THE DIFFERENCE BETWEEN CONSECUTIVE PRIMES IN AN ARITHMETIC PROGRESSION

A. KUMCHEV

ABSTRACT. We obtain new results concerning the simultaneous distribution of prime numbers in arithmetic progressions and in short intervals. For example, we show that there is an absolute constant $\delta > 0$ such that 'almost all' arithmetic progressions $a \bmod q$ with $q \le x^{\delta}$ and $(a, q) = 1$ contain prime numbers from the interval $(x - x^{0.53}, x]$.

1. Introduction

It is an old problem in number theory to find the least $h = h(x)$ such that the interval $(x - h, x]$ contains a prime number for all sufficiently large x. The first non-trivial result in this direction was obtained by Hoheisel [8] who proved that if $h = x^{1-(3300)^{-1}}$,

$$\pi(x) - \pi(x - h) \sim h(\log x)^{-1} \quad \text{as } x \to \infty, \tag{1.1}$$

$\pi(x)$ being the number of primes $\le x$. There have been numerous improvements on Hoheisel's result, and it is now known that (1.1) holds for $x^{7/12} \le h \le x$, as proven by Heath-Brown [7]. Furthermore, by pursuing a lower bound of the form

$$\pi(x) - \pi(x - h) \gg h(\log x)^{-1} \tag{1.2}$$

instead of an asymptotic formula, one can introduce sieve ideas and prove the existence of prime numbers in intervals $(x - h, x]$ of even smaller length. The best known result in this direction is due to Baker, Harman and Pintz [3] who established (1.2) with $h = x^{0.525}$.

In this paper, we seek similar results for primes in arithmetic progressions. Let $\pi(x; q, a)$ be the number of primes $p \le x$ with $p \equiv a \pmod q$. If $q \le (\log x)^A$ for some fixed $A > 0$, the machinery used to prove results of the forms (1.1) or (1.2) can be adjusted to produce similar estimates for $\pi(x; q, a) - \pi(x - h; q, a)$. For example, Baker, Harman and Pintz [2, Theorem 3] showed that if $q \le (\log x)^A$, $(a, q) = 1$ and $x^{11/20+\varepsilon} \le h \le x(\log x)^{-1}$, then

$$\pi(x; q, a) - \pi(x - h; q, a) \gg \frac{h}{\phi(q) \log x}. \tag{1.3}$$

For larger moduli, however, such estimates seem to be beyond the reach of present methods. The first result for such moduli was obtained by Jutila [11]. Let $\Lambda(n)$ be

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von Mangoldt's function. Jutila proved that

$$\sum_{q\le Q}\max_{(a,q)=1}\max_{h\le z}\max_{x/2\le y<x}\left|\sum_{\substack{y-h<n\le y\\ n\equiv a \pmod q}}\Lambda(n)-\frac{h}{\phi(q)}\right|\ll\frac{z}{(\log x)^A} \tag{1.4}$$

for any $A>0$ and $Q\le z^{2/5}x^{-1/4-\varepsilon}$. Subsequently, a number of authors improved on Jutila's result, showing that (1.4) holds for smaller values of z and/or larger values of Q (see [10, 14, 15, 16, 18, 19]). Currently, (1.4) is known if

$$x^{3/5+\varepsilon}\le z\le x,\quad Q\le zx^{-1/2}(\log x)^{-B(A)}, \tag{1.5}$$

or

$$x^{7/12+\varepsilon}\le z\le x,\quad Q\le zx^{-11/20-\varepsilon}. \tag{1.6}$$

The former result was obtained independently by Perelli, Pintz and Salerno [15] and by Timofeev [19], while the latter is due to Timofeev [19].

It follows from (1.4) that the analogue of (1.1) for primes in arithmetic progressions is true for 'almost all' moduli $q\le Q$ (i.e., for all but $O\big(Q(\log x)^{-A}\big)$ moduli $q\le Q$). Furthermore, (1.4) includes (via the term $q=1$) a statement of the form (1.1), so any version of (1.4) with $z=x^{7/12-\varepsilon}$ would represent an improvement on the aforementioned result by Heath-Brown [7]. Such a result seems to be out of the reach of present methods, but we will show in this paper that one can adjust the ideas used to obtain results of the form (1.2) to prove that (1.3) holds for 'almost all' moduli $q\le Q$ when z and Q are subject to (1.9) below. Our method shares many features with the method used in [1, 2, 3] to deal with primes in short intervals, but unlike [3] we cannot reach intervals of length $x^{0.525}$ due to the lack of an analogue for Dirichlet L-functions of Watt's mean-value theorem [20].

Given an arithmetic function $f(n)$, let us define

$$E_f(y,h;q,a)=\sum_{\substack{y-h<n\le y\\ n\equiv a \pmod q}}f(n)-\frac{hh_0^{-1}}{\phi(q)}\sum_{y-h_0<n\le y}f(n), \tag{1.7}$$

where $x/2\le y<x$ and $h_0=x\exp\left(-3(\log x)^{1/3}\right)$. Then, our main result reads as follows.

Theorem. *There is an arithmetic function $\lambda(n)$ with the following properties:*

(i) *if n is an integer in $[2,x)$, then*

$$\lambda(n)\le\begin{cases}1 & \text{if } n \text{ is prime,}\\ 0 & \text{otherwise;}\end{cases}$$

(ii) *if $x/2\le y<x$ and $h_0=x\exp\left(-3(\log x)^{1/3}\right)$, then*

$$\sum_{y-h_0<n\le y}\lambda(n)\gg\frac{h_0}{\log x}; \tag{1.8}$$

(iii) *there is an absolute constant $\delta>0$ such that if $E_\lambda(y,h;q,a)$ is defined by (1.7) with $f(n)=\lambda(n)$, and if*

$$x^{0.53}\le z\le x,\quad Q\le zx^{-0.53+\delta}, \tag{1.9}$$

then for any $A > 0$

$$\sum_{q \le Q} \max_{(a,q)=1} \max_{h \le z} \max_{x/2 \le y < x} |E_\lambda(y,h;q,a)| \ll \frac{z}{(\log x)^A}, \tag{1.10}$$

with an implied constant depending at most on A.

Remark 1. The absolute constant δ in the statement of the Theorem can be taken equal to 0.002 based on work by Lewis [12].

Remark 2. Our methods also yield a function $\lambda(n)$ which is an upper bound for the characteristic function of the set of primes and satisfies an inequality opposite to (1.8), and for which (1.10) holds whenever $Q \le zx^{-1/2-\varepsilon}$ We, however, do not pursue this, as it is superseded by the Brun–Tichmarsh inequality; see, for example, [5, Section 3.4].

Notation. Throughout the paper ε will be a fixed sufficiently small positive number. We reserve the letter p, with or without indices, for prime numbers; q is reserved to denote moduli of arithmetic progressions or Dirichlet characters. The letter B denotes an absolute constant, not necessarily the same in each occurrence; similarly, η denotes a positive constant, sufficiently small in terms of ε, whose value can change between two appearances. We write $n \sim N$ for the condition $N/2 \le n < N$ and $n \asymp N$ for the condition $c_1 N \le n < c_2 N$, where c_1 and c_2 are some absolute constants. We shall use the following arithmetic functions:

$d(n)$**:** the number of positive divisors of n;
$\mu(n)$**:** the Möbius function;
$\psi(n,w)$**:** for real $w \ge 2$, we define

$$\psi(n,w) = \begin{cases} 1 & \text{if } n \text{ has no prime divisor } \le w, \\ 0 & \text{otherwise;} \end{cases} \tag{1.11}$$

$\chi_0(n)$**:** the trivial character with $\chi_0(n) = 1$ for all n.

We also write $\mathscr{L} = \log x$ and $\Psi(T) = \min\left(zx^{-1/2}, x^{1/2}T^{-1}\right)$. We use $\sum_{\chi \bmod q}$ and $\sum^*_{\chi \bmod q}$ to denote, respectively, summations over all and over the primitive Dirichlet characters modulo q; when the modulus q is clear from the context, we write simply $\sum_{\chi}$ and $\sum^*_{\chi}$. Also, if χ is a Dirichlet character, we define $\delta(\chi)$ to be equal to 1 or 0 according as χ is principal or not.

2. Dirichlet polynomials

In this section we collect some general results on mean and large values of Dirichlet polynomials

$$N(s,\chi) = \sum_{n \asymp N} b_n \chi(n) n^{-s}, \tag{2.1}$$

where χ is a Dirichlet character. Notice that we use the same letter to denote the polynomial and its length—this will allow us to shorten the statements of some lemmas.

If $\mathfrak{T}$ is a set of triples (t,q,χ) with $|t| \le T$, $q \sim Q$ and χ a primitive character modulo q, we say that $\mathfrak{T}$ is *well-spaced* if $|t - t'| \ge 1$ for every two distinct triples

(t,q,χ) and (t',q,χ) in $\mathfrak{T}$. Similarly, we say that a set of real numbers $\mathfrak{T}$ is *well-spaced* if $|t-t'| \geq 1$ for every two distinct elements t and t' of $\mathfrak{T}$. Since we want to work with both kinds of well-spaced sets simultaneously, we identify a number t in a well-spaced set of the latter kind with the triple $(t,1,\chi_0)$, and we say that a set $\mathfrak{T}$ of such triples *corresponds to principal characters*.

Given a well-spaced set $\mathfrak{T}$ we define the following norms on the Dirichlet polynomials $N(s,\chi)$ of the form (2.1):

$$\|N\|_p = \begin{cases} \Big(\sum_{(t,q,\chi)\in\mathfrak{T}} \big|N(\tfrac12+it,\chi)\big|^p \Big)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup_{(t,q,\chi)\in\mathfrak{T}} \big|N(\tfrac12+it,\chi)\big| & \text{if } p = \infty. \end{cases}$$

Lemma 1. *Let $\mathfrak{T}$ be well-spaced and let $N(s,\chi)$ be defined by* (2.1)*. Then,*

$$\|N\|_2^2 \ll (N + Q^2T)G\mathscr{L}, \tag{2.2}$$

where

$$G = G(N) = \sum_{n\asymp N} |b_n|^2 n^{-1}.$$

Furthermore, if

$$\big|N(\tfrac12+it,\chi)\big| \geq V$$

for all $(t,q,\chi)\in\mathfrak{T}$, then the cardinality of $\mathfrak{T}$ is bounded by

$$|\mathfrak{T}| \ll \big(NV^{-2} + G^2Q^2TNV^{-6}\big)G\mathscr{L}^2. \tag{2.3}$$

Proof. Inequality (2.2) is a special case of [13, Theorem 7.3]. The second part of the lemma can be deduced from [13, Theorem 8.2] by Huxley's dissection method (see [9, Section 2], for example). □

The next lemma is an analogue of [2, Theorem 4] and [3, Lemma 9]. We omit its proof, since it is similar to the proofs of those results (using Lemma 1 in place of [2, Lemma 1]). Similarly, Lemma 3 is a version of [2, Lemma 3].

Lemma 2. *Let $\mathfrak{T}$ be a well-spaced set and define*

$$M(s,\chi) = \sum_{m\asymp M} a_m\chi(m)m^{-s}, \quad N(s,\chi) = \sum_{n\asymp N} b_n\chi(n)n^{-s}, \tag{2.4}$$

$$K(s,\chi) = \sum_{k\asymp K} c_k\chi(k)k^{-s},$$

where $KMN \asymp x$ and the coefficients a_m, b_n, c_k satisfy

$$|a_m| \leq d(m)^B, \quad |b_n| \leq d(n)^B, \quad |c_k| \leq d(k)^B. \tag{2.5}$$

Suppose that $Q \leq zx^{-\theta-\eta}$ where $1/2+\varepsilon \leq \theta \leq 7/12$. Let g be a positive integer, and suppose that

$$1 \leq M/N \ll x^{2\theta-1}, \quad K \ll x^{\gamma}, \tag{2.6}$$

where

$$\gamma = \min\left(4\theta-2, \frac{(8g-4)\theta-(4g-3)}{4g-1}, \frac{24g\theta-(12g+1)}{4g-1}\right). \tag{2.7}$$

Suppose also that for any $A>0$ $K(s,\chi)$ satisfies

$$\|K\|_\infty \ll QK^{1/2}\mathscr{L}^{-A}. \tag{2.8}$$

Then, for any $A > 0$

$$\Psi(T)\|KMN\|_1 \ll Qz\mathscr{L}^{-A}, \tag{2.9}$$

with an implied constant depending at most on A, g, ε and η. Furthermore, without assumption (2.8)*, we have*

$$\Psi(T)\|KMN\|_1 \ll Qz\mathscr{L}^{B}. \tag{2.10}$$

Lemma 3. *Let $\mathfrak{T}$ be a well-spaced set, let $K(s,\chi)$, $M(s,\chi)$ and $N(s,\chi)$ be defined by* (2.4) *with coefficients a_m, b_n, c_k satisfying* (2.5)*, and let $R(s,\chi)$ be defined similarly. Suppose that $KMNR \asymp x$ and that $K(s,\chi)$, $N(s,\chi)$ and $R(s,\chi)$ satisfy condition* (2.8)*. Suppose also that $Q \le zx^{-\theta-\eta}$ where $1/2+\varepsilon \le \theta \le 7/12$. Then, for any $A > 0$,*

$$\Psi(T)\|KMNR\|_1 \ll Qz\mathscr{L}^{-A}, \tag{2.11}$$

provided that any of the following sets of conditions holds:

(i) $M \gg x^{1-\theta}$, $N \gg x^{(1-\theta)/2}$, $R \gg x^{(1-\theta)/4}$, $K \gg x^{2(1-\theta)/7}$;
(ii) $M \gg x^{1-\theta}$, $N \gg x^{(1-\theta)/2}$, $R \gg x^{(1-\theta)/3}$, $K \gg x^{2(1-\theta)/11}$;
(iii) $M \gg x^{1-\theta}$, $N \gg x^{(1-\theta)/3}$, $R^2K \gg x^{1-\theta}$, $K \gg x^{2(1-\theta)/5}$;
(iv) $M \gg x^{1-\theta}$, $N \ll x^{(1-\theta)/3}$, $R \ll x^{(1-\theta)/3}$, $NR \gg x^{4(1-\theta)/7}$, $KNR \gg x^{14(1-\theta)/13}$.

Lemma 4. *Let $\mathfrak{T}$ be a well-spaced set and define $M(s,\chi)$, $N(s,\chi)$ by* (2.4) *with $MN \asymp x$ and with coefficients a_m, b_n satisfying* (2.5)*. Suppose that*

$$Q \le zx^{-1/2-\eta}, \quad \max(M,N) \le zx^{-\eta}.$$

Then, for any $A > 0$

$$\Psi(T)\|MN\|_1 \ll Qz\mathscr{L}^{-A} + z\mathscr{L}^{B}, \tag{2.12}$$

with an implied constant depending at most on A and η.

Proof. We start by dividing $\mathfrak{T}$ into $O(\mathscr{L}^2)$ subsets $\mathfrak{S}$ such that

$$\left|M(\tfrac{1}{2}+it,\chi)\right| \asymp M^{\sigma_1}, \quad \left|N(\tfrac{1}{2}+it,\chi)\right| \asymp N^{\sigma_2}, \tag{2.13}$$

for all $(t,q,\chi) \in \mathfrak{S}$. By (2.2),

$$|\mathfrak{S}| \ll \left(M^{1-2\sigma_1} + Q^2TM^{-2\sigma_1}\right)\mathscr{L}^B,$$
$$|\mathfrak{S}| \ll \left(N^{1-2\sigma_2} + Q^2TN^{-2\sigma_2}\right)\mathscr{L}^B.$$

Hence,

$$\begin{aligned}|\mathfrak{S}| &\ll \left(M^{1-2\sigma_1} + Q^2TM^{-2\sigma_1}\right)^{1/2}\left(N^{1-2\sigma_2} + Q^2TN^{-2\sigma_2}\right)^{1/2}\mathscr{L}^B \\ &\ll M^{-\sigma_1}N^{-\sigma_2}\bigl(x^{1/2} + QT^{1/2}\max(M,N)^{1/2} + Q^2T\bigr)\mathscr{L}^B.\end{aligned}$$

The lemma follows from this inequality and (2.13). □

Lemma 5. *Let $\mathfrak{T}$ be a well-spaced set and define*

$$K(s,\chi) = \sum_{k\sim K} \chi(k)k^{-s}, \tag{2.14}$$

where $K \ll QT$. Suppose that $Q \le T$ and, if $\mathfrak{T}$ corresponds to principal characters, suppose also that $T/2 \le |t| \le T$ for all $(t,1,\chi_0) \in \mathfrak{T}$. Then,

$$\|K\|_4^4 \ll Q^2T\mathscr{L}^8. \tag{2.15}$$

Proof. We recall the truncated Perron formula

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT} u^s \frac{ds}{s} = E(u) + O\left(\frac{u^b}{T|\log u|}\right), \tag{2.16}$$

where $b > 0$ and $E(u)$ is 0 or 1 according as $0 < u < 1$ or $u > 1$. Assuming, as we can, that $(K - \frac{1}{4}) \in \mathbb{Z}$, we deduce from (2.16) that

$$K(s,\chi) = \frac{1}{2\pi i}\int_{c-iT_1}^{c+iT_1} L(w+s,\chi)\frac{(2K)^w - K^w}{w}dw + O(K^{1/2}T^{-1}\mathscr{L}).$$

where $s = \frac{1}{2} + it$, $c = \frac{1}{2} + \mathscr{L}^{-1}$ and $T_1 = 2T$. We now use the rectangular contour with vertices $\pm iT_1$, $c \pm iT_1$ to move the integration to the line $\operatorname{Im} w = 0$. By [17, Theorem 3] with $\eta = \mathscr{L}^{-1}$,

$$L(\sigma + it, \chi) \ll (q(|t|+2))^{(1-\sigma)/2}\mathscr{L},$$

whenever $0 \le \sigma \le 1$ and χ is a primitive character modulo q. Hence, the contribution from the horizontal segments is

$$\ll \max_{0\le\sigma\le c}\left((QT)^{(1-2\sigma)/4}K^\sigma T^{-1}\mathscr{L}\right) \ll \left(T^{-1/2} + K^{1/2}T^{-1}\right)\mathscr{L} \ll \mathscr{L}.$$

Moreover, the integrand is regular inside the contour except for a possible simple pole at $w = \frac{1}{2} - it$ with residue $\ll \delta(\chi)K^{1/2}(1+|t|)^{-1}$. Thus,

$$\left|K(\tfrac{1}{2} + it, \chi)\right| \ll J_1(t,\chi) + \mathscr{L} + \frac{\delta(\chi)K^{1/2}}{1+|t|}, \tag{2.17}$$

where for $v \ge 0$,

$$J_v(t,\chi) = \int_{-T_1}^{T_1} \left|L(\tfrac{1}{2} + i(t+\tau),\chi)\right|^v \frac{d\tau}{1+|\tau|}.$$

We now raise (2.17) to fourth power, sum the resulting estimate over the elements of $\mathfrak{T}$, and use Hölder's inequality to get

$$\|K\|_4^4 \ll \mathscr{L}^3 \sum_{(t,q,\chi)\in\mathfrak{T}} J_4(t,\chi) + |\mathfrak{T}|\mathscr{L}^4 + K^2 \sum_{(t,q,\chi)\in\mathfrak{T}} \frac{\delta(\chi)}{1+|t|^4}.$$

Since

$$\sum_{(t,q,\chi)\in\mathfrak{T}} J_4(t,\chi) \ll \mathscr{L}\sum_{q\sim Q}\sum_{\chi}{}^{*}\int_{-2T_1}^{2T_1}\left|L(\tfrac{1}{2} + iu,\chi)\right|^4 du,$$

we can now refer to [13, Theorem 10.1] to complete the proof. □

Lemma 6. *Let χ be a Dirichlet character modulo q and define $K(s,\chi)$ by (2.14). Then,*

$$K\left(\tfrac{1}{2} + it, \chi\right) \ll \delta(\chi)K^{1/2}\tau^{-1} + K^{-1/2}(q\tau)^{1/2}\log(q\tau),$$

where $\tau = |t| + 2$.

Proof. This follows from [4, Theorem 1] by partial summation. □

Lemma 7. *Let $\mathfrak{T}$ be a well-spaced set, let $K(s,\chi)$ be defined by (2.14), and let $M(s,\chi)$, $N(s,\chi)$ be defined by (2.4) with coefficients a_m, b_n subject to (2.5). Suppose that $KMN \asymp x$, $Q \le \min\left(zx^{-\theta-\eta}, T\right)$ where $1/2 + \varepsilon \le \theta \le 7/12$. Suppose also that*

$$\max\left(M, Qx^{1-\theta}\right)\max\left(N, Q^{1/2}x^{(1-\theta)/2}\right) \le Q^{3/2}x^{(1+\theta)/2}. \tag{2.18}$$

Furthermore, if $\mathfrak{T}$ corresponds to principal characters, suppose that $MN \geq x^{\eta}$ and that $|t| \geq \exp\left(\mathscr{L}^{1/3}\right)$ for all triples $(t, 1, \chi_0) \in \mathfrak{T}$. Then, for any $A > 0$

$$\Psi(T)\|KMN\|_1 \ll Qz\mathscr{L}^{-A},$$

the implied constant depending at most on A, ε and η.

Proof. First, suppose that $\mathfrak{T}$ does not correspond to principal characters. If $K \leq QT$, Lemma 5 yields $\|K\|_4^4 \ll Q^2T\mathscr{L}^8$, whence

$$\begin{aligned}\|KMN\|_1 &\leq \|M\|_2\|N\|_4\|K\|_4 = \|M\|_2\|N^2\|_2^{1/2}\|K\|_4 \\ &\ll \left(Q^2T + M\right)^{1/2}\left(Q^2T + N^2\right)^{1/4}\left(Q^2T\right)^{1/4}\mathscr{L}^B.\end{aligned}$$

If $K > QT$, combining Lemmas 1 and 6, we get

$$\begin{aligned}\|KMN\|_1 &\leq \|K\|_\infty\|M\|_2\|N\|_2 \\ &\ll \left(Q^2T + M\right)^{1/2}\left(Q^2T + N\right)^{1/2}\mathscr{L}^B.\end{aligned}$$

Now, suppose that $\mathfrak{T}$ corresponds to principal characters. We split it into $O(\mathscr{L})$ subsets $\mathfrak{S}$ such that $U \leq |t| \leq 2U$ for all $(t, 1, \chi_0) \in \mathfrak{S}$. We estimate the contribution of each individual $\mathfrak{S}$ as before. If $K \leq U$, by Lemma 5,

$$\begin{aligned}\|KMN\|_1 &\leq \|M\|_2\|N\|_4\|K\|_4 = \|M\|_2\|N^2\|_2^{1/2}\|K\|_4 \\ &\ll \left(U + M\right)^{1/2}\left(U + N^2\right)^{1/4}U^{1/4}\mathscr{L}^B \\ &\ll \left(T + M\right)^{1/2}\left(T + N^2\right)^{1/4}T^{1/4}\mathscr{L}^B.\end{aligned}$$

On the other hand, if $K > U$, Lemmas 1 and 6 give

$$\begin{aligned}\|KMN\|_1 &\leq \|K\|_\infty\|M\|_2\|N\|_2 \\ &\ll \left(U + M\right)^{1/2}\left(U + N\right)^{1/2}(K^{1/2}U^{-1} + 1)\mathscr{L}^B \\ &\ll x^{1/2}T_0^{-1/3} + \left(T + M\right)^{1/2}\left(T + N\right)^{1/2}\mathscr{L}^B,\end{aligned}$$

where $T_0 = \exp\left(\mathscr{L}^{1/3}\right)$. □

The next lemma is a consequence of the Siegel–Walfisz theorem. Its proof can be found in [2, Lemma 5].

Lemma 8. *Let χ be a character modulo q, $q \leq \mathscr{L}^C$ with C fixed, and let t be real with*

$$\delta(\chi)\exp\left(\mathscr{L}^{1/4}\right) \leq |t| \leq x^B. \tag{2.19}$$

Then, for any $A > 0$

$$\sum_{k \sim K} \psi(k, w)\chi(k)k^{-1/2-it} \ll K^{1/2}\mathscr{L}^{-A},$$

where $w \geq \exp\left(\mathscr{L}^{9/10}\right)$, $\psi(k, w)$ is defined by (1.11), and the implied constant depends at most on A and C.

3. Sieve estimates

Given an arithmetic function $f(k)$ and a Dirichlet character $\chi(k)$, define

$$E_f(y,h;\chi) = \sum_{y-h<k\le y} f(k)\chi(k) - \delta(\chi) h h_0^{-1} \sum_{y-h_0<k\le y} f(k). \tag{3.1}$$

In this section we prove inequalities of the form

$$\sum_{q\sim Q} \sideset{}{^*}\sum_{\chi} \max_{h\le z} \max_{y\sim x} |E_f(y,h;\chi)| \ll Qz\mathscr{L}^{-A}, \tag{3.2}$$

when $f(k)$ is a special kind of convolution.

Lemma 9. *Let*

$$c(k) = \sum_{\substack{mn\ell=k\\ m\asymp M, n\asymp N}} a_m b_n \psi(\ell,w), \tag{3.3}$$

where $w = \exp\left(\mathscr{L}^{9/10}\right)$, $\psi(\ell,w)$ *is defined by* (1.11), *and the coefficients* a_m, b_n *are subject to* (2.5). *Suppose that* $Q \le zx^{-\theta-\eta}$ *with* $1/2+\varepsilon \le \theta \le 7/12$. *Suppose also that* M, N *satisfy*

$$\max\left(M, Qx^{1-\theta}\right) \max\left(N, Q^{1/2}x^{(1-\theta)/2}\right) \le Q^{3/2}x^{(1+\theta)/2}. \tag{3.4}$$

Then, (3.2) *holds for* $f(k) = c(k)$ *and any* $A > 0$*; the implied constant depends at most on* A, ε *and* η.

Proof. We start by writing

$$c(k) = \sum_{m,n} \sum_{\substack{mn\ell=k\\ d|\ell, d|P(w)}} a_m b_n \mu(d), \tag{3.5}$$

where we have suppressed the conditions on m and n for brevity. We aim to express the right side of (3.5) as a linear combination of $O(\mathscr{L})$ similar convolutions for each of which (3.2) follows from Lemmas 4 or 7. We proceed differently according as $Q = 1$, $1 < Q \le \mathscr{L}^{A+B}$, or $Q \ge \mathscr{L}^{A+B}$.

Case 1. $Q \ge \mathscr{L}^{A+B}$. By a simple splitting-up argument, it suffices to consider functions $c(k)$ for which $d \sim D$ on the right side of (3.5). We consider three subcases.

Case 1.1. $MD \le Qx^{1-\theta}$. Introducing in (3.5) new summation variables $u = md$, $v = \ell d^{-1}$, we can write $c(k)$ in the form

$$c(k) = \sum_{\substack{nuv=k\\ u\asymp U, n\asymp N}} a'_u b_n, \tag{3.6}$$

where $U = MD$ and the new coefficients a'_u satisfy (2.5). Assuming, as we can, that both y and $y-h$ belong to $\mathbb{Z}+\frac{1}{2}$, we combine (3.6) and (2.16) to get

$$\sum_{y-h<k\le y} c(k)\chi(k) = \frac{1}{2\pi i}\int_{\frac12-ix}^{\frac12+ix} S(s,\chi)\frac{y^s-(y-h)^s}{s}ds + O\big(x^\eta\big), \tag{3.7}$$

where

$$S(s,\chi) = N(s,\chi)U(s,\chi)V(s,\chi),$$

with

$$U(s,\chi) = \sum_{u \asymp MD} a'_u \chi(u) u^{-s}, \quad V(s,\chi) = \sum_{v \asymp x/(MDN)} \chi(v) v^{-s},$$

and $N(s,\chi)$ given by (2.4). Hence,

$$\max_{h,y} |E_c(y,h;\chi)| \ll \mathscr{L} \max_{Q \le T \le x} \Psi(T) \int_{-T}^{T} \big| S(\tfrac{1}{2}+it,\chi) \big| dt + x^{\eta}. \tag{3.8}$$

We now apply Lemma 7 to $S(s,\chi)$ with $U(s,\chi)$ and $V(s,\chi)$ in place of $M(s,\chi)$ and $K(s,\chi)$ in that lemma. We get

$$\max_{Q \le T \le x} \Psi(T) \sum_{q \sim Q} \sideset{}{^*}\sum_{\chi} \int_{-T}^{T} \big| S(\tfrac{1}{2}+it,\chi) \big| dt \ll Qz\mathscr{L}^{-A}, \tag{3.9}$$

which in combination with (3.8) completes the proof of (3.2).

Case 1.2. $M \ge Qx^{1-\theta}$. We use (3.7) with

$$S(s,\chi) = M(s,\chi)V(s,\chi),$$

where $M(s,\chi)$ is defined by (2.4) and $V(s,\chi)$ corresponds to a new summation variable $v = nd\ell$. Using (3.4), we can readily verify the hypotheses of Lemma 4, so from (2.12),

$$\max_{Q \le T \le x} \Psi(T) \sum_{q \sim Q} \sideset{}{^*}\sum_{\chi} \int_{-T}^{T} \big| S(\tfrac{1}{2}+it,\chi) \big| dt \ll Qz\mathscr{L}^{-A},$$

which completes the proof.

Case 1.3. $M \le Qx^{1-\theta} \le MD$. In this case, our idea is to use that d is composed of small primes to write $d = d_1 d_2$ with d_1 such that $Qx^{1-\theta-\eta} \ll Md_1 \ll Qx^{1-\theta}$; we then proceed as in Case 1.2 with $M(s,\chi)$ and $V(s,\chi)$ corresponding the new variables $u = md_1$ and $v = nd_2\ell$.

We have $c(k) = \sum_{j=1}^{J} c_j(k)$, with $J \le \mathscr{L}$ and

$$c_j(k) = \sum_{m,k} \sum_{\substack{p_1,\dots,p_j \\ mnp_1\cdots p_j\ell = k}} a_m b_n (-1)^j,$$

where $p_1, \dots, p_j$ are subject to

$$p_j < \cdots < p_1 < w, \quad D < p_1 \cdots p_j \le 2D, \quad p_1 \cdots p_j M \ge Qx^{1-\theta}.$$

Let r be the greatest integer such that $p_r \cdots p_j M \ge Qx^{1-\theta}$. We would like to take $d_2 = p_1 \cdots p_r$; notice that with this choice we have

$$Qx^{1-\theta} w^{-1} < Md_1 \le Qx^{1-\theta}.$$

The resulting coefficients, however, are not of the form $\sum_{uv=k} a'_u b'_v$ because of the conditions

$$D < p_1 \cdots p_j \le 2D, \quad p_r \cdots p_j M > Qx^{1-\theta}, \quad p_{r+1} < p_r, \tag{3.10}$$

so Lemma 4 is not (directly) applicable. We overcome this obstacle by means of Perron's formula.

Let us consider, for example, the last inequality in (3.10). By (2.16), we have

$$c_j(k) = \frac{1}{2\pi i} \sideset{}{^\dagger}\sum_{mn\cdots=k} (-1)^j a_m b_n \int_{\alpha-iX}^{\alpha+iX} \left(\frac{p_r - 1/2}{p_{r+1}} \right)^{\xi} \frac{d\xi}{\xi} + O(x^\eta X^{-1}),$$

where $\alpha = \mathscr{L}^{-1}$, $X = x^3$, and $\sum^{\dagger}_{mn\cdots=k}$ denotes summation over the same variables and conditions as in $c_j(k)$ except for the condition $p_{r+1} < p_r$, which has been removed. Hence,

$$\begin{aligned} |E_{c_j}(y,h;\chi)| &\ll \int_{-X}^{X} |E_{c^\dagger}(y,h;\chi)| \frac{dt}{|\alpha+it|} + x^{-1} \\ &\ll |E_{c^\dagger}(y,h;\chi)| \mathscr{L} + x^{-1}, \end{aligned}$$

where

$$c^\dagger(k) = \sideset{}{^\dagger}\sum_{mn\cdots=k} (-1)^j a_m b_n (p_r - \tfrac{1}{2})^{\alpha+it_0} p_{r+1}^{\alpha-it_0}.$$

Clearly, we can remove the other inequalities in (3.10) from the summation conditions in $c^\dagger(k)$ by the same technique. This yields a function $c^\ddagger(k) = \sum_{uv=k} a'_u b'_v$ for which we can prove (3.2) as intended (i.e., using (2.16) followed by Lemma 4).

Case 2. $Q = 1$. By [7, Lemma 15],

$$\sum_{d|\ell, d|P(w)} \mu(d) = \sum_{\substack{d|\ell, d|P(w) \\ d<\gamma}} \mu(d) + O\Bigg(\sum_{\substack{d|\ell, d|P(w) \\ \gamma\le d<w\gamma}} 1 \Bigg). \tag{3.11}$$

Substituting (3.11) with $\gamma = x^\eta$ into (3.5), we get

$$\sum_{y-h<k\le y} c(k) = \sum_{y-h<k\le y} c_1(k) + O\Bigg(\sum_{y-h<k\le y} c_2(k) \Bigg),$$

where

$$\begin{aligned} c_1(k) &= \sum_{m,n} \sum_{\substack{d|P(w) \\ d<\gamma}} \sum_{mnd\ell=k} a_m b_n \mu(d), \\ c_2(k) &= \sum_{m,n} \sum_{\substack{d|P(w) \\ \gamma\le d<w\gamma}} \sum_{mnd\ell=k} |a_m b_n|. \end{aligned}$$

It therefore suffices to show that if $h \le z$, $y \sim x$, then

$$\sum_{y-h<k\le y} c_j(k) = hh_0^{-1} \sum_{y-h_0<k\le y} c_j(k) + O\big(z\mathscr{L}^{-A}\big) \quad (j=1,2), \tag{3.12}$$

$$\sum_{y-h_0<k\le y} c_2(k) \ll h_0 \mathscr{L}^{-A}. \tag{3.13}$$

Let us consider (3.12). If $MN \le x^\eta$, we have

$$\begin{aligned} \sum_{y-h<k\le y} c_1(k) &= h \sum_{m,n,d} \frac{a_m b_n \mu(d)}{mnd} + O\big(x^\eta\big) \\ &= hh_0^{-1} \sum_{y-h_0<k\le y} c_1(k) + O\big(x^\eta\big), \end{aligned}$$

and an obvious modification holds for $c_2(k)$. Thus, we can focus on the case $MN \geq x^\eta$. As in Case 1.1, we have

$$\sum_{y-h<k\leq y} c_j(k) = \frac{1}{2\pi i}\int_{\frac12 - ix}^{\frac12 + ix} S_j(s,\chi_0)\frac{y^s-(y-h)^s}{s}ds + O\big(x^\eta\big), \tag{3.14}$$

where $S_j(s,\chi)$ has been defined analogously to the polynomial $S(s,\chi)$ appearing in (3.7). If $MN \geq x^\eta$, Lemma 7 yields

$$\max_{T_0\leq T\leq x}\Psi(T)\int_{T_0}^{T}\big|S_j(\tfrac12+it,\chi_0)\big|dt \ll z\mathscr{L}^{-A},$$

where $T_0 = \exp\big(\mathscr{L}^{1/3}\big)$. Hence,

$$\sum_{y-h<k\leq y} c_j(k) = \frac{h}{2\pi i}\int_{\frac12 - iT_0}^{\frac12 + iT_0} S_j(s,\chi_0)y^{s-1}ds + O\big(z\mathscr{L}^{-A}\big), \tag{3.15}$$

and similarly

$$\sum_{y-h_0<k\leq y} c_j(k) = \frac{h_0}{2\pi i}\int_{\frac12 - iT_0}^{\frac12 + iT_0} S_j(s,\chi_0)y^{s-1}ds + O\big(h_0\mathscr{L}^{-A}\big). \tag{3.16}$$

Combining (3.15) and (3.16), we complete the proof of (3.12). As to (3.13), by [3, Lemma 7] we have

$$\sum_{y-h_0<k\leq y} c_2(k) \ll h_0\sum_{m,n}\frac{|a_m b_n|}{mn}\sum_{\substack{d|P(w)\\ \gamma<d\leq w\gamma}}\frac{1}{d} \ll h_0\exp\big(-\mathscr{L}^{1/10}\big).$$

Case 3. $1 < Q \leq \mathscr{L}^{A+B}$. Using (3.11) with $\gamma = x^\eta$, we get

$$\sum_{y-h<k\leq y} c(k)\chi(k) = \sum_{y-h<k\leq y} c_1(k)\chi(k) + O\bigg(\sum_{y-h<k\leq y} c_2(k)\bigg),$$

with $c_1(k)$ and $c_2(k)$ as in Case 2. The contribution from $c_2(k)$ can be estimated via (3.12) and (3.13) with $A+B$ in place of A, and the contribution from $c_1(k)$ can be estimated as in Case 1.1. □

Lemma 10. *Let $c(k)$ be defined by (3.3) with a_m, b_n subject to (2.5). Let $Q \leq zx^{-\theta-\eta}$ with $1/2+\varepsilon \leq \theta \leq 7/12$, let $M \ll x^{1/2}$, and let h be the least positive integer such that*

$$M \geq x^{1/2-h(2\theta-1)}. \tag{3.17}$$

Define

$$M^* = \max\Big(x^{2h(1-\theta)}M^{-1}, x^{2(h-1)\theta}M\Big)^{1/(2h-1)}, \tag{3.18}$$

and suppose that

$$NM^* \ll x^{(1+\theta)/2}. \tag{3.19}$$

Suppose also that

$$w \leq \min\Big(x^\gamma, (x^\theta M^{-1})^{2/(2h-1)}\Big), \tag{3.20}$$

where γ is defined by (2.7). Then, (3.2) holds for $f(k) = c(k)$ and any $A > 0$; the implied constant depends at most on A, g, h, ε and η.

Note that $x^{1-\theta} \le M^* \le x^{1/2}$, so the upper bound for N in (3.19) is $\gg x^{\theta/2}$.

Proof. We recall Buchstab's identity in the form

$$\psi(n, w_1) = \psi(n, w_2) - \sum_{\substack{pm=n \\ w_2 \le p < w_1}} \psi(m, p), \tag{3.21}$$

where $2 \le w_2 < w_1$. Using (3.21), we can write

$$c(k) = c_0'(k) - c^*(k),$$

where

$$c_0'(k) = \sum_{mn\ell=k} a_m b_n \psi(\ell, w_0),$$
$$c^*(k) = \sum_{\substack{mnp_1\ell_1=k \\ w_0 \le p_1 < w}} a_m b_n \psi(\ell_1, p_1),$$

with $w_0 = \exp\left(\mathscr{L}^{9/10}\right)$. We split $c^*(n)$ into two subsums:

$$c^*(k) = c_0''(k) + c_1(k),$$

where $c_0''(k)$ and $c_1(k)$ are subject to $mp_1^{1/2} > x^{1-\theta}$ and $mp_1^{1/2} \le x^{1-\theta}$, respectively. We now use (3.21) to decompose $c_1(n)$. In general, if

$$c_j(k) = \sum_{\substack{mnp_1\cdots p_j\ell_j=k \\ w_0 \le p_j < \cdots < p_1 < w}} a_m b_n \psi(\ell_j, p_j)$$

is subject to

$$mp_1 \cdots p_{j-1}p_j^{1/2} \le x^{1-\theta}, \tag{3.22}$$

then (3.21) gives

$$c_j(k) = c_j'(k) - c_j''(k) - c_{j+1}(k)$$

where $c_j'(k)$ is $c_j(k)$ with $\psi(\ell_j, w_0)$ in place of $\psi(\ell_j, p_j)$ and $c_j''(k)$ is $c_{j+1}(k)$ with condition (3.22) replaced by

$$mp_1 \cdots p_{j-1}p_j^{1/2} \le x^{1-\theta} < mp_1 \cdots p_j p_{j+1}^{1/2}. \tag{3.23}$$

Clearly, this process has to stop after at most $\mathscr{L}$ steps.

Before proceeding further, we take a moment to note two consequences of (3.22) and (3.23). Inequality (3.22) implies that

$$p_j \le (x^{1-\theta}M^{-1})^{2/(2j-1)}.$$

If $j \ge h$, we can infer from this bound and (3.22) that

$$mp_1 \cdots p_j \le x^{1-\theta}(x^{1-\theta}M^{-1})^{1/(2h-1)} \le M^*,$$

while if $j < h$, we have

$$mp_1 \cdots p_j \le 2M(x^{\theta}M^{-1})^{2(h-1)/(2h-1)} \le 2M^*.$$

Similarly, we get

$$M^2p_1^2 \cdots p_j^2 p_{j+1} \ll \begin{cases} x^{2-2\theta}(x^{1-\theta}M^{-1})^{4/(2h-1)} & \text{if } j \ge h, \\ M^2 w^{2h-1} & \text{if } j < h. \end{cases}$$

As both expressions on the right side of the latter inequality are $\ll x^{2\theta}$ (by (3.17) and (3.20), respectively), we conclude that (3.23) implies

$$\max\left(\frac{x}{M^2p_1^2\cdots p_j^2p_{j+1}}, \frac{M^2p_1^2\cdots p_j^2p_{j+1}}{x}\right) \ll x^{2\theta-1}. \tag{3.24}$$

We now show that (3.2) holds for $f(k) = c'_j(k)$ and $f(k) = c''_j(k)$. We just proved that $mp_1\cdots p_j \le 2M^*$. Hence, if $f(k) = c'_j(k)$, (3.2) follows from Lemma 9 with a_m replaced by

$$a'_u = \sum_{\substack{mp_1\cdots p_j = u\\ m,p_1,\dots,p_j:(3.22)\\ w_0\le p_j<\cdots<p_1<w}} a_m.$$

We now turn to $c''_j(k)$. As in the proof of Lemma 9, (3.2) can be reduced to mean-value estimates similar to (3.9). In this case, we would like to work with

$$S(s,\chi) = M_1(s,\chi)N_1(s,\chi)K(s,\chi),$$

where $M_1(s,\chi)$ corresponds to the variable $m_1 = mp_1\cdots p_j$, $K(s,\chi)$ corresponds to the summation over p_{j+1}, and $N_1(s,\chi)$ corresponds to the product of the remaining variables. To do so, however, we first need to disentangle p_{j+1} from the other variables. This can be done via Perron's formula.

As in Case 1.3 in the proof of Lemma 9, we can use (2.16) and a standard splitting-up argument to write $c''_j(k)$ as the sum of $O(\mathscr{L}^3)$ functions of the form

$$\frac{1}{2\pi i}\int_{\alpha-ix^3}^{\alpha+ix^3} c_\ddagger(k;\xi)\frac{d\xi}{\xi} + O(x^{-2}), \tag{3.25}$$

where $\alpha = \mathscr{L}^{-1}$ and

$$c_\ddagger(k;\xi) = \sum_{mnp_1\cdots p_{j+1}\ell_{j+1}=k} a(m,p_1,\dots,p_j)b(n,\ell_{j+1})p_{j+1}^{-\xi}.$$

Here, the coefficients $a(m,p_1,\dots,p_j)$ and $b(n,\ell_{j+1})$ satisfy bounds of the form (2.5), and the summation variables are subject to the conditions in $c''_j(k)$, minus $p_{j+1} < p_j$, and plus

$$Mp_1\cdots p_j \asymp M_1, \quad p_j \sim P_j, \quad p_{j+1}\sim K,$$

where

$$w_0 \le K < 2P_j \le 2w, \quad M_1^2P_j^{-1} \ll x^{1-\theta} \ll M_1^2K.$$

Removing more summation conditions leads to a multiple integral in (3.25) and changes the coefficient of p_{j+1} in $c_\ddagger(k;\xi)$ to $p_{j+1}^{-L(\xi_1,\dots)}$, where $L(\xi_1,\dots)$ is a linear form in the integration variables. The result looks messier than (3.25), but can be dealt with just as easily. Thus, we shall assume through the rest of the proof that $p_{j+1} < p_j$ is the only condition that needs to be removed.

If $Q > 1$, it suffices to show that

$$\max_{Q\le T\le x}\Psi(T)\int_{\alpha-ix^3}^{\alpha+ix^3}\sum_{q\sim Q}\sideset{}{^*}\sum_{\chi}\int_{-T}^{T}\left|S_\xi(\tfrac12+it,\chi)\right|dt\frac{|d\xi|}{|\xi|} \ll Qz\mathscr{L}^{-A},$$

where

$$S_\xi(s,\chi) = \sum_{m_1\asymp M_1}\sum_{p\sim K}\sum_{n_1\asymp x/(M_1K)} a^*_{m_1}b^*_{n_1}p^{-\xi}\chi(m_1n_1p)(m_1n_1p)^{-s},$$

with $m_1 = mp_1 \cdots p_j$, $p = p_{j+1}$, $n_1 = n\ell_{j+1}$, and with coefficients

$$|a^*_{m_1}| \ll d(m_1)^B, \quad |b^*_{n_1}| \ll d(n_1)^B.$$

We now invoke Lemma 2 with $M(s,\chi)$, $N(s,\chi)$ and $K(s,\chi)$ corresponding to the summations over m_1, n_1 and p, respectively. These polynomials satisfy hypotheses (2.6) of Lemma 2 because of (3.20) and (3.24), so the proof will be completed if we show that

$$\sum_{p \sim K} \chi(p) p^{-\frac{1}{2}-it-\xi} \ll QK^{1/2}\mathscr{L}^{-A-B}.$$

This bound follows from Lemma 8 and partial summation if $Q \le \mathscr{L}^{A+B}$, and is trivial otherwise.

On the other hand, if $Q = 1$, we need to show that

$$\max_{T_0 \le T \le x} \Psi(T) \int_{\alpha - ix^3}^{\alpha + ix^3} \int_{T_0}^{T} \big| S_\xi(\tfrac{1}{2}+it,\chi_0) \big| dt \frac{|d\xi|}{|\xi|} \ll z\mathscr{L}^{-A}, \tag{3.26}$$

where $T_0 = \exp\left(\mathscr{L}^{1/3}\right)$ and $S_\xi(s,\chi)$ is as above. The portion of the integral in (3.26) for which $|t + \operatorname{Im}\xi| \ge \exp(\mathscr{L}^{1/4})$ can be estimated via Lemma 2 as before (we need the extra restriction because of the left-hand side inequality in (2.19)). Hence, to finish the proof, we need to bound the portion of the integral on the left side of (3.26) for which

$$|t + \tau| \le \exp\left(\mathscr{L}^{1/4}\right). \tag{3.27}$$

It does not exceed

$$\max_{T_0 \le T \le x} \Psi(T) \int_{T_0}^{T} \big| S_{\xi_0}(\tfrac{1}{2}+it,\chi_0) \big| \int_{(3.27)} \frac{d\tau}{|\tau|} dt, \tag{3.28}$$

where ξ_0 is a fixed number of the form $\alpha + i\tau_0$. The inner integral is $\ll \exp(-\mathscr{L}^{1/4})$ and Lemma 2 yields

$$\max_{T_0 \le T \le x} \Psi(T) \int_{T_0}^{T} \big| S_{\xi_0}(\tfrac{1}{2}+it,\chi_0) \big| dt \ll z\mathscr{L}^{B}.$$

Thus, (3.28) is $\ll z\mathscr{L}^{-A}$, which completes the proof of (3.26). □

4. Proof of the Theorem

4.1. **From** (3.2) **to** (1.10)**.** In this section we demonstrate that, modulo two mild arithmetic constraints, any arithmetic function $f(n)$ which satisfies (3.2) for all $A > 0$ and $Q \le Q_0$ also satisfies

$$\sum_{q \le Q_0} \max_{(a,q)=1} \max_{h \le z} \max_{x/2 \le y < x} |E_f(y,h;q,a)| \ll \frac{z}{(\log x)^A}, \tag{4.1}$$

where $E_f(y,h;q,a)$ is defined by (1.7). By the orthogonality of the Dirichlet characters modulo q, we have

$$E_f(y,h;q,a) = \frac{1}{\phi(q)} \sum_{\chi} \bar{\chi}(a) E_f(y,h;\chi),$$

whenever $(a,q)=1$; here $E_\lambda(y,h;\chi)$ is defined by (3.1). Hence, the left side of (4.1) does not exceed

$$\sum_{q\le Q_0}\frac{1}{\phi(q)}\sum_{\chi}\max_{y,h}|E_f(y,h;\chi)|. \tag{4.2}$$

For a non-principal character χ mod q, let χ^* denote the primitive character (say, mod g, $g|q$) inducing χ; for χ principal, we write $\chi^*=\chi_0$ and $g=1$. We then have

$$E_f(y,h;\chi)=E_f(y,h;\chi^*)+O(R_1(\chi)+R_2(\chi)),$$

where

$$R_1(\chi)=\sum_{\substack{y-h<n\le y\\(n,q)>1,(n,g)=1}}|f(n)|,\qquad R_2(\chi)=\delta(\chi)hh_0^{-1}\sum_{\substack{y-h_0<n\le y\\(n,q)>1}}|f(n)|.$$

Suppose that $|f(n)|\ll d(n)^B$ and that

$$f(n)=0 \text{ if } n \text{ has a prime divisor } \ p<D. \tag{4.3}$$

Using (4.3), we obtain

$$\begin{aligned}
&\sum_{q\le Q_0}\frac{1}{\phi(q)}\sum_{\chi}\max_{h,y}R_1(\chi)\\
&\ll\sum_{D\le d\le Q_0}\sum_{\substack{q\le Q_0\\q\equiv0\ (\mathrm{mod}\ d)}}\frac{1}{\phi(q)}\sum_{\substack{g|q\\(g,d)=1}}\sideset{}{^*}\sum_{\chi\bmod g}\max_{h,y}\sum_{y-h<md\le y}x^\eta\\
&\ll x^\eta\sum_{D\le d\le Q_0}\left(\frac{z}{d}+1\right)\sum_{q\le Q_0d^{-1}}\frac{1}{\phi(qd)}\sum_{\chi\bmod q}1\\
&\ll x^\eta\sum_{D\le d\le Q_0}\left(\frac{z}{d}+1\right)\frac{Q_0}{d\phi(d)}\ll zx^\eta Q_0D^{-2}+x^\eta Q_0D^{-1}.
\end{aligned}$$

Furthermore, a similar (and simpler) argument gives

$$\sum_{q\le Q_0}\frac{1}{\phi(q)}\sum_{\chi}\max_{h,y}R_2(\chi)\ll zx^\eta D^{-1}.$$

Thus, (4.2) is bounded above by

$$\sum_{q\le Q_0}\frac{1}{\phi(q)}\sum_{\chi}\max_{y,h}|E_f(y,h;\chi^*)|+zx^{-\eta},$$

provided that

$$D\ge x^\eta\quad\text{and}\quad Q_0\le\min\left(D^2x^{-\eta},Dzx^{-\eta}\right). \tag{4.4}$$

Using the elementary bound

$$\sum_{\substack{q\le Q\\q\equiv0\ (\mathrm{mod}\ d)}}\frac{1}{\phi(q)}\ll\frac{\mathscr{L}}{\phi(d)},$$

we conclude that if $f(n)$ is as above and if (4.4) holds, then the left side of (4.1) is

$$\ll\mathscr{L}\sum_{q\le Q_0}\frac{1}{\phi(q)}\sideset{}{^*}\sum_{\chi}\max_{h,y}|E_f(y,h;\chi)|+zx^{-\eta},$$

and therefore if (3.2) holds for any $A > 0$ and $Q \le Q_0$, we also obtain (4.1) for all $A > 0$.

4.2. **The contsruction of $\lambda(n)$.** We derive $\lambda(n)$ from $\psi(n, x^{1/2})$ using Buchstab's identity (3.21). Applying (3.21) several times, we shall write $\psi(n, x^{1/2})$ in the form

$$\psi(n, x^{1/2}) = \sum_{j=1}^{k} c_j(n) - \sum_{j=k+1}^{\ell} c_j(n), \tag{4.5}$$

with non-negative arithmetic functions $c_j(n)$. The actual decomposition will depend on a parameter $\theta \in (\frac{1}{2}+\varepsilon, \frac{7}{12}]$ and the functions $c_j(n)$ will possess the following properties:

1) $c_j(n) \ll d(n)^B$ $(1 \le j \le \ell)$;
2) $c_j(n) = 0$ if n has a prime divisor $p < x^{2\theta-1}$ $(1 \le j \le \ell)$;
3) given any $A > 0$ we have
$$\sum_{q \sim Q} \sideset{}{^*}\sum_{\chi} \max_{h \le z} \max_{y \sim x} |E_{c_j}(y, h; \chi)| \ll Qz\mathscr{L}^{-A},$$
for $Q \le zx^{-\theta}$ and $j = 1, \dots, r, k+1, \dots, \ell$, with $r < k$;
4) if $y \sim x$, $h_0 = x \exp\left(-3(\log x)^{1/3}\right)$, and $\theta \ge 0.53 - \delta$,
$$\sum_{y-h_0 < n \le y} \sum_{j=r+1}^{k} c_j(n) \le (\beta + o(1)) \frac{h_0}{\log x},$$
where $\beta < 1$ is an absolute constant.

We define $\lambda(n)$ by

$$\lambda(n) = \sum_{j=1}^{r} c_j(n) - \sum_{j=k+1}^{\ell} c_j(n). \tag{4.6}$$

Since $\psi(n, x^{1/2})$ vanishes on $[2, x^{1/2})$ and equals the characteristic function of the prime numbers on $[x^{1/2}, x)$, this function has property (i) from the statement of the Theorem. Also, by (4.5)

$$\sum_{y-h_0<n\le y} \lambda(n) = \sum_{y-h_0<n\le y} \left(\psi(n, x^{1/2}) - \sum_{j=r+1}^{k} c_j(n) \right),$$

so property (ii) follows from the Prime Number Theorem and 4) above, provided that $\theta \ge 0.53 - \delta$. Finally, the discussion in Section 4.1 and 1)–3) above imply (1.10) for

$$Q \le \min\left(x^{4\theta-2-\eta}, zx^{-\theta-\eta}\right). \tag{4.7}$$

Observe that, in view of the work by Timofeev [19] and Perelli, Pintz and Salerno [15] mentioned in the Introduction, we need to consider only the case $z \le x^{3/5+\eta}$. In that case, the right side of (4.7) equals $zx^{-\theta}$ for $\theta \ge 0.52 + \eta$, so property (iii) from the statement of the Theorem follows by taking $\theta = 0.53 - \delta$.

Thus, to complete the proof it remains to find an identity of the form (4.5) with functions $c_j(n)$ subject to 1)–4) above. We shall use a variant of the identity applied by Harman, Lewis and the author [6, Section 6] to study the distribution of prime ideals of imaginary quadratic fields in small regions.

Given an integer $m < x^{1/2}$, we write

$$w(m) = \min\left(x^{\gamma}, (x^{\theta} m^{-1})^{2/(2h-1)}\right),$$

where γ is defined by (2.7) with $g = 4$ (which is the optimal choice for g when $169/321 \le \theta \le 103/193$); we also set $w(m) = 0$ if $m \ge x^{1/2}$. We start the decomposition by applying (3.21) twice to obtain

$$\begin{aligned}\psi(n, x^{1/2}) &= \psi(n, w_0) - \sum_{\substack{n=mp\\ w_0 \le p < x^{1/2}}} \psi(m, w(p)) + \sum_{\substack{n=mp_1p_2\\ w(p_1)\le p_2 < p_1 < x^{1/2}}} \psi(m, p_2)\\ &= c_1(n) - c_{k+1}(n) + b_1(n) \quad \text{say;}\end{aligned}$$

here $w_0 = x^{2\theta-1}$. As the right side of (3.20) is $\ge x^{2\theta-1}$, Lemma 10 yields (3.2) for $c_1(n)$ and $c_{k+1}(n)$ as well as for the portion of $b_1(n)$ with $p_2 < w(p_1p_2)$, which we denote by $c_2(n)$.

We split the remainder of $b_1(n)$ into subsums $b_2(n), \dots, b_8(n)$ subject to the conditions

$b_2(n)$**:** (4.8), $p_1p_2^2 \ge x$;
$b_3(n)$**:** (4.8), $p_1p_2^2 < x$, $p_2 > x^{\theta/2}$;
$b_4(n)$**:** (4.8), $p_1 \le x^{1/4}$;
$b_5(n)$**:** (4.8), $x^{1/4} < p_1 \le x^{2/5}$, $p_1p_2^3 < x$;
$b_6(n)$**:** (4.8), $x^{2/5} < p_1 < x^{1/2}$, $p_1 > p_2^2$;
$b_7(n)$**:** (4.8), $p_2 \le x^{\theta/2}$, $p_1^2p_2 < x \le p_1p_2^3$;
$b_8(n)$**:** (4.8), $p_2 \le x^{\theta/2}$, $p_1p_2^2 < x \le p_1^2p_2$, $p_1 < p_2^2$;

where

$$w(p_1, p_2) := \max\left(w(p_1), w(p_1p_2)\right) \le p_2 < p_1 < x^{1/2}. \tag{4.8}$$

We set $c_{r+1}(n) = b_2(n)$ and $c_{r+2} = b_3(n)$. Since $b_2(n)$ and $b_3(n)$ counts almost-primes p_1p_2 and $p_1p_2p_3$, respectively, the Prime Number Theorem yields

$$\sum_{y-h_0<n\le y} b_2(n) \ll \frac{h_0}{\log^2 x} \tag{4.9}$$

and

$$\sum_{y-h_0<n\le y} b_3(n) = \frac{\beta_3 h_0}{\log x} + O\left(\frac{h_0}{\log^2 x}\right), \tag{4.10}$$

where

$$\beta_3 = \beta_3(\theta) = \int_{\theta/2}^{1-\theta} \int_{\theta/2}^{\min(u_1,(1-u_1)/2)} \frac{du_1 du_2}{u_1u_2(1-u_1-u_2)}.$$

We shall refer to β_3 as the *loss from* $b_3(n)$.

We can decompose two more times each of $b_4(n), \dots, b_8(n)$. We have

$$\begin{aligned}b_j(n) &= \sum_{n=p_1p_2m} \psi(m, w(p_1,p_2)) - \sum_{\substack{n=p_1p_2p_3m\\ w(p_1,p_2)\le p_3<p_2}} \psi(m, p_3)\\ &= b_{j,1}(n) - b_{j,2}(n) \quad \text{say,}\end{aligned}$$

where p_1 and p_2 are subject to the summation conditions in $b_j(n)$. Since $p_1 < x^{1/2}$ and $p_2 \le x^{\theta/2}$, we can apply Lemma 10 to $b_{j,1}(n)$ (with m and n corresponding to the summations over p_1 and p_2, respectively). Hence, we can include $b_{j,1}(n)$ among

$c_2(n), \dots, c_r(n)$. We decompose $b_{j,2}(n)$ further. Let $b_{j,3}(n)$ be the portion of $b_{j,2}(n)$ subject to

$$p_1p_3 < x^{1/2} \quad \text{or} \quad p_1^*p_2p_3 \le x^{(1+\theta)/2}, \tag{4.11}$$

where p_1^* is defined by (3.18), and let $b_{j,4}(n)$ be the remainder of $b_{j,2}(n)$. We write

$$\begin{aligned} b_{j,3}(n) &= \sum_{n=p_1p_2p_3m} \psi(m,w_0) - \sum_{\substack{n=p_1p_2p_3p_4m \\ w_0 \le p_4 < p_3}} \psi(m,p_4) \\ &= b_{j,5}(n) - b_{j,6}(n) \quad \text{say,} \end{aligned}$$

where p_1, p_2 and p_3 are subject to all the summation conditions in $b_{j,3}(n)$. In order to decompose $b_{j,4}(n)$ we reverse the roles of m and p_1. We have

$$\begin{aligned} b_{j,4}(n) &= \sum_{\substack{n=p_1p_2p_3m \\ p_1,p_2,p_3,m:(j,4)}} \psi(m,p_3) = \sum_{\substack{n=kp_2p_3m \\ k,p_2,p_3,m:(j,4)}} \psi(m,p_3)\psi(k,(x/p_2p_3m)^{1/2}) \\ &= \sum_{\substack{n=kp_2p_3m \\ k,p_2,p_3,m:(j,4)}} \psi(m,p_3)\psi(k,w_0) - \sum_{\substack{n=kp_2p_3p_4m \\ kp_4,p_2,p_3,m:(j,4) \\ w_0 \le p_4 < (x/p_2p_3m)^{1/2}}} \psi(m,p_3)\psi(k,p_4) \\ &= b_{j,7}(n) - b_{j,8}(n) \quad \text{say.} \end{aligned}$$

We include $b_{j,5}(n)$ and $b_{j,7}(n)$ among $c_{k+2}(n), \dots, c_\ell(n)$. We prove that they satisfy (3.2) by applying Lemma 10—in the application to $b_{j,5}(n)$ we have m, n corresponding to p_1p_3, p_2 or p_1, p_2p_3 depending on whether the first or second condition in (4.11) holds; in the application to $b_{j,7}(n)$ we have m, n corresponding to p_2m and p_3 (this choice is admissible since $p_1p_3 \ge x^{1/2} \Rightarrow p_2m < x^{1/2}$).

We now turn to $b_{j,6}(n)$ and $b_{j,8}(n)$. First of all, we can prove (3.2) for subsums of $b_{j,6}(n)$ and $b_{j,8}(n)$ in which we can group the variables $p_1, \dots, p_4, k, m$ to produce four new variables with sizes satisfying any of conditions (i)–(iv) of Lemma 3. Indeed, if $c_j(n)$ is such a subsum, we can show that it satisfies (3.2) by recalling (3.7)–(3.9) and estimating (3.9) using Lemma 3 instead of Lemma 7; the required bounds of the form (2.8) follow from Lemma 8 and unwanted interdependencies between the new variables can be removed via Perron's formula, as in the final stage of the proof of Lemma 10. We can therefore include such portions of $b_{j,6}(n)$ and $b_{j,8}(n)$ among $c_2(n), \dots, c_r(n)$. Let $b_{j,9}(n)$ be the remainder of $b_{j,6}(n)$ and $b_{j,8}(n)$. We split $b_{j,9}(n)$ into two—a part allowing two further decompositions along the same lines as $b_j(n)$ and the remainder. We include the latter among $c_{r+3}(n), \dots, c_k(n)$ and decompose the former. The decomposition yields more terms $c_j(n)$ that can be treated via Lemma 10 and a six-dimensional sum similar to $b_{j,6}(n)$ and $b_{j,8}(n)$. For a significant part of this six-dimensional sum we can prove (3.2) via Lemma 3 as discribed above; again, this subsum becomes one of $c_2(n), \dots, c_r(n)$. The remaining part of the six-dimensional sum is treated similarly to $b_{j,9}(n)$—a part of it goes to $c_{r+3}(n), \dots, c_k(n)$ and another part (possibly empty) can be decomposed two more times to yield sums accessible via Lemma 10 and an eight-dimensional sum, which (if it appears) we include among $c_{r+3}(n), \dots, c_k(n)$.

The final step of the proof is to show that after implementing the above program we are left with coefficients $c_{r+1}(n), \dots, c_k(n)$ satisfying 4) above. We have

$$\sum_{y-h_0<n\le y} c_j(n) = \frac{\delta_j h_0}{\log x} + O\left(\frac{h_0}{\log^2 x}\right), \tag{4.12}$$

where δ_j is a constant. The proofs of (4.12) for different values of j rely on the repeated use of the Prime Number Theorem, partial summation and the approximate formula

$$\sum_{u_1 < n \le u} \psi(n, w) = \frac{u - u_1}{\log w} \omega \left(\frac{\log u}{\log w} \right) + O\left(\frac{u - u_1}{\log^2 u} + u e^{-\sqrt{\log u}} \right),$$

where $u/2 \le u_1 < u$, $u^{\varepsilon} \le w \le u^{1-\varepsilon}$ and $\omega(t)$ is Buchstab's function defined as the continuous solution of the differential delay equation

$$\begin{cases} w(t) = 1/t & \text{if } 1 \le t \le 2, \\ (tw(t))' = w(t-1) & \text{if } t > 2. \end{cases}$$

The constants δ_j in (4.12) are expressed as multidimensional integrals involving $\omega(t)$ and similar in appearance to β_3 above (in fact, by (4.10), $\beta_3 = \delta_{r+2}$). In general, these integrals are very difficult or even impossible to calculate exactly, but we are content with precise enough upper bounds, which we can obtain using numerical integration. As the arising integrals are exactly the same as those appearing in the numeric part of a recent work by Harman, Lewis and the author [6], we can quote the bounds obtained in that paper. On writing β_i for the loss from all $c_j(n)$, $r < j \le k$, descending from $b_i(n)$, we have the following upper bounds for the losses from $b_2(n), \dots, b_8(n)$ when $\theta = 0.53$.

j	2	3	4	5	6	7	8
$\beta_j \le$	0	0.20	0.07	0.19	0.25	0.12	0.12

As the bounds are continuous in θ, it follows from here that $\sum \beta_j < 1$ for $\theta \ge 0.53 - \delta$, where δ is a positive absolute constant. The value $\delta = 0.002$ quoted in the Remark following the statement of the Theorem is the consequence of the more precise estimation of those integrals achieved by Lewis [12], where he showed that, in fact, $\sum \beta_j < 1$ for $\theta \ge 0.528$. □

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References

[1] R. C. Baker and G. Harman, *The difference between consecutive primes*, Proc. London Math. Soc. (3) **72** (1996), 261–280.

[2] R. C. Baker, G. Harman, and J. Pintz, *The exceptional set for Goldbach's problem in short intervals*, Sieve Methods, Exponential Sums and their Applications in Number Theory, Cambridge University Press, 1997, pp. 1–54.

[3] ———, *The difference between consecutive primes II*, Proc. London Math. Soc. (3) **83** (2001), 532–562.

[4] A. Fujii, P. X. Gallagher, and H. L. Montgomery, *Some hybrid bounds for character sums and Dirichlet L-functions*, Topics in Number Theory, Colloq. Math. Soc. Janos Bolyai, vol. 13, North–Holland, 1976, pp. 41–57.

[5] H. Halberstam and H.-E. Richert, *Sieve Methods*, Academic Press, London, 1974.

[6] G. Harman, A. V. Kumchev, and P. A. Lewis, *The distribution of prime ideals of imaginary quadratic fields*, Trans. Amer. Math. Soc. **356** (2004), 599–620.

[7] D. R. Heath-Brown, *The number of primes in a short interval*, J. Reine Angew. Math. **389** (1988), 22–63.
[8] G. Hoheisel, *Primzahlprobleme in der Analysis*, Sitz. Preuss. Akad. Wiss. **2** (1930), 1–13.
[9] M. N. Huxley, *On the difference between consecutive primes*, Invent. Math. **15** (1972), 164–170.
[10] M. N. Huxley and H. Iwaniec, *Bombieri's theorem in short intervals*, Mathematika **22** (1975), 188–194.
[11] M. Jutila, *A statistical density theorem for L-functions with applications*, Acta Arith. **16** (1969), 207–216.
[12] P. A. Lewis, *Finding information about Gaussian primes using sieves*, Ph.D. thesis, Cardiff University, 2001.
[13] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Springer-Verlag, 1971.
[14] Y. Motohashi, *On a mean value theorem for the remainder term in the prime number theorem for short arithmetic progressions*, Proc. Japan Acad. Ser. A Math. Sci. **47** (1971), 653–657.
[15] A. Perelli, J. Pintz, and S. Salerno, *Bombieri's theorem in short intervals*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **11** (1984), 529–539.
[16] ———, *Bombieri's theorem in short intervals. II*, Invent. Math. **79** (1985), 1–9.
[17] H. Rademacher, *On the Phragmen–Lindelöf theorem and some applications*, Math. Z. **72** (1959/60), 192–204.
[18] S. J. Ricci, *Mean-value theorems for primes in short intervals*, Proc. London Math. Soc. (3) **37** (1978), 230–242.
[19] N. M. Timofeev, *Distribution of arithmetic functions in short intervals in the mean with respect to progressions*, Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), 341–362, in Russian.
[20] N. Watt, *Kloosterman sums and a mean value for Dirichlet polynomials*, J. Number Theory **53** (1995), 179–210.

Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 3G3
E-mail address: kumchev@math.toronto.edu