Fouvry [3]. Retaining the notation from §7.1, we define the function

$$\Lambda^-(m_1,m_2) = \lambda_1(m_1)\psi(m_2,X) - \lambda_2(m_1)\lambda_1^*(m_2).$$

We can now restate the sieve inequality underlying (7.19) as

$$\psi(m_1,X)\psi(m_2,X) \ge \Lambda^-(m_1,m_2).$$

Furthermore, by interchanging the roles of λ_2 and λ_3 in §7.1, we obtain

$$\psi(m_1, X)\psi(m_2, X) \leq \Lambda^+(m_1, m_2),$$

where

$$\Lambda^+(m_1, m_2) = \lambda_1(m_1)\psi(m_2, X) + \lambda_3(m_1)\lambda_1^*(m_2).$$

With the functions $\Lambda^\pm$ in hand, we can state our version of the vector sieve as follows:

$$\prod_{j=1}^{4} \psi(m_j, X) \geq \Lambda^+(\mathbf{m})\Lambda^-(\tilde{\mathbf{m}}) + \Lambda^-(\mathbf{m})\Lambda^+(\tilde{\mathbf{m}}) - \Lambda^+(\mathbf{m})\Lambda^+(\tilde{\mathbf{m}}), \tag{7.29}$$

where $\mathbf{m} = (m_1, m_2)$ and $\tilde{\mathbf{m}} = (m_3, m_4)$. Accordingly, we set

$$F_6 = F_6(\alpha) = f_1 f_2 \left[2(g_1 f_2 + g_3 g_1^*)(g_1 f_2 - g_2 g_1^*) - (g_1 f_2 + g_3 g_1^*)^2\right],$$

where $g_i = g(\alpha; \lambda_i)$ and $g_1^* = g(\alpha; \lambda_1^*)$. We can deal with $F_6(\alpha)$ similarly to $F_5(\alpha)$ (using Lemma 6.2 instead of Lemma 6.1). The only significant change occurs in the final step of the evaluation of the singular integral. Because of the change in the sifting functions, the function $\ell_1 - \ell_2\ell_1^*$ appearing in (7.28) has to be replaced by

$$\ell = \ell(\rho) = 2(\ell_1 - \ell_2\ell_1^*)(\ell_1 + \ell_3\ell_1^*) - (\ell_1 + \ell_3\ell_1^*)^2,$$

where ℓ_3 is defined analogously to ℓ_1 and ℓ_2 (i.e., we extend (7.23) to $i = 3$). Correspondingly, we need to change the value of the parameter ρ so that $\ell(\rho) > 0$. Resorting to numerical integration, we find that we can choose $\rho = 3/35 + 10^{-6}$, which leads to the desired conclusion. □

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