ON A BINARY DIOPHANTINE INEQUALITY INVOLVING PRIME POWERS

A. KUMCHEV AND M. B. S. LAPORTA

1. Introduction

Given $c > 1$, let $H(c)$ denote the least number s such that, for every fixed $\varepsilon > 0$ and for all real $N \geq N_0(\varepsilon)$, the inequality

$$\left|p_1^c + \ldots + p_s^c - N\right| < \varepsilon \ , \tag{1.1}$$

has solutions in prime numbers $p_1, \ldots, p_s$. In 1952 Piatetski-Shapiro [12] established the existence of $H(c)$ for every (non-integer) $c > 1$ and proved that

$$\limsup_{c \to +\infty} \frac{H(c)}{c \log c} \leq 4 \ .$$

For c close to 1, the celebrated three primes theorem by Vinogradov [15] suggests that one should expect $H(c) \leq 3$. While in [12] the upper bound $H(c) \leq 5$ is provided for $1 < c < 3/2$, in 1992 Tolev [14] showed that (1.1) is solvable for $s = 3$ if $1 < c < 15/14$. (He also showed that one may let ε be a function of N which tends to zero as N tends to infinity.) Subsequently, several authors sharpened Tolev's result improving on the range for c (see [2, 7, 8]). The most recent improvement is due to the first author [7]; he used Harman's sieve [3, 4] to show that Tolev's theorem holds for $1 < c < 61/55$.

Our purpose in this paper is to provide an improvement on a recent result [9] of the second author regarding the solvability of inequality (1.1) for $s = 2$. In [9] it is proved that, for *almost all* $y \in [N, 2N)$ (i.e., the Lebesgue measure of the set of the exceptions is $o(N)$), the following inequality has solutions in primes p_1, p_2:

$$\left|p_1^c + p_2^c - y\right| < \varepsilon \tag{1.2}$$

where $\varepsilon = N^{1-15/(14c)} \log^C N$ with $1 < c < 15/14$ and for some positive constant C (and, in fact, a more careful examination of the proof of [9, Theorem 2] reveals that one can obtain the same conclusion with 9/8 in place of 15/14 after a little bit of extra work). Here we use the method of

Date: Draft from September 11, 2000.

[7] in order to show that Laporta's result holds for $1 < c < 6/5$ if one is content with an upper bound for the measure of the exceptional set which is slightly weaker than the one in [9]. More precisely, we prove

Theorem 1. *Let c be fixed with $1 < c < 6/5$ and $\delta > 0$ be a fixed number sufficiently small in terms of c. Let also N be a sufficiently large real number, let $\varepsilon \geq N^{1-6/(5c)+\delta}$, and let g be a function satisfying*

$$\lim_{N\to\infty} g(N) = \infty.$$

There exists a measurable set $\mathcal{H}$ having Lebesgue measure

$$|\mathcal{H}| \ll g(N)^2 N (\log N)^{-1/2}$$

such that for each $y \in [N, 2N) \setminus \mathcal{H}$, the number $R(y)$ of the solutions of (1.2) *satisfies*

$$R(y) \gg \frac{\varepsilon N^{2/c-1}}{\log^2 N}. \tag{1.3}$$

The implied constants may depend only on c and δ.

In [7, Theorem 2] it is also shown that one should expect $H(c) \leq 3$ at least for $1 < c < 3/2$. We apply a similar probabilistic argument to prove the following result for the binary case.

Theorem 2. *For almost all (in the sense of Lebesgue measure) $c \in (1, 2)$, one can find an N_0 such that if $N \geq N_0$ there is a set $\mathcal{H}$ having Lebesgue measure $|\mathcal{H}| \ll N(\log N)^{-1/2}$ and such that, for each $y \in [N, 2N) \setminus \mathcal{H}$ and each $\varepsilon \geq N^{1-2/c}(\log N)^{10}$, the number $R(y, c)$ of the solutions of* (1.2) *satisfies* (1.3).

The Piatetski-Shapiro inequality (1.1) is considered as a variant of the Waring-Goldbach problem which asks for the possibility of expressing natural numbers as sums of kth powers of primes, with a bounded number of summands. It is worthwhile to recall that, for $k = 2$, every sufficiently large natural number (which has to satisfy some natural congruence conditions) is the sum of at most five prime squares (see [5]). Moreover, the problem can be solved with a smaller number of summands by showing that three prime squares are sufficient to represent almost all sufficiently large natural numbers. Then our Theorem 2 provides a result (at least for c close to 2 from below) which is somehow better than the one we would expect from the current knowledge concerning the Goldbach-Waring problem for squares. This invites to extend investigations on inequality (1.1) to the case when c is close to any positive integer. The authors hope to present an account of the developments arising from the latter remark in a future paper.

Notation. Throughout the paper p, q, r, with or without subscripts, always denote primes; d, k, ℓ, m, n denote integers. We choose $X = \frac{1}{2}N^{1/c}$; $\eta = \delta^2$ where δ is the number from the statement of Theorem 1. We write $m \sim M$ if m runs through the interval $[M, 2M)$. Also, we will use a kernel K having the following properties:

1. There is a constant $C(\eta) > 0$, depending only on η, such that
$$|K(x)| \le C(\eta) \min\left(\varepsilon, |x|^{-1}|\varepsilon x|^{-10/\eta}\right).$$
2. Both K and its Fourier transform
$$\widehat{K}(y) = \int_{\mathbb{R}} K(x)e(-xy)\,dx$$
are non-negative; henceforth, $e(x) = e^{2\pi ix}$.
3. If χ is the characteristic function of the interval $(-\varepsilon, \varepsilon)$, then, for all real x, one has
$$\tfrac{1}{3}\chi(4x) \le \widehat{K}(x) \le \chi(x).$$

One can construct such a function by dilating the kernel described in [1, Lemma 1]. Finally, throughout the paper, we say that an assertion holds for 'almost all' $y \in [N, 2N)$, if given a sufficiently large real N one can find a measurable set $\mathcal{H}$ with Lebesgue measure $|\mathcal{H}| \ll g(N)^2 N(\log N)^{-1/2}$ and such that the assertion is true for all $y \in [N, 2N) \setminus \mathcal{H}$.

2. The Sieve Method

Writing
$$P(z) = \prod_{p<z} p,$$
for any sequence of integers $\mathcal{E}$ weighted by the numbers $w(n)$, $n \in \mathcal{E}$, we set
$$S(\mathcal{E}, z) = \sum_{\substack{n\in\mathcal{E}\\ (n,P(z))=1}} w(n),$$
and denote by $\mathcal{E}_d$ the subsequence of elements $n \in \mathcal{E}$ with $n \equiv 0 \pmod{\mathrm{d}}$. For every $y \in [N, 2N)$, we define $\mathcal{A} = \mathcal{A}(y)$ to be the sequence of integers $n \in [X, 2X)$ weighted by
$$w(n) = w(n, y) = \sum_{p\sim X} \widehat{K}(p^c + n^c - y).$$
Since
$$R(y) \ge \sum_{p_1,p_2\sim X} \widehat{K}(p_1^c + p_2^c - y) = S(\mathcal{A}, (2X)^{1/2}),$$

in order to prove Theorem 1, it suffices to show that the estimate

$$S(\mathcal{A}, (2X)^{1/2}) \gg \frac{\varepsilon X^{2-c}}{\log^2 X} \tag{2.1}$$

holds for almost all $y \in [N, 2N)$.

In sieve theory, a common technique for obtaining sharp lower bounds is to combine upper bounds and (not as sharp) known lower bounds by means of combinatorial identities like *Buchstab's*

$$S(\mathcal{E}, z_1) = S(\mathcal{E}, z_2) - \sum_{z_2 \le p < z_1} S(\mathcal{E}_p, p). \tag{2.2}$$

Our proof of (2.1) uses a version of this idea developed by Harman [3, 4]. Let $\mathcal{B}$ be the set of integers in $[X, 2X)$. We use arithmetic information in the form of asymptotic formulas

$$\sum_{m \sim M} a(m)\, S(\mathcal{A}_m, z(m)) = \lambda \sum_{m \sim M} a(m)\, S(\mathcal{B}_m, z(m)) + \text{ error terms,} \tag{2.3}$$

where λ is suitably chosen. We expect the error terms here to be always 'small', but of course, we can prove this only for certain values of M and $z(m)$. We apply (2.2) repeatedly to express the left-hand side of (2.1) as a linear combination of sifting functions. This decomposition of $S(\mathcal{A}, (2X)^{1/2})$ is done so that we have asymptotic formulas for most of the arising sifting functions and suitable upper and lower bounds for the few sifting functions for which we have no asymptotic formula. Combining the estimates for all the terms in the decomposition, we obtain the lower bound.

Throughout the rest of this section we set up the decomposition. We set $A = X^{8/75}$, $B = X^{1/5}$, $C = X^{11/41}$, $D = X^{1/3}$, and $F = X^{11/25}$. Applying (2.2), we find

$$\begin{aligned} S(\mathcal{A}, (2X)^{1/2}) &= S(\mathcal{A}, A) - \sum_{A \le p < B} S(\mathcal{A}_p, p) \\ &\quad - \sum_{B \le p \le C} S(\mathcal{A}_p, p) - \sum_{C < p < D} S(\mathcal{A}_p, p) \\ &\quad - \sum_{D \le p \le F} S(\mathcal{A}_p, p) - \sum_{F < p < \sqrt{2X}} S(\mathcal{A}_p, p) \\ &= S_1 - S_2 - S_3 - S_4 - S_5 - S_6, \qquad \text{say.} \end{aligned} \tag{2.4}$$

We shall give further decomposition for S_2 and S_4. Another application of (2.2) gives

$$\begin{aligned} S_2 &= \sum_{A\le p<B} S(\mathcal{A}_p, A) - \sum_{\substack{A\le q<p<B\\ pq\le C}} S(\mathcal{A}_{pq}, q) \\ &\quad - \sum_{\substack{A\le q<p<B\\ C<pq<D}} S(\mathcal{A}_{pq}, q) - \sum_{\substack{A\le q<p<B\\ pq\ge D}} S(\mathcal{A}_{pq}, q) \\ &= S_7 - S_8 - S_9 - S_{10}, \quad \text{say.} \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} S_4 &= \sum_{C<p<D} S(\mathcal{A}_p, A) - \sum_{\substack{C<p<D\\ A\le q<B}} S(\mathcal{A}_{pq}, q) \\ &\quad - \sum_{\substack{C<p<D\\ B\le q\le C}} S(\mathcal{A}_{pq}, q) - \sum_{C<q<p<D} S(\mathcal{A}_{pq}, q) \\ &= S_{11} - S_{12} - S_{13} - S_{14}, \quad \text{say.} \end{aligned}$$

Consider now S_9. Let $S_9^{\dagger}$ and $S_9^{\ddagger}$ be the subsums of S_9 in which $pq^2 \le F$ and $pq^2 > F$, respectively. Using (2.2) twice more, we decompose $S_9^{\dagger}$ further:

$$\begin{aligned} S_9^{\dagger} &= \sum_{p,q:(\dagger)} S(\mathcal{A}_{pq}, A) - \sum_{\substack{p,q:(\dagger)\\ A\le r<q}} S(\mathcal{A}_{pqr}, A) \\ &\quad + \sum_{\substack{p,q:(\dagger)\\ A\le s<r<q}} S(\mathcal{A}_{pqrs}, s) \\ &= S_{15} - S_{16} + S_{17}, \quad \text{say;} \end{aligned}$$

here r and s are primes and $(\dagger)$ stands for the summation conditions in $S_9^{\dagger}$. Finally, we deal with S_6. It counts almost primes having two prime factors,

$$S_6 = \sum_{\substack{pq\sim X\\ F<p\le q}} w(pq).$$

It turns out to be more convenient to switch the sifting process from the product pq to the prime variable appearing in the definition of $w(n)$. In order to do so, we write S_6 as $S(\mathcal{A}^*, (2X)^{1/2})$ where $\mathcal{A}^* = \mathcal{A}^*(y)$ is the set of integers in $[X, 2X)$ weighted by

$$w^*(n) = \sum_{\substack{pq\sim X\\ F<p\le q}} \widehat{K}(n^c + (pq)^c - y).$$

Let S_i^* denote a sum similar to S_i in which $\mathcal{A}$ has been replaced by $\mathcal{A}^*$. We decompose $S(\mathcal{A}^*, (2X)^{1/2})$ via (2.4) and then deal with S_2^* exactly as we did with S_2, but we do not decompose S_4^* and S_6^* further. Combining all the decompositions above, we obtain the identity

$$\begin{aligned} S(\mathcal{A}, (2X)^{1/2}) = {} & S_1 - S_3 - S_5 - S_7 + S_8 + S_9^{\ddagger} + S_{10} \\ & - S_{11} + S_{12} + S_{13} + S_{14} + S_{15} - S_{16} \\ & + S_{17} - S_1^* + S_3^* + S_4^* + S_5^* + S_6^* + S_7^* \\ & - S_8^* - S_9^{\ddagger *} - S_{10}^* - S_{15}^* + S_{16}^* - S_{17}^*. \end{aligned} \tag{2.5}$$

In Section 4, we will establish asymptotic formulas of the form (2.3) for almost all $y \in [N, 2N)$ (cf. Lemmas 7 and 8). Those results will allow us to evaluate the terms on the right of (2.5). The most difficult part of the proofs of the asymptotic formulas is the estimation of certain double exponential sums; these are dealt with in the next section. In Section 5, we combine (2.5) with Lemmas 7 and 8, and after some numerical work, we are able to show that the lower bound resulting from the above procedure is non-trivial.

3. Exponential Sums

In this section we gather some estimates for exponential sums of the form

$$U(x) = \sum_{m \sim M} \sum_{\ell \sim L} a(m) b(\ell) e(x(m\ell)^c) \tag{3.1}$$

where $a(m)$, $b(\ell)$ are complex numbers of modulus ≤ 1 and $ML \asymp X$.

Lemma 1. *Let $1 < c < 6/5$ and let $X^{1-c-\eta} \le |x| \le X^{6/5-c+\eta}$. Let also $U(x)$ be defined by* (3.1). *We then have*

$$|U(x)| \ll X^{9/10+\eta}, \tag{3.2}$$

whenever

$$X^{1/5} \ll L \ll X^{11/41}, \tag{3.3}$$

or

$$X^{1/3} \ll L \ll X^{11/25}. \tag{3.4}$$

Proof. Inequalities (3.4) and the restriction $c < 6/5$ are the assumptions under which [13, Theorem 9] provides the upper bound (3.2). The proof of

(3.3) is easier and, in fact, repeats that of [7, Lemma 6]. By the Weyl–van der Corput inequality,

$$|U|^2 \ll \frac{X^2}{Q} + \frac{X}{Q} \sum_{q \le Q} \sum_{\ell \sim L} \left| \sum_{m \sim M} e(f(m)) \right|,$$

where $f(m) = x((\ell+q)^c - \ell^c)m^c$ and $Q \le L$ is a parameter at our disposal. We choose $Q = X^{1/5-2\eta}$. Then, by using an exponent pair (κ, λ) and the hypotheses about x, we get

$$|U|^2 \ll X^{9/5+2\eta} + X^{1+\lambda+2\kappa/5} L^{1-\lambda}.$$

Thus, the lemma follows provided that

$$L \ll X^{1-(2\kappa+1)/5(1-\lambda)}.$$

Using $(\kappa, \lambda) = BABABA^3B(0,1) = (\frac{13}{49}, \frac{57}{98})$, we can now infer (3.2) from (3.3). ∎

Lemma 2. *Let $1 < c < 6/5$ and let $X^{1-c-\eta} \le |x| \le X^{6/5-c+\eta}$. Let also $U(x)$ be defined by (3.1) with $b(\ell) = 1$ for all ℓ, $\ell \sim L$. We then have*

$$|U(x)| \ll X^{9/10+\eta}, \tag{3.5}$$

whenever

$$L \gg X^{1/2}. \tag{3.6}$$

Proof. For $L \ge X^{3/5}$, the desired bound follows by applying the exponent pair $(\frac{1}{6}, \frac{2}{3})$ to the summation over ℓ and then summing the resulting estimate over m. We now treat the case $X^{1/2} \le L \le X^{3/5}$. We first apply Cauchy's inequality and Weyl's lemma with $Q = X^{1/5}$ to obtain

$$|U|^2 \ll \frac{X}{Q} \sum_{q \le Q} \sum_{m \sim M} \sum_{\ell} e(x((\ell+q)^c - \ell^c)m^c) + X^{9/5}, \tag{3.7}$$

where ℓ runs through a subinterval of $[L, 2L)$. Denote the sum over (m, ℓ) by $U_1(q)$. Applying the truncated Poisson formula and partial summation to the variables ℓ and m successively we find

$$|U_1(q)| \ll XF^{-1} \left| \sum_{\mu,\nu} e(f(\mu,\nu)) \right| + E,$$

where for $F = |x|qX^cL^{-1}$, $\mu \cong FM^{-1}$, $\nu \cong FL^{-1}$,

$$E = X^\eta \left(XF^{-1/2} + M + F^{1/2} \right),$$

and $f(\mu,\nu)$ is a C^∞ function satisfying (here $B_{i,j}$ is a constant depending on i and j)

$$\frac{\partial^{i+j}}{\partial\mu^i\partial\nu^j} f(\mu,\nu) = B_{i,j}(xq)^{1/(2-2c)}\nu^{1/2-j}\mu^{c/(2c-2)-i}\left(1+O\left(\frac{q}{L}\right)\right)$$
$$\asymp F\mu^{-i}\nu^{-j}.$$

Substituting the estimate for $U_1(q)$ in (3.7), we get

$$|U|^2 \ll \frac{X^2}{Q}\sum_{q\le Q} F^{-1}\left|\sum_{\mu,\nu} e(f(\mu,\nu))\right| + X^{9/5}. \tag{3.8}$$

Estimating the sum over (μ,ν) in (3.8) via Kolesnik's AB-theorem [10, Lemma 9], we complete the proof of the lemma. ■

Lemma 3. *Let $1 < c < 3/2$ and let $X^{3/2-2c-2\eta} \le |x| \le X^{1-c-\eta}$. Let also $U(x)$ be defined by* (3.1). *Then,*

$$|U(x)| \ll X^{1-\eta/3}, \tag{3.9}$$

whenever

$$X^\eta \ll L \ll X^{1/2}. \tag{3.10}$$

Furthermore, if $b(\ell) = 1$ for all ℓ, $\ell \sim L$, (3.9) *holds for*

$$L \gg X^\eta. \tag{3.11}$$

Proof. We first consider (3.9) under the assumption (3.10). By Cauchy's inequality and a Weyl shift,

$$|U(x)|^2 \ll \frac{X^2}{Q} + \frac{X}{Q}\sum_{q\le Q}\sum_{\ell\sim L}\left|\sum_{m\sim M} e(f(m))\right|,$$

where $f(m) = x((\ell+q)^c - \ell^c)m^c$ and $Q \le L$ is a parameter at our disposal. We now estimate the sum over m via the Kuz'min–Landau inequality and obtain

$$|U(x)|^2 \ll X^2Q^{-1} + |x|^{-1}X^{2-c}Q^{-1}L\log X. \tag{3.12}$$

This is justified, provided that $|f'(m)| \le 1/2$. To complete the proof of (3.9) we choose

$$Q = \begin{cases} X^{2\eta/3} & \text{if } |x| \ge X^{1-c-2\eta}, \\ \min\left(L, |x|^{-1}X^{1-c-\eta}\right) & \text{if } |x| < X^{1-c-2\eta}. \end{cases}$$

Note that in both cases

$$|f'(m)| \asymp |x|qX^{c-1} \ll X^{-\eta/3} = o(1),$$

and the expression

$$X^{2-2\eta/3} + X^{1/2+c+3\eta}$$

is an upper bound for the right side of (3.12).

We now consider the case when all the coefficients $b(\ell)$ are equal to one. For $L \le X^{1-\eta}$ we can apply the first part of the lemma (with M and L interchanged if $L \ge X^{1/2}$). If $L \ge X^{1-\eta}$, we can prove (3.9) by applying the exponent pair $(\frac{1}{2}, \frac{1}{2})$ to the summation over ℓ. This gives

$$\begin{aligned} |U(x)| &\ll M\left((|x|X^{c-1}M)^{1/2}L^{1/2} + (|x|X^{c-1}M)^{-1}\right) \\ &\ll X^{1/2+\eta} + X^{c-1/2+2\eta}. \end{aligned}$$

This completes the proof of (3.9). ∎

4. Mean Square Estimates

We start this section by introducing some further notation. We will use the following exponential sums and integrals

$$S(x) = \sum_{p\sim X} e(xp^c), \quad S_1(x) = \sum_{\substack{pq\sim X \\ X^{0.44}<p\le q}} e(x(pq)^c),$$

$$I_0(x) = \int_X^{2X} \frac{e(xt^c)}{\log t} dt, \quad I_1(x) = \sum_{X^{0.44}<p<\sqrt{2X}} \frac{1}{p}\int_X^{2X} \frac{e(xt^c)}{\log(t/p)}\, dt.$$

Also, we set $\tau = X^{1-c-\eta}$ and define

$$W_j(n,y) = \int_{-\tau}^{\tau} I_j(x)K(x)e((n^c-y)x)dx \qquad (j=0,1).$$

Finally, X^σ denotes a function of the form $e^{a(\log X)^{1/4}}$ with some unspecified constant $a>0$; in particular, we may write $X^{-\sigma}(\log X)^A \ll X^{-\sigma}$ instead of

$$e^{-a(\log X)^{1/4}}(\log X)^A \ll e^{-\frac{a}{2}(\log X)^{1/4}}.$$

Lemma 4. *Let $1<c<6/5$. Then,*

$$\int_N^{2N} \Big| \sum_{m\sim M}\sum_{\ell\sim L} a(m)b(\ell)\big(w(m\ell,y) - W_0(m\ell,y)\big)\Big|^2 dy \ll \varepsilon^2 X^{4-c-\sigma}, \tag{4.1}$$

whenever L satisfies the inequalities (3.3) or (3.4). Moreover, the same inequality holds when $b(\ell)=1$ for all ℓ and L satisfies (3.6).

Proof. Let $U(x)$ be given by (3.1). By the Fourier inversion formula,

$$\sum_{m\sim M}\sum_{\ell\sim L} a(m)b(\ell)w(m\ell,y) = \int_{\mathbb{R}} S(x)U(x)K(x)e(-yx)dx.$$

Denote the last integral by $D(X,y)$. It suffices to establish the estimate

$$\int_N^{2N}\left|D(X,y)-\int_{-\tau}^{\tau} I_0(x)U(x)K(x)e(-yx)dx\right|^2 dy \ll \varepsilon^2 X^{4-c-\sigma}. \tag{4.2}$$

We partition the line into three subsets:

$$E_1=(-\tau,\tau), \quad E_2=\{x:\ \tau\le|x|\le H\}, \quad E_3=\mathbb{R}\setminus(E_1\cup E_2),$$

where $H=X^{6/5-c+\eta}$, and write

$$D_i(X,y)=\int_{E_i} S(x)U(x)K(x)e(-yx)dx \qquad (i=1,2,3).$$

In view of the rapid decay of K, we readily have

$$\int_N^{2N} |D_3(X,y)|^2 dy \ll 1. \tag{4.3}$$

We now show that

$$\int_N^{2N}\left|D_1(X,y)-\int_{E_1} I_0(x)U(x)K(x)e(-yx)dx\right|^2 dy \ll \varepsilon^2 X^{4-c-\sigma}, \tag{4.4}$$

$$\int_N^{2N} |D_2(X,y)|^2 dy \ll \varepsilon^2 X^{4-c-\eta}. \tag{4.5}$$

The first step towards both of these is based upon Plancherel's theorem. For any function $F\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, one has

$$\begin{aligned}&\int_N^{2N}\left|\int_{\mathbb{R}} F(x)e(-yx)dx\right|^2 dy\\ &\le \int_{\mathbb{R}}\left|\int_{\mathbb{R}} F(x)e(-yx)dx\right|^2 dy = \int_{\mathbb{R}}\big|F(x)\big|^2 dx.\end{aligned} \tag{4.6}$$

Let $\mathbf{1}_{\mathfrak{B}}$ denote the indicator function of a measurable set $\mathfrak{B}$. Applying (4.6) to the function $F=(S-I_0)UK\mathbf{1}_{E_1}$, we find that the left side of (4.4) is

$$\begin{aligned}&\le \int_{E_1} |(S(x)-I_0(x))U(x)K(x)|^2\,dx\\ &\ll \varepsilon^2 \sup_{x\in E_1}|S(x)-I_0(x)|^2\int_{E_1}|U(x)|^2dx.\end{aligned}$$

Thus, (4.4) follows from the estimates

$$\sup_{x\in E_1} |S(x)-I_0(x)| \ll X^{1-\sigma}, \qquad \int_{E_1} |U(x)|^2 dx \ll X^{2-c}(\log X)^4. \tag{4.7}$$

The latter can be proven similarly to the bounds in [14, Lemma 7]. To prove the first inequality in (4.7) we appeal to partial summation and the asymptotic relation

$$\sum_{X\le p<Y} (\log p)e(xp^c) = \int_X^Y e(xt^c)dt + O(X^{1-\sigma}), \tag{4.8}$$

valid for $|x| \le \tau$ and $X < Y \le 2X$. This can be established similarly to [14, Lemma 14].

Finally, we prove (4.5). If $F = SUK\mathbf{1}_{E_2}$, we deduce from (4.6) that

$$\int_N^{2N} \big|D_2(X,y)\big|^2 dy \ll \varepsilon^2 \max_{x\in E_2} |U(x)|^2 \sum_{0\le h\le H} \int_h^{h+1} |S(x)|^2 dx.$$

By [14, Lemma 7], $\int_h^{h+1} |S(x)|^2 dx \ll X$, so (4.5) follows from the upper bound

$$\max_{x\in E_2} |U(x)| \ll X^{9/10+\eta}$$

upon choosing η sufficiently small in terms of δ. Hence, we can complete the proof of the lemma by referring to (3.2), if L satisfies (3.3) or (3.4), or to (3.5), if L satisfies (3.6). ■

Lemma 5. *Let $1 < c < 6/5$. Then,*

(4.9)
$$\int_N^{2N} \bigg| \sum_{m\sim M} \sum_{\ell\sim L} a(m)b(\ell)\big(w^*(m\ell,y) - W_1(m\ell,y)\big)\bigg|^2 dy \ll \varepsilon^2 X^{4-c-\sigma},$$

whenever L satisfies the inequalities (3.3) *or* (3.4). *Moreover, the same formula holds when $b(\ell) = 1$ for all ℓ and L satisfies* (3.6).

Proof. This is similar to the proof of Lemma 4. One wants to prove an analogue of (4.2) with $S_1(x)$ in place of $S(x)$ and $I_1(x)$ in place of $I_0(x)$. The corresponding analogues of (4.3) and (4.5) can be established exactly as in the previous proof, but we need to approach the inequality

$$\int_N^{2N} \bigg| \int_{E_1} (S_1(x) - I_1(x))U(x)K(x)e(-xy)dx \bigg|^2 dy \ll \varepsilon^2 X^{4-c-\sigma}, \tag{4.10}$$

slightly differently. For, using (4.8) with X/p in place of X and xp^c in place of x, we now have the approximation

$$S_1(x) = I_1(x) + O(X^{1-\sigma}) \tag{4.11}$$

only for $|x| \le \tau_1 := X^{3/2-2c-2\eta}$. Let

$$E_{1,1} = (-\tau_1, \tau_1) \quad \text{and} \quad E_{1,2} = E_1 \setminus E_{1,1}.$$

Using (4.11), we can show similarly to (4.4) that

$$\int_N^{2N} \left| \int_{E_{1,1}} (S_1(x) - I_1(x))U(x)K(x)e(-xy)dx \right|^2 dy \ll \varepsilon^2 X^{4-c-\sigma}.$$

To finish the proof we show that

$$\int_N^{2N} \left| \int_{E_{1,2}} (S_1(x) - I_1(x))U(x)K(x)e(-xy)dx \right|^2 dy \ll \varepsilon^2 X^{4-c-\eta/2}.$$

This inequality is similar to (4.5). Indeed, by virtue of the estimates (which can be established similarly to those in [14, Lemma 7])

$$\int_{E_1} |S_1(x)|^2 dx \ll X^{2-c}(\log X)^2 \quad \text{and} \quad \int_{E_1} |I_1(x)|^2 dx \ll X^{2-c}(\log X)^2,$$

one can adapt the proof of (4.5) so that it suffices to have the bound

$$\max_{x \in E_{1,2}} |U(x)| \ll X^{1-\eta/3},$$

which, under the given hypotheses, is provided by Lemma 3. ■

Lemma 6. *Let $1 < c < 6/5$, $ML \asymp X$, and L satisfies one of the inequalities (3.3) or (3.4). Let I, J be integers and $\mathfrak{I}_i$, $\mathfrak{J}_j$ be intervals for $1 \le i \le I$, $1 \le j \le J$. Write*

$$a(m, \ell) = \sum_{\substack{rp_1 \cdots p_I = \ell \\ p_1 < p_2 < \cdots < p_I \\ p_i \in \mathfrak{I}_i}} c(\ell) \sum_{\substack{lq_1 \cdots q_J = m \\ q_1 < q_2 < \cdots < q_J \\ q_j \in \mathfrak{J}_j}} d(m)$$

with $|c(\ell)|$, $|d(m)| \le 1$ and $p_1, \dots, p_I$ and $q_1, \dots, q_J$ satisfying $O(1)$ joint conditions of the form

$$p_u \le q_v \quad \text{or} \quad q_v \le p_u$$

or

$$\prod_{u \in \mathcal{U}} p_u \prod_{v \in \mathcal{V}} q_v \le Q \quad \text{or} \quad \prod_{u \in \mathcal{U}} p_u \ge \prod_{v \in \mathcal{V}} q_v$$

or similar (for given $\mathcal{U} \subset \{1, \dots, I\}$*,* $\mathcal{V} \subset \{1, \dots, J\}$*,* $Q \le X$*). Then,*

$$\int_N^{2N} \Big| \sum_{m\sim M} \sum_{\ell \sim L} a(m,\ell)\big(w(m\ell,y) - W_0(m\ell,y)\big)\Big|^2 dy \ll \varepsilon^2 X^{4-c-\sigma}.$$

Furthermore, the result still holds if we replace $w(n)$ *by* $w^*(n)$ *and* $W_0(n)$ *by* $W_1(n)$*.*

Proof. One needs to combine Lemmas 4 and 5 with Perron's formula [4, (15)]. The details can be found in [4, Lemma 1] or in [7, Lemma 11]. ■

Let

$$I(x) = \int_X^{2X} e(xt^c)\, dt,$$

and define

$$J_0(X,y) = \int_{\mathbb{R}} I_0(x) I(x) K(x) e(-yx)\, dx,$$
$$J_1(X,y) = \int_{\mathbb{R}} I_1(x) I(x) K(x) e(-yx)\, dx.$$

We shall also use the *Buchstab function* $\omega(x)$ which is the continuous solution of the differential-difference equation

$$\begin{cases} \omega(x) = 1/x & \text{if } 1 < x \le 2, \\ (x\omega(x))' = \omega(x-1) & \text{if } x > 2. \end{cases}$$

It will enter the discussion through the asymptotic relation

$$\sum_{\substack{X \le n < Y \\ (n,P(z))=1}} 1 = \frac{Y-X}{\log z}\, \omega\left(\frac{\log X}{\log z}\right) + O\left(X(\log X)^{-2}\right), \tag{4.12}$$

which is valid when $X < Y \le 2X$ and $z \ge X^a$ for some constant $a > 0$.

Lemma 7. *Let* $1 < c < 6/5$ *and* $u \ge 1$*, and suppose that, for some* L *satisfying* (3.3) *or* (3.4)*, there exists a set* $\mathcal{D} \subset \{1, \dots, u\}$ *with*

$$\prod_{j\in\mathcal{D}} p_j \asymp L.$$

Then,

$$\int_N^{2N} \Big| \sum_{p_1,\dots,p_u} \Big(S(\mathcal{A}_{p_1\cdots p_u}, p_1) - \frac{J_0(X,y)}{X} S(\mathcal{B}_{p_1\cdots p_u}, p_1)\Big)\Big|^2 dy \ll \varepsilon^2 X^{4-c} (\log X)^{-4.5}. \tag{4.13}$$

Here the summation is over primes $p_1, \dots, p_u \ge X^{8/75}$ satisfying $p_j > p_1$, together with $O(1)$ further conditions of the type

$$p_j \le p_l \quad \textit{or} \quad Q \le \prod_{j \in \mathcal{F}} p_j \le R$$

for some $\mathcal{F} \subset \{1, \dots, u\}$ and $R \le X$. Furthermore, the result still holds if we replace $\mathcal{A}$ by $\mathcal{A}^*$ and J_0 by J_1, and/or if, instead of L, X/L satisfies (3.3) *or* (3.4).

Proof. We have

$$\sum_{p_1, \dots, p_u} S(\mathcal{A}_{p_1 \cdots p_u}, p_1) = \sum_{p_1, \dots, p_u} \sum_{\substack{n \sim X/p_1 \cdots p_u \\ (n, P(p_1)) = 1}} w(np_1 \cdots p_u).$$

Upon writing the product $np_1 \cdots p_u$ as $m\ell$ where

$$\ell = \prod_{j \in \mathcal{D}} p_j \quad \text{and} \quad m = \Big(\prod_{j \notin \mathcal{D}} p_j \Big) \cdot n,$$

we can now use Lemma 6 to replace $\sum S(\mathcal{A}_{p_1 \cdots p_u}, p_1)$ in (4.13) by

$$\int_{-\tau}^{\tau} I_0(x) U(x) K(x)\, e(-yx)\, dx \tag{4.14}$$

where

$$U(x) = \sum_{p_1, \dots, p_u} \sum_{\substack{n \sim X/p_1 \cdots p_u \\ (n, P(p_1)) = 1}} e(x(np_1 \cdots p_u)^c).$$

(Note that $|U(x)| \ll X(\log X)^{-1}$.) Then, using (4.6) and the bound (cf. [11, Lemma 5.1])

$$|I_0(x)| \ll \frac{1}{|x| X^{c-1} \log X}, \tag{4.15}$$

we obtain

$$\int_N^{2N} \left| \int_{|x| \ge \tau_1} I_0(x) U(x) K(x) e(-yx) dx \right|^2 dy \ll \frac{\varepsilon^2 X^{4-c}}{(\log X)^{4.5}};$$

here $\tau_1 = X^{-c}(\log X)^{1/2}$. Hence, we may replace (4.14) by a similar integral with τ_1 in place of τ. Similarly, we may replace in (4.13) $J_0(X, y)$ by

$$\int_{-\tau_1}^{\tau_1} I_0(x) I(x) K(x) e(-yx) dx.$$

Therefore, it suffices to show that, for $|x| < \tau_1$, we have the approximation

$$U(x) = I(x)X^{-1} \sum_{p_1,\dots,p_u} S(\mathcal{B}_{p_1\cdots p_u}, p_1) + O\left(X(\log X)^{-3/2}\right);$$

this follows from (4.12) and partial summation. ∎

Lemma 8. *Let $1 < c < 6/5$ and $M \le X^{11/25}$. Suppose further that $a(m)$ are real numbers such that $a(m) \ll 1$ and $a(m) = 0$ unless all prime divisors of m are $\ge X^{8/75}$. Then,*

$$\int_N^{2N} \Bigg| \sum_{m\sim M} a(m)\Big(S(\mathcal{A}_m, X^{8/75}) - \frac{J_0(X,y)}{X} S(\mathcal{B}_m, X^{8/75})\Big)\Bigg|^2 dy \ll \varepsilon^2 X^{4-c}(\log X)^{-4.5}.$$

Furthermore, the result still holds if we replace $\mathcal{A}$ by $\mathcal{A}^*$ and J_0 by J_1.

Proof. The transition from $\sum a(m)S(\mathcal{A}_m, X^{8/75})$ to the integral (4.14) in which now

$$U(x) = \sum_{m\sim M} \sum_{\substack{n\sim X/m \\ (n,P(X^{8/75}))=1}} a(m)e(x(mn)^c)$$

is made by using the Eratosthenes–Legendre sieve as in [4, Lemma 2] or in [7, Lemma 13]. Then, we argue as in the proof of the previous lemma. ∎

5. The Proof of Theorem 1

5.1. **The lower bound.** We are now in position to complete the proof of Theorem 1 by combining (2.5) with Lemmas 7 and 8. Using these lemmas (and occasionally trivial estimates), we can evaluate the terms on the right side of (2.5) for almost all $y \in [N, 2N)$. For example, by Lemma 8, the measure of the set of values of y for which the inequality (for the quantities $A, B, \dots$ see Section 2)

$$\left| S_7 - \frac{J_0(X,y)}{X} \sum_{A\le p<B} S(\mathcal{B}_p, A)\right| < \frac{\varepsilon X^{2-c}}{g(N)(\log X)^2} \tag{5.1}$$

fails is $\ll g(N)^2 N(\log N)^{-1/2}$, i.e., (5.1) holds for almost all $y \in [N, 2N)$. We can apply similarly Lemma 8 to S_1, S_{11}, S_{15}, S_{16}, S_1^*, S_7^*, S_{15}^*, S_{16}^*; and Lemma 7 applies to S_3, S_5, S_8, S_{10}, S_{13}, S_3^*, S_5^*, S_8^*, S_{10}^*.

Let T_j be a sum similar to S_j or S_j^* in which the set $\mathcal{A}$ or $\mathcal{A}^*$ has been replaced by $\mathcal{B}$ (e.g., T_7 is the sum appearing on the left side of (5.1)), and write

$$\lambda_j = J_j(X,y)/X \qquad (j = 0, 1).$$

We have now shown that, for almost all $y \in [N, 2N)$,

$$S(\mathcal{A}, (2X)^{1/2}) = \lambda_0\big(T_1 - T_3 - T_5 - T_7 + T_8 + T_{10} - T_{11} + T_{13} + T_{15} - T_{16}\big) - \lambda_1\big(T_1 - T_3 - T_5 - T_7 + T_8 + T_{10} + T_{15} - T_{16}\big) + S_9^{\ddagger} + S_{12} + S_{14} + S_{17} + S_4^* + S_6^* - S_9^{\ddagger *} - S_{17}^* + O\Big(\frac{\varepsilon X^{2-c}}{g(N)(\log X)^2}\Big). \tag{5.2}$$

Next, we observe that, for almost all $y \in [N, 2N)$, we can obtain asymptotic formulas for S_{17} and S_{17}^*. Indeed, the summation conditions in these sums are such that either $pqr \geq X^{1/3}$ and pqr satisfies (3.4), or $rs \leq X^{2/9}$ and rs satisfies (3.3). Thus, Lemma 7 can be used to evaluate S_{17} and S_{17}^*. Lemma 7 works also for the following subsums of S_{12} and S_{14}:

$$S_{12}' := \sum_{\substack{C<p<D \\ A\leq q<B \\ D\leq pq\leq F}} S(\mathcal{A}_{pq}, q) \quad \text{and} \quad S_{14}' := \sum_{\substack{C<q<p<D \\ pq\geq X/D}} S(\mathcal{A}_{pq}, q).$$

Using that $S_{12} \geq S_{12}'$, $S_{14} \geq S_{14}'$, and $S_9^{\ddagger}$, S_4^* and S_6^* are non-negative, we infer from (5.2) that, for almost all $y \in [N, 2N)$,

$$S(\mathcal{A}, (2X)^{1/2}) \geq \lambda_0\big(T_1 - T_3 - T_5 - T_7 + T_8 + T_{10} - T_{11} + T_{12}' + T_{13} + T_{14}' + T_{15} - T_{16} + T_{17}\big) - \lambda_1\big(T_1 - T_3 - T_5 - T_7 + T_8 + T_{10} + T_{15} - T_{16} + T_{17}\big) - S_9^{\ddagger *} + O\Big(\frac{\varepsilon X^{2-c}}{g(N)(\log X)^2}\Big).$$

Applying the decomposition for $S(\mathcal{A}, (2X)^{1/2})$ from Section 2 to the sifting function $S(\mathcal{B}, (2X)^{1/2})$, we see that the sum in the first parentheses equals

$$S(\mathcal{B}, (2X)^{1/2}) + T_6 - T_9^{\ddagger} - T_{12}'' - T_{14}'',$$

where

$$T_{12}'' := T_{12} - T_{12}' \quad \text{and} \quad T_{14}'' := T_{14} - T_{14}'.$$

Similarly, using the decomposition for $S(\mathcal{A}^*, (2X)^{1/2})$, we find that the sum in the second parentheses is

$$S(\mathcal{B}, (2X)^{1/2}) + T_4 + T_6 - T_9^{\ddagger}.$$

Finally, we use that $S_9^{\ddagger *}$ does not exceed

$$S_{18}^* := \sum_{\substack{A \le q < p < B \\ C < pq < D \\ pq^2 > F}} S(\mathcal{A}_{pq}^*, A),$$

which can be evaluated using Lemma 8. Therefore, for almost all $y \in [N, 2N)$,

$$\begin{aligned} S(\mathcal{A}, (2X)^{1/2}) \ge \lambda_0 \big(& S(\mathcal{B}, (2X)^{1/2}) + T_6 - T_9^{\ddagger} - T_{12}'' - T_{14}'' \big) \\ & - \lambda_1 \big(S(\mathcal{B}, (2X)^{1/2}) + T_4 + T_6 - T_9^{\ddagger} + T_{18} \big) \\ & + O\Big(\frac{\varepsilon X^{2-c}}{g(N)(\log X)^2} \Big). \end{aligned} \tag{5.3}$$

In order to simplify the notation, we now drop the superscripts in T_{12}'', T_{14}'', and $T_9^{\ddagger}$. We observe that, for almost all $y \in [N, 2N)$,

$$J_1(X, y) = \log\left(\tfrac{14}{11}\right) \cdot J_0(X, y) + O\Big(\frac{\varepsilon X^{2-c}}{g(N)(\log X)} \Big).$$

This follows easily from the inequality

$$\left| I_1(x) - \log\left(\tfrac{14}{11}\right) I_0(x) \right| \ll X(\log X)^{-2}.$$

Using (4.12), the Prime Number Theorem and partial summation, we also obtain that

$$S(\mathcal{B}, (2X)^{1/2}) = \frac{X}{\log X} + O\Big(X(\log X)^{-2} \Big),$$

$$T_j = \frac{f_j X}{\log X} + O\Big(X(\log X)^{-2} \Big),$$

where the f_j's are the following constants:

$$f_4 = \int_2^{30/11} \omega(t)dt, \quad f_6 = \log\left(\tfrac{14}{11}\right),$$

$$f_{18} = \frac{75}{8} \iint_{\mathcal{D}_{18}} \omega\left(\frac{1-u-v}{8/75} \right) \frac{dudv}{uv},$$

$$f_j = \iint_{\mathcal{D}_j} \omega\left(\frac{1-u-v}{v} \right) \frac{dudv}{uv^2} \qquad (j = 9, 12, 14).$$

Here, the domains of integration $\mathcal{D}_j$ are:

$$\mathcal{D}_9 = \mathcal{D}_{18} = \left\{ (u,v) : \tfrac{8}{75} < v < u < \tfrac{1}{5}, \ \tfrac{11}{41} < u + v < \tfrac{1}{3}, \ u + 2v > \tfrac{11}{25} \right\},$$

$$\mathcal{D}_{12} = \left\{ (u,v) : \tfrac{8}{75} < v < \tfrac{1}{5}, \ \tfrac{11}{41} < u < \tfrac{1}{3}, \ u + v > \tfrac{11}{25} \right\},$$

$$\mathcal{D}_{14} = \left\{ (u,v) : \tfrac{11}{41} < v < u < \tfrac{1}{3}, \ u + v < \tfrac{14}{25} \right\}.$$

Substituting the above approximations into (5.3), we deduce, for almost all $y \in [N, 2N]$, that

$$S(\mathcal{A}, (2X)^{1/2}) \geq \frac{J_0(X, y)}{\log X}\Big(1 - f_9 - f_{12} - f_{14} - \log\left(\tfrac{14}{11}\right)\left(f_4 + f_6 - f_9 + f_{18}\right)\Big) + O\Big(\frac{\varepsilon X^{2-c}}{g(N)(\log X)^2}\Big).$$

Hence, Theorem 1 will follow if we show that

$$J_0(X, y) \gg \varepsilon X^{2-c}(\log X)^{-1} \tag{5.4}$$

and

$$1 - f_9 - f_{12} - f_{14} - \log\left(\tfrac{14}{11}\right)\left(f_4 + f_6 - f_9 + f_{18}\right) > 0. \tag{5.5}$$

The proof of (5.4) is similar to that of [14, Lemma 6], and (5.5) will be established in Section 5.2. Theorem 1 is proved.

5.2. **The numeric work.** First of all, we notice that

$$\omega(t) = \begin{cases} 1/t & 1 \leq t \leq 2, \\ (1 + \log(t-1))/t & 2 \leq t \leq 3, \end{cases}$$

and $\omega(t) \leq 0.5644$ for $t \geq 3$; for the latter estimate see [6, Lemma 8]. Hence,

$$f_4 = \int_2^{30/11} \frac{1 + \log(t-1)}{t} dt \leq 0.3983.$$

Note that the numeric integration here and elsewhere in this section is safe as we use it to calculate univariate integrals only.

For $(u, v) \in \mathcal{D}_9$, we have $u + 4v < 1$, which implies that $(1 - u - v)/v > 3$. Thus,

$$f_9 \leq 0.5644 \iint_{\mathcal{D}_9} \frac{dudv}{uv^2}.$$

Since $\mathcal{D}_9$ can be represented as $\mathcal{D}_{9,1} \cup \mathcal{D}_{9,2}$ where

$$\mathcal{D}_{9,1} = \left\{(u, v) : \tfrac{11}{75} < u < \tfrac{1}{6},\ \tfrac{11}{50} - u/2 < v < u\right\},$$
$$\mathcal{D}_{9,2} = \left\{(u, v) : \tfrac{1}{6} < u < \tfrac{1}{5},\ \tfrac{11}{50} - u/2 < v < \tfrac{1}{3} - u\right\},$$

direct computation of the last integral gives

$$f_9 \leq 0.5644 \times \left(\frac{50}{11}\log\left(\frac{5}{3}\right) - \frac{9}{11} - 3\log\left(\frac{3}{2}\right)\right) \leq 0.1622.$$

Similarly

$$f_{18} \leq 5.292\left[\int_{11/75}^{1/6} \log\left(\frac{2u}{\frac{11}{25} - u}\right)\frac{du}{u} + \int_{1/6}^{1/5} \log\left(\frac{\frac{2}{3} - 2u}{\frac{11}{25} - u}\right)\frac{du}{u}\right] \leq 0.217.$$

Thus,

$$1 - f_9 - \log\left(\tfrac{14}{11}\right)\left(f_4 + f_6 - f_9 + f_{18}\right) > 0.6265. \tag{5.6}$$

We now consider f_{12}. We have $\mathcal{D}_{12} = \mathcal{D}_{12,1} \cup \mathcal{D}_{12,2}$ where

$$\mathcal{D}_{12,1} = \left\{(u,v) : \tfrac{11}{41} < u < \tfrac{1}{3},\ \tfrac{11}{25} - u < v < \tfrac{1}{4}(1-u)\right\},$$
$$\mathcal{D}_{12,2} = \left\{(u,v) : \tfrac{11}{41} < u < \tfrac{1}{3},\ \tfrac{1}{4}(1-u) < v < \tfrac{1}{5}\right\}.$$

For $(u,v) \in \mathcal{D}_{12,1}$, we have $u + 4v < 1$, so once again

$$\iint_{\mathcal{D}_{12,1}} \omega\left(\frac{1-u-v}{v}\right)\frac{dudv}{uv^2} \le 0.5644 \iint_{\mathcal{D}_{12,1}} \frac{dudv}{uv^2}$$
$$= 0.5644 \times \left(\frac{25}{11}\log 2 - 4\log\left(\frac{15}{11}\right)\right) \le 0.189.$$

If $(u,v) \in \mathcal{D}_{12,2}$, we have

$$\omega\left(\frac{1-u-v}{v}\right) = \frac{1 + \log\big((1-u)/v - 2\big)}{(1-u-v)/v} \le \frac{1+\log 2}{(1-u-v)/v},$$

so we obtain

$$\iint_{\mathcal{D}_{12,2}} \omega\left(\frac{1-u-v}{v}\right)\frac{dudv}{uv^2} \le 1.6932 \iint_{\mathcal{D}_{12,2}} \frac{dudv}{uv(1-u-v)}$$
$$= 1.6932 \int_{11/41}^{1/3} \frac{\log\big(3/(4-5u)\big)}{u(1-u)} du \le 0.0963.$$

Finally, we have

$$\mathcal{D}_{14} = \left\{(u,v) : \tfrac{7}{25} < u < \tfrac{299}{1025},\ \tfrac{11}{41} < v < \tfrac{14}{25} - u\right\}$$
$$\cup \left\{(u,v) : \tfrac{11}{41} < u < \tfrac{7}{25},\ \tfrac{11}{41} < v < u\right\},$$

and for $(u,v) \in \mathcal{D}_{14}$,

$$\omega\left(\frac{1-u-v}{v}\right) = \frac{v}{1-u-v}.$$

Hence,

$$f_{14} = \int_{7/25}^{299/1025} \frac{1}{u(1-u)} \log\left(\frac{(14-25u)(30-41u)}{121}\right) du$$
$$+ \int_{11/41}^{7/25} \frac{1}{u(1-u)} \log\left(\frac{u(30-41u)}{11(1-2u)}\right) du \le 0.0041.$$

We now have that

$$f_{12} + f_{14} < 0.2894.$$

Combined with (5.6), this inequality shows that the left-hand side of (5.5) is $> \frac{1}{3}$.

Remark. Note that the limit of the method is set by the condition $c < 6/5$ imposed during the application of [13, Theorem 9] in the proof of Lemma 1. In other words, the arithmetic information (2.3) 'fails' before the sieve machinery starts deriving trivial conclusions. This is not typical for the method—in most applications the sieve fails before the analytic part of the argument (see the comments in Harman [4, p. 256]). In fact, the authors are convinced that improvement on the term $(XM_1^6M_2^6)^{1/8}$ in [13, Theorem 9] will imply an immediate improvement on Theorem 1.

6. Proof of Theorem 2

For $1 < c < 2$, let $X = X(c) = \frac{1}{2}N^{1/c}$, $\varepsilon(c) = N^{1-2/c}(\log N)^{10}$. We define

$$R^*(y,c) = \sum_{p_1,p_2\sim X} \widehat{K}(p_1^c + p_2^c - y) = \int_{\mathbb{R}} S(x,c)^2K(x,c)e(-yx)dx,$$

$$J^*(y,c) = \int_{\mathbb{R}} I_0(x,c)^2K(x,c)e(-yx)dx \gg \varepsilon(c)X^{2-c}(\log X)^{-2},$$

where $S(x,c)$, $K(x,c)$ and $I_0(x,c)$ are the functions $S(x)$, $K(x)$ and $I_0(x)$ from the previous sections. The theorem will follow if we prove that, for any fixed $\rho \in (0,\frac{1}{2})$,

$$\int_{1+\rho}^{2-\rho}\int_N^{2N} \big|R^*(y,c) - J^*(y,c)\big|^2 dy\, dc \ll N(\log N)^{14}. \tag{6.1}$$

We first note that we can show analogously to (4.4) that the inequality

$$\int_N^{2N}\left|\int_{-\tau(c)}^{\tau(c)} \big(S(x,c)^2 - I_0(x,c)^2\big)K(x,c)e(-xy)dx\right|^2 dy \ll N$$

holds uniformly for $c \in (1+\rho, 2-\rho)$; here $\tau(c) = X^{1-c-\eta}$ with $\eta < \frac{1}{3}\rho$. Also, in view of (4.6) and (4.15), we have

$$\int_N^{2N}\left|\int_{|x|>\tau(c)} I_0(x,c)^2K(x,c)e(-xy)dx\right|^2 dy \ll 1.$$

We now proceed to establish the estimate

$$\int_{1+\rho}^{2-\rho}\int_N^{2N}\left|\int_{|x|>\tau(c)} S(x,c)^2K(x,c)e(-xy)dx\right|^2 dy\, dc \ll N(\log N)^{14}. \tag{6.2}$$

Splitting up the interval $(1+\rho, 2-\rho)$ into $O(\log N)$ subintervals (a,b) where $b = a + (\log N)^{-1}$ and using (4.6) once again, we see that it suffices to prove that

$$\int_a^b \int_{|x|>\tau(c)} |S(x,c)|^4 K(x,c)^2 dx\, dc \ll N(\log N)^{13}, \tag{6.3}$$

whenever $1+\rho < a < 2-\rho$, $b = a + (\log N)^{-1}$ (so, we have $N^{1/a} \asymp N^{1/b}$). The left-hand side of (6.3) is

$$\ll \varepsilon(b) \int_{\tau(b)}^{\infty} |S(x,c_0)|^2 \left(\int_a^b |S(x,c)|^2 dc \right) K(x,c_0) dx,$$

where $c_0 \in (a,b)$. To estimate the inner integral we use the upper bound

$$\int_a^b |S(x,c)|^2 dc \ll N^{1/a} + N^{2/a-1}|x|^{-1}$$

which can be proven using standard techniques (cf. [14, Lemma 7]). Hence, the left-hand side of (6.3) is

$$\begin{aligned} &\ll \varepsilon(b) \int_{\tau(b)}^{\infty} |S(x,c_0)|^2 \big(N^{1/a} + N^{2/a-1}|x|^{-1}\big) K(x,c_0) dx, \\ &\ll \varepsilon(b) \bigg(N^{2/a-1} \varepsilon(c_0) \int_{\tau(b)}^{1} |S(x,c_0)|^2 |x|^{-1} dx \\ &\qquad + N^{1/a} \int_0^{\infty} |S(x,c_0)|^2 K(x,c_0) dx \bigg). \end{aligned}$$

Using the mean value estimate (see [14, Lemma 7])

$$\int_n^{n+1} |S(x,c)|^2 dx \ll N^{1/c},$$

we now infer that

$$\int_a^b \int_{|x|>\tau(c)} |S(x,c)|^4 K(x,c)^2 dx\, dc \ll \frac{\varepsilon^2(b) N^{3/b-1}}{\tau(b)} + \varepsilon(b) N^{2/b}.$$

By the choice of $\varepsilon(b)$, $\tau(b)$ and η, the last expression is $O(N(\log N)^{10})$ and, therefore, the proof of (6.1) is complete.

References

[1] J. Brüdern and A. Kumchev, Diophantine approximation by cubes of primes and an almost prime II, *Illinois J. Math.* (to appear).

[2] Y.-C. Cai, On a diophantine inequality involving prime numbers, *Acta Math. Sinica* **39** (1996), 733–742 (in Chinese).

[3] G. Harman, On the distribution of αp modulo one, *J. London Math. Soc.* (2) **27** (1983), 9–18.

[4] G. Harman, On the distribution of αp modulo one II, *Proc. London Math. Soc.* (3) **72** (1996), 241–260.

[5] L.-K. Hua, *Additive Theory of Prime Numbers*, Providence, Rhode Island: American Math. Soc., 1965.
[6] C.-H. Jia, On the Piatetski-Shapiro–Vinogradov theorem, *Acta Arith.* **73** (1995), 1–28.
[7] A. Kumchev, A diophantine inequality involving prime powers, *Acta Arith.* **89** (1999), 311–330.
[8] A. Kumchev and T. Nedeva, On an equation with prime numbers, *Acta Arith.* **83** (1998), 117–126.
[9] M. B. S. Laporta, On a binary diophantine inequality involving prime numbers, *Acta Math. Hungar.* **83** (1999), 205-213.
[10] H.-Q. Liu, On the number of abelian groups of a given order, *Acta Arith.* **59** (1991), 261–277.
[11] H. L. Montgomery, *Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis*, CBMS Regional Conf. Ser. in Math. **84**, Amer. Math. Soc., Providence, RI, 1994.
[12] I.I. Piatetski-Shapiro, On a variant of Waring-Goldbach's problem, *Mat. Sb.* **30** (72) (1), (1952), 105-120 (in Russian).
[13] P. Sargos and J. Wu, Multiple exponential sums with monomials and their applications in number theory, *Acta Math. Hungar.* **88** (2000), (to appear).
[14] D. I. Tolev, On a diophantine inequality involving prime numbers, *Acta Arith.* **61** (1992), 289-306.
[15] I. M. Vinogradov, Representation of an odd number as the sum of three primes, *Dokl. Akad. Nauk SSSR* **15** (1937), 291–294, (in Russian).

Department of Mathematics
University of South Carolina
Columbia, SC 29208
U.S.A.

koumtche@math.sc.edu

Dipartimento di Matematica e Appl. "R. Caccioppoli"
Complesso Universitario di Monte S. Angelo
Via Cinthia
80126 Napoli
Italy

laporta@matna2.dma.unina.it