Diophantine Approximation by Cubes of Primes and an Almost Prime II

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Abstract

Let $\lambda_1, \dots, \lambda_4$ be non-zero with λ_1/λ_2 irrational and negative, and let $\mathcal{S}$ be the set of values attained by the form

$$\lambda_1 x_1^3 + \cdots + \lambda_4 x_4^3$$

when x_1 has at most 3 prime divisors and the remaining variables are prime. We prove that most real numbers are close to an element of $\mathcal{S}$.

1 Introduction

Let $\lambda_1, \dots, \lambda_4$ be non-zero with λ_1/λ_2 irrational and negative. Recently, the second author [8] showed that most real numbers are close to values taken by the form

$$\lambda_1 x^3 + \lambda_2 p_1^3 + \cdots + \lambda_4 p_3^3$$

when $p_1, \dots, p_3$ are primes and x is a P_6-number (henceforth, a number is called a *P_r-number* if it has at most r prime factors, counted with multiplicities). Although this result is close to the best we can hope for, it is not quite satisfactory, since in the related problem for representing integers as sums of three cubes of primes and a cube of an almost prime, the first author [1] and K. Kawada [7] obtained similar results in which the almost prime is a P_4- and a P_3-number, correspondingly. In the present paper we fix this defect by proving the following theorem.

Theorem 1. *Let $\lambda_1, \dots, \lambda_4$ be non-zero real numbers with λ_1/λ_2 irrational and negative. Let $\mathcal{E}(N) = \mathcal{E}(N, \delta)$ denote the Lebesgue measure of the set of real numbers η for which $|\eta| \le N$ and the inequality*

$$\left| \lambda_1 x^3 + \lambda_2 p_1^3 + \cdots + \lambda_4 p_3^3 - \eta \right| < N^{-\delta} \tag{1.1}$$

has no solutions in primes $p_1, \dots, p_3$ and a P_3-number x. Then, there exists an absolute constant $\delta > 0$ such that one can find arbitrarily large values of N for which $\mathcal{E}(N) \ll N \exp\left(-(\log N)^{1/4}\right)$.

One can easily infer from Theorem 1 that if $\lambda_1, \dots, \lambda_8$ are non-zero and λ_1/λ_2 is as above, the values taken by the form

$$\lambda_1 x^3 + \lambda_2 p_1^3 + \cdots + \lambda_8 p_7^3$$

when x is a P_3-number and $p_1, \dots, p_7$ are primes form a set dense in $\mathbb{R}$. The proof is analogous to that of [8, Theorem 2] and is essentially an application of the pigeonhole principle.

The primary reason that [8, Theorem 1] is weaker than the result in [1] is the use of the diminishing ranges lemmas due to Davenport and Roth [5], which are weaker than the corresponding results for equations based on the work of R. C. Vaughan [11]. Applying a technique of the first author [2], we obtain an analogue for inequalities of Vaughan's result that reads as follows.

Theorem 2. *Let λ and μ be fixed non-zero real numbers. Also, let $X \geq 1$ and $Y = X^{5/6}$. Denote by $\mathcal{S}$ the number of solutions of the diophantine inequality*

$$\left|\lambda(x_1^3 - x_2^3) + \mu(y_1^3 + y_2^3 - y_3^3 - y_4^3)\right| < 1/2 \tag{1.2}$$

subject to

$$X < x_i \leq 2X, \quad Y < y_i \leq 2Y. \tag{1.3}$$

Then, for every $\varepsilon > 0$,

$$\mathcal{S} \ll X^{1+\varepsilon} Y^2.$$

In Section 2, we shall apply the linear sieve in order to derive Theorem 1 from Propositions 1 and 2 below. In Section 3, we prove Theorem 2, as well as other results of the same nature, which we then use in Section 4 to establish the propositions and so complete the proof of Theorem 1.

2 Outline of the proof of Theorem 1

Without loss of generality we can assume that $\lambda_1 > 0$ and $\lambda_2 < 0$. Also, since the positive and the negative values of η can be treated similarly, we shall consider only the case $\eta > 0$. Let $\varepsilon > 0$ be sufficiently small in terms of δ and let a/q be a convergent to the continued fraction of λ_1/λ_2 with $q \geq q_0(\delta, \varepsilon)$. Then, choose N so that

$$N^{1/8+16\delta+20\varepsilon} \leq q \leq N^{1/2-6\delta-10\varepsilon}.$$

It suffices to show that, for M with $N^{1-\varepsilon} < M \leq N$, the measure $\mathcal{E}(M, N)$ of the set of real numbers $\eta \in (M, 2M]$ for which (1.1) is not solvable satisfies

$$\mathcal{E}(M, N) \ll N \exp\left(-(\log N)^{1/4}\right). \tag{2.1}$$

If λ_1/λ_2 is also algebraic, one can argue somewhat differently. If q_v is the sequence of denominators of the convergents to λ_1/λ_2 in ascending order, then by Roth's theorem on diophantine approximation, one has $q_{v+1} \ll q_v^{1+\varepsilon}$. Hence, if now N is sufficiently large, one can choose a q such that the above inequality holds; thus N is not restricted to certain intervals in this case.

We start by restricting the variables in (1.1) to certain ranges. Define X, X_1, and Y by

$$\lambda_1 X^3 = |\lambda_2| X_1^3 = \frac{1}{2} M, \quad Y = X^{5/6}, \tag{2.2}$$

and write $\tau = N^{-\delta}$, $L = \log X$. Then, let $\mathcal{R}(\eta)$ be the number of solutions of (1.1) with the variable x being a P_3-number in the range $X < x \le 2X$ and $p_1, \dots, p_3$ being primes subject to

$$X_1 < p_1 \le 2X_1, \quad Y < p_2, p_3 \le 2Y. \tag{2.3}$$

We will approach (2.1) via the version of the circle method due to Davenport and Heilbronn [4] combined with the linear sieve. Let K be a function whose Fourier transform $\widehat{K}$ satisfies the inequality $\widehat{K} \le \chi$, χ being the characteristic function of the interval $(-1, 1)$ (see Lemma 1 below for the definition of the particular function K we use). We have

$$\mathcal{R}(\eta) \ge \sum_{x, p_1, \dots, p_3} \widehat{K}\left(\frac{\lambda_1 x^3 + \lambda_2 p_1^3 + \cdots + \lambda_4 p_3^3 - \eta}{\tau} \right), \tag{2.4}$$

where the summation is over P_3-numbers $x \in (X, 2X]$ and primes $p_1, \dots, p_3$ satisfying (2.3). We then sift the right-hand side of (2.4). Let

$$z = X^{1/7}, \quad \Pi(z) = \prod_{p<z} p, \quad V(z) = \prod_{p<z} \left(1 - \frac{1}{p}\right),$$

and, for $m \in \mathbb{N}$, define $\Omega(m)$ as the number of prime divisors of m counted with multiplicities (in particular, m is a P_r-number if $\Omega(m) \le r$). Let $\mathcal{R}_1(\eta)$ be a sum analogous to that appearing on the right side of (2.4) in which the condition $\Omega(x) \le 3$ has been replaced with $(x, \Pi(z)) = 1$, and let $\mathcal{R}_2(\eta)$ be a similar sum in which the condition $\Omega(x) \le 3$ is replaced by $(x, \Pi(z)) = 1$, $\Omega(x) \ge 4$ and the prime p_1 by an integer y with $(y, \Pi(z)) = 1$. We can deduce from (2.4) that

$$\mathcal{R}(\eta) \ge \mathcal{R}_1(\eta) - \mathcal{R}_2(\eta). \tag{2.5}$$

Let ϕ and Φ be the standard functions of the linear sieve defined as the solutions of the simultaneous differential-difference equations

$$\begin{cases} \Phi(u) = 2e^{\gamma}/u, \quad \phi(u) = 0 & , 0 < u \le 2, \\ (u\Phi(u))' = \phi(u-1), \quad (u\phi(u))' = \Phi(u-1) & , u \ge 2; \end{cases}$$

here γ is Euler's constant. For a squarefree d, let $\mathcal{R}_1(\eta, d)$ be the sum analogous to $\mathcal{R}_1(\eta)$ with the condition $(x, \Pi(z)) = 1$ replaced by $x \equiv 0 \pmod{d}$, and suppose that we can find a positive quantity $J_1(\eta) \ge X^{\varepsilon}$ and a $D > z$ such that

$$\sum_{d \le D} \xi_d \left(\mathcal{R}_1(\eta, d) - \frac{1}{d} J_1(\eta) \right) \ll J_1(\eta) L^{-2} \tag{2.6}$$

for any choice of the complex numbers ξ_d, $|\xi_d| \le 1$. Then, the lower bound linear sieve [6, (8.5.2)] yields

$$\mathcal{R}_1(\eta) \ge J_1(\eta)V(z)\left(\phi\left(\frac{\log D}{\log z}\right) + O\left(L^{-1/14}\right)\right). \tag{2.7}$$

Similarly, if $\mathcal{R}_2(\eta, d)$ is the sum $\mathcal{R}_2(\eta)$ with the condition $y \equiv 0\,(\mathrm{mod}\ d)$ in place of $(y, \Pi(z)) = 1$, and if we can find a function $J_2(\eta) \ge X^{\varepsilon}$ and a $D > z$ with

$$\sum_{d \le D} \xi_d \left(\mathcal{R}_2(\eta, d) - \frac{1}{d}J_2(\eta)\right) \ll J_2(\eta)L^{-2} \tag{2.8}$$

whenever the coefficients ξ_d have absolute values ≤ 1, then the upper bound sieve [6, (8.5.1)] gives

$$\mathcal{R}_2(\eta) \le J_2(\eta)V(z)\left(\Phi\left(\frac{\log D}{\log z}\right) + O\left(L^{-1/14}\right)\right). \tag{2.9}$$

Hence, if we could choose $J_1(\eta)$ and $J_2(\eta)$ so that (2.6) and (2.8) hold with $D = X^{0.333}$ and, in addition,

$$J_2(\eta) \le J_1(\eta)\left(0.182 + O\left(L^{-1}\right)\right), \tag{2.10}$$

it would follow from (2.5), (2.7), and (2.9) that

$$\mathcal{R}(\eta) \gg J_1(\eta)L^{-1}.$$

However, proving an asymptotic formula like (2.6) or (2.8) is beyond the scope of the present methods. What we *can* prove is that these two inequalities hold on average over η. More precisely, we can establish the following two propositions.

Proposition 1. *Let $\theta < \frac{1}{3}$ and $D = X^{\theta}$. Also, let ξ_d be complex numbers of modulus ≤ 1 and let $J_1(\eta)$ be given by (4.2) below. Then,*

$$\int_M^{2M} \left|\sum_{d \le D} \xi_d \left(\mathcal{R}_1(\eta, d) - \frac{1}{d}J_1(\eta)\right)\right|^2 d\eta \ll \tau^2 XY^4 \exp\big(-(\log N)^{1/4}\big). \tag{2.11}$$

Proposition 2. *Let $\theta < \frac{1}{3}$ and $D = X^{\theta}$. Also, let ξ_d be complex numbers of modulus ≤ 1 and let $J_2(\eta)$ be given by (4.9) below. Then,*

$$\int_M^{2M} \left|\sum_{d \le D} \xi_d \left(\mathcal{R}_2(\eta, d) - \frac{1}{d}J_2(\eta)\right)\right|^2 d\eta \ll \tau^2 XY^4 \exp\big(-(\log N)^{1/4}\big). \tag{2.12}$$

Since both $J_1(\eta)$ and $J_2(\eta)$ will have orders of magnitude $\tau X^{-1}Y^2L^{-3}$, these propositions imply that the measures of the sets $\mathcal{E}_1(M, N)$ and $\mathcal{E}_2(M, N)$ of values of η for which (2.6) and (2.8) fail do not exceed $O\big(N\exp\big(-(\log N)^{1/4}\big)\big)$ and, therefore, complete the proof of (2.1).

3 Counting solutions of diophantine inequalities

3.1 Auxiliary results

Our first lemma constructs a function K that combines the properties of the two kernels most commonly used in the present context: $(\sin \pi x/\pi x)^2$ and the function constructed by Davenport [3, Lemma 4]. The former, as Davenport mentioned in [3, p. 85], 'leads to unnecessary complications' (due to its relatively slow decay). Davenport's kernel, however, changes sign and, in some situations (e.g., if one wants to use convexity estimates), this also is a problem. The function K built in the following lemma has all the properties of that in [3] and, in addition, is non-negative.

Lemma 1. *Let $A > 0$ be fixed. There is an even real-valued function $K \in L^1(\mathbb{R})$ with the properties:*

(1°) *There is a constant $C(A) > 0$, depending only on A, such that*

$$|K(x)| \le \frac{C(A)}{(1+|x|)^A}.$$

(2°) *Both K and its Fourier transform*

$$\widehat{K}(y) = \int_{\mathbb{R}} K(x) e(-xy)\, dx$$

are non-negative.

(3°) *If χ is the characteristic function of the interval $(-1, 1)$, then, for all real x, one has*

$$\tfrac{1}{3}\chi(4x) \le \widehat{K}(x) \le \chi(x).$$

Proof. Actually, we only need to modify Davenport's proof of [3, Lemma 4]. Let r be the integer with $2r < A \le (2r + 2)$. In the notation of [3], choose $\delta = (6r)^{-1}$ and $\psi_0(y) = \max(0, 1 - |y|)$ and define K as the inverse Fourier transform of $\psi_{2r}(\frac{4}{3}y)$. Then, *mutatis mutandis*, Davenport's argument yields the properties of $\widehat{K}$. Also, using [10, (11.3)] in place of [3, (20)], we obtain the explicit formula

$$K(x) = K_r(x) = \frac{3}{4}\left(\frac{\sin 3\pi x/4}{3\pi x/4}\right)^2 \left(\frac{\sin \pi x/4r}{\pi x/4r}\right)^{2r},$$

from which the desired properties of K follow readily. ■

Our next lemma contains a device developed recently by the first author [2, Lemma 3].

Lemma 2. *Let $Q \ge 1$, $N \ge 1$ and define the set of* major arcs $\mathfrak{M}$ *as the union of all intervals*

$$\mathfrak{M}(q, a) = \{\alpha : \ |q\alpha - a| \le Q/N\},$$

where a and q are integers satisfying $1 \le q \le Q$ and $(a,q)=1$. Let $\Psi \in L^1(\mathbb{T})$ be defined by

$$\Psi(\alpha) = \begin{cases} (q+N|q\alpha - a|)^{-1} & \text{for } \alpha \in \mathfrak{M}(q,a), \\ 0 & \text{for } \alpha \notin \mathfrak{M}, \end{cases} \tag{3.1}$$

and let $K \in L^1(\mathbb{R})$ be a continuous function whose Fourier transform vanishes for $|\alpha| \ge A$. Finally, let F be a function given by

$$F(\alpha) = \sum_{v \in \mathcal{V}} a_v e(-\alpha v)$$

where a_v are real coefficients and $\mathcal{V}$ is a finite set of real numbers contained in the interval $[-V,V]$ for some $V \ge 1$. Then, for any $\varepsilon > 0$,

$$\int_{\mathfrak{M}} \Psi(\alpha) F(\alpha) K(\alpha)\, d\alpha \ll (QNV)^{\varepsilon} N^{-1} \left(\sum_{v \in \mathcal{V}} |a_v| + Q \sum_{|v| \le A} |a_v| \right).$$

The implied constant depends on ε and K.

3.2 Proof of Theorem 2

Let $\mathcal{S}_0$ and $\mathcal{S}_1$ denote the numbers of solutons of (1.2), (1.3) subject to $x_1 = x_2$ and $x_1 > x_2$, respectively; by symmetry, $\mathcal{S} = \mathcal{S}_0 + 2\mathcal{S}_1$. Since, by a standard divisor argument, $\mathcal{S}_0 \ll X^{1+\varepsilon} Y^2$, it remains to estimate $\mathcal{S}_1$. Write $x_1 = x + h$, $x_2 = x$. Then, (1.2) becomes

$$\left| \lambda h(3x^2 + 3hx + h^2) + \mu(y_1^3 + y_2^3 - y_3^3 - y_4^3) \right| < 1/2 \tag{3.2}$$

and the unknowns satisfy the inequalities

$$1 \le h \le CX^{1/2}, \quad X < x \le 2X, \quad Y < y_i \le 2Y, \tag{3.3}$$

where $C > 0$ depends only on λ and μ.

We now express the number of solutions of (3.2) subject to (3.3) by a Fourier integral. Let K_0 be the function obtained from Lemma 1 with $A = 2$ and let $K(x) = 2K(2x)$. Write $H = CX^{1/2}$ and define

$$F(\alpha) = \sum_{h \le H} \sum_{X < x \le 2X} e(\alpha h(3x^2 + 3hx + h^2)), \quad g(\alpha) = \sum_{Y < y \le 2Y} e(\alpha y^3).$$

It then follows that

$$\mathcal{S}_1 \ll \int_{\mathbb{R}} F(\lambda\alpha) |g(\mu\alpha)|^4 K(\alpha)\, d\alpha \ll \int_{\mathbb{R}} F(\alpha) |g(\mu_1\alpha)|^4 K_1(\alpha)\, d\alpha, \tag{3.4}$$

where $\mu_1 = \mu/\lambda$ and $K_1(\alpha) = K(\alpha/\lambda)$. Let $\mathfrak{M}$ be the set of major arcs defined in Lemma 2 corresponding to $N = HX^2 \asymp Y^3$ and $Q = X$. By Cauchy's inequality and an estimate of Vaughan [12, Lemma 3.1], we now have

$$F(\alpha) \ll HX^{1+\varepsilon} \Psi(\alpha)^{1/2} + X^{1+\varepsilon}, \tag{3.5}$$

where Ψ is given by (3.1). Substituting (3.5) into (3.4), we get

$$\begin{aligned} \mathcal{S}_1 \ll HX^{1+\varepsilon} \int_{\mathfrak{M}} \Psi(\alpha)^{1/2} |g(\mu_1\alpha)|^4 K_1(\alpha)\, d\alpha \\ + X^{1+\varepsilon} \int_{\mathbb{R}} |g(\mu_1\alpha)|^4 K_1(\alpha)\, d\alpha. \end{aligned} \tag{3.6}$$

The second integral is bounded by the number of solutions of

$$\left|\mu(y_1^3 + y_2^3 - y_3^3 - y_4^3)\right| < 2$$

with $Y < y_i \leq 2Y$, and is therefore $O(Y^{2+\varepsilon})$. So, the second term in (3.6) contributes $\ll X^{1+2\varepsilon}Y^2$. By Cauchy's inequality, the first integral in (3.6) is

$$\ll \left(\int_{\mathfrak{M}} \Psi(\alpha) |g(\mu_1\alpha)|^2 K_1(\alpha)\, d\alpha \right)^{1/2} \times \left(\int_{\mathbb{R}} |g(\mu_1\alpha)|^6 K_1(\alpha)\, d\alpha \right)^{1/2}.$$

By convexity and Hua's lemma [10, Lemma 2.5],

$$\int_{\mathbb{R}} |g(\mu_1\alpha)|^6 K_1(\alpha)\, d\alpha \ll Y^{7/2+\varepsilon}. \tag{3.7}$$

Also, Lemma 2 with $A = 2$, $Q = X$, $V = N = Y^3$, and $F(\alpha) = |g(\mu_1\alpha)|^2$ yields

$$\int_{\mathfrak{M}} \Psi(\alpha) |g(\mu_1\alpha)|^2 K_1(\alpha)\, d\alpha \ll Y^{-3+\varepsilon} \left(XY + Y^2 \right) \ll X^{1+\varepsilon} Y^{-2}.$$

Thus, the first term in (3.6) is $O(X^{2+2\varepsilon}Y^{3/4}) = O(XY^2)$. ∎

3.3 Further lemmas on diophantine inequalities

In this section we prove analogues of [1, Lemma 3] and [1, (5.9)]; the reader should also compare them with [8, Lemmas 1 and 2]. The following result is weaker than [1, (5.9)] just by a factor of X^ε.

Lemma 3. *Let λ and μ be fixed non-zero real numbers. Also, let $X \geq 1$ and $Y = X^{5/6}$, and denote by $\mathcal{S}$ the number of solutions of the diophantine inequality*

$$\left|\lambda(x_1^3 - x_2^3) + \mu(y_1^3 + y_2^3 + y_3^3 - y_4^3 - y_5^3 - y_6^3)\right| < 1/2 \tag{3.8}$$

with $X < x_i \leq 2X$, $Y < y_i \leq 2Y$. Then, for every $\varepsilon > 0$,

$$\mathcal{S} \ll X^{-1+\varepsilon} Y^6.$$

Proof. The proof is similar to that of Theorem 2. The number of solutions of (3.8) with $x_1 = x_2$ is $\ll X\mathcal{S}_0$, where $\mathcal{S}_0$ is the number of solutions of the inequality

$$\left|\mu(y_1^3 + y_2^3 + y_3^3 - y_4^3 - y_4^3 - y_6^3)\right| < 1/2.$$

Replacing $\mathcal{S}_0$ by a Fourier integral, we can use (3.7) to estimate it and, hence, the solutions of (3.8) with $x_1 = x_2$ contribute $O(X^{1+\varepsilon}Y^{7/2}) = O(X^{-1}Y^6)$. Now, let $\mathcal{S}_1$ be the number of solutions of (3.8) with $x_1 > x_2$ and keep all the notation used in the estimation of the corresponding quantity in Section 3.2. Similarly to (3.6), we obtain

$$\begin{aligned} \mathcal{S}_1 \ll HX^{1+\varepsilon} &\int_{\mathfrak{M}} \Psi(\alpha)^{1/2}|g(\mu_1\alpha)|^6 K_1(\alpha)\,d\alpha \\ &+X^{1+\varepsilon}\int_{\mathbb{R}} |g(\mu_1\alpha)|^6 K_1(\alpha)\,d\alpha. \end{aligned} \tag{3.9}$$

Again, we can use (3.7) to estimate the last integral and see that the contribution from the second term in (3.9) is $O(X^{-1}Y^6)$. By Cauchy's inequality, the first integral on the right-hand side of (3.9) is

$$\ll \left(\int_{\mathfrak{M}} \Psi(\alpha)|g(\mu_1\alpha)|^4 K_1(\alpha)\,d\alpha\right)^{1/2} \times \left(\int_{\mathbb{R}} |g(\mu_1\alpha)|^8 K_1(\alpha)\,d\alpha\right)^{1/2}.$$

By Hua's lemma, the second integral is $O(Y^{5+\varepsilon})$ and the first one can be estimated via Lemma 2, the resulting estimate being $O(Y^{1+\varepsilon})$. Hence, the contribution from the first term in (3.9) is

$$\ll X^{3/2+\varepsilon}Y^{3+\varepsilon} \ll X^{-1+2\varepsilon}Y^6. \qquad \blacksquare$$

The next lemma is our version of [1, Lemma 3]. Note that, when $W \asymp X$, it is weaker than the expected estimate just by a factor of X^ε, and it yields a bound that is $O(X^{1-\varepsilon}Y^4)$, if $\mathcal{W}$ is 'thin', e.g., if $W \ll X^{1-3\varepsilon}$. The proof repeats that of [8, Lemma 2] with Theorem 2 and Lemma 3 in place of [8, Lemma 1], so we omit it.

Lemma 4. *Let λ, μ, and κ be fixed non-zero real numbers. Also, let $X \geq 1$, $X_1 \asymp X$, $Y = X^{5/6}$, and let $\mathcal{W}$ be a subset of $(X_1, 2X_1] \cap \mathbb{Z}$ with W elements. Denote by $\mathcal{S}(\mathcal{W})$ the number of solutions of the diophantine inequality*

$$\left|\lambda(x_1^3 - x_2^3) + \mu(w_1^3 - w_2^3) + \kappa(y_1^3 + y_2^3 - y_3^3 - y_4^3)\right| < 1/2$$

with $w_i \in \mathcal{W}$, $X < x_i \leq 2X$, $Y < y_i \leq 2Y$. Then, for every $\varepsilon > 0$,

$$\mathcal{S}(\mathcal{W}) \ll X^{43/12+\varepsilon}W^{3/4}.$$

4 The proof of Theorem 1 completed

4.1 Proof of Proposition 1

We start with the definition of some exponential sums and integrals. Let

$$f(\alpha) = \sum_{d\leq D}\sum_{X<dn\leq 2X} \xi_d\, e(\alpha(dn)^3),$$

$$g(\alpha) = \sum_{X_1<p\leq 2X_1} e(\alpha p^3), \quad h(\alpha) = \sum_{Y<p\leq 2Y} e(\alpha p^3),$$

the summations in g and h being over primes only. Also write

$$v(\beta,\Xi) = \int_{\Xi}^{2\Xi} e(\beta t^3)\,dt \quad\text{and}\quad w(\beta,\Xi) = \frac{1}{3}\int_{\Xi}^{2\Xi} \frac{e(\beta t^3)}{\log t}\,dt.$$

If K is the function defined in Lemma 1 with $A = 4/\varepsilon$ and $K_\tau(\alpha) = \tau K(\tau\alpha)$, upon using Fourier inversion, we have

$$\sum_{d\le D} \xi_d \mathcal{R}_1(\eta, d) = \int_{\mathbb{R}} F(\alpha) K_\tau(\alpha) e(-\alpha\eta)\,d\alpha, \tag{4.1}$$

where

$$F(\alpha) = f(\lambda_1\alpha) g(\lambda_2\alpha) h(\lambda_3\alpha) h(\lambda_4\alpha).$$

Also, define $J_1(\eta)$ by

$$J_1(\eta) = \int_{\mathbb{R}} F_1(\alpha) K_\tau(\alpha) e(-\alpha\eta)\,d\alpha, \tag{4.2}$$

where

$$F_1(\alpha) = v(\lambda_1\alpha, X) w(\lambda_2\alpha, X_1) w(\lambda_3\alpha, Y) w(\lambda_4\alpha, Y),$$

and let

$$F_2(\alpha) = F_1(\alpha) \sum_{d\le D} \xi_d/d.$$

The left side of (2.11) then does not exceed

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \big(F(\alpha) - F_2(\alpha)\big) K_\tau(\alpha) e(-\alpha\eta)\,d\alpha \right|^2 d\eta,$$

which, by Plancherel's theorem, is equal to

$$\int_{\mathbb{R}} \big|F(\alpha) - F_2(\alpha)\big|^2 K_\tau(\alpha)^2\,d\alpha.$$

Hence, (2.11) will follow from the inequality

$$\int_{\mathbb{R}} \big|F(\alpha) - F_2(\alpha)\big|^2 K_\tau(\alpha)\,d\alpha \ll \tau X Y^4 \exp\big(-(\log N)^{1/4}\big). \tag{4.3}$$

Writing $\omega = D^{-1}X^{-2-\varepsilon}$ and $H = \tau X^\varepsilon$, we dissect the real line into the following three subsets

$$\mathfrak{M} = (-\omega, \omega), \quad \mathfrak{m} = \{\alpha:\ \omega \le |\alpha| \le H\}, \quad \mathfrak{t} = \{\alpha:\ |\alpha| > H\}.$$

It is easy to see that

$$\int_{\mathfrak{t}} \big|F(\alpha) - F_2(\alpha)\big|^2 K_\tau(\alpha)\,d\alpha \ll \tau Y^4, \quad \text{say.} \tag{4.4}$$

On the ‘major arc’ $\mathfrak{M}$, we can approximate f, g, and h by exponential integrals. The Poisson summation formula [10, Lemma 4.2] yields

$$f(\lambda_1\alpha) = v(\lambda_1\alpha, X) \sum_{d \le D} \xi_d/d + O(D),$$

and, by a standard technique based on the explicit formula for $\psi(x)$ as a sum over the zeros of $\zeta(s)$ and zero-density estimates,

$$\begin{aligned} g(\lambda_2\alpha) &= w(\lambda_2\alpha, X_1) + O\big(X \exp\big(-(\log N)^{1/4}\big)\big), \\ h(\lambda_j\alpha) &= w(\lambda_j\alpha, Y) + O\big(Y \exp\big(-(\log N)^{1/4}\big)\big) \qquad (j = 3, 4). \end{aligned}$$

Combining these approximations and the mean-value estimates

$$\int_{-\omega}^{\omega} |g(\lambda_2\alpha)|^2 d\alpha \ll X^{-1}L^2, \qquad \int_{-\omega}^{\omega} |v(\lambda_1\alpha, X)|^2 d\alpha \ll X^{-1}L,$$

we now have

$$\int_{\mathfrak{M}} \big|F(\alpha) - F_2(\alpha)\big|^2 K_\tau(\alpha)\, d\alpha \ll \tau X Y^4 \exp\big(-(\log N)^{1/4}\big). \tag{4.5}$$

Thus, it remains to bound the contribution of the ‘minor arcs’ $\mathfrak{m}$. Since for $\alpha \in \mathfrak{m}$ (cf. [10, Lemma 6.2])

$$v(\alpha, \Xi),\ w(\alpha, \Xi) \ll \Xi^{-2}|\alpha|^{-1}, \tag{4.6}$$

we obtain easily that

$$\int_{\mathfrak{m}} \big|F_2(\alpha)\big|^2 K_\tau(\alpha)\, d\alpha \ll \tau X^{1-\varepsilon} Y^4. \tag{4.7}$$

Finally, we also have

$$\int_{\mathfrak{m}} \big|F(\alpha)\big|^2 K_\tau(\alpha)\, d\alpha \ll \tau X^{1-\varepsilon} Y^4. \tag{4.8}$$

The proof of this inequality is the most difficult part of the proof. It is, however, almost identical to the proof of the corresponding estimate in [8] with Theorem 2 and Lemmas 3 and 4 in place of Lemmas 1 and 2 in that paper, so we omit it.

Combining (4.4), (4.5), (4.7), and (4.8), we complete the proof of (4.3). ■

4.2 Proof of Proposition 2

The approach is similar to that in the previous section. Let h, v, w, and K_τ be the same as before and change X to X_1 in the definition of f. Also, define the exponential sums

$$g(\alpha) = \sum_{\substack{X < x \le 2X \\ (x, \Pi(z)) = 1 \\ 4 \le \Omega(x) \le 6}} e(\alpha x^3) \quad \text{and} \quad g_r(\alpha) = \sum_{\substack{X < x \le 2X \\ (x, \Pi(z)) = 1 \\ \Omega(x) = r}} e(\alpha x^3) \quad (r = 4, 5, 6).$$

Then, (4.1) holds with $\mathcal{R}_2(\eta, d)$ in place of $\mathcal{R}_1(\eta, d)$ and F defined by

$$F(\alpha) = g(\lambda_1\alpha) f(\lambda_2\alpha) h(\lambda_3\alpha) h(\lambda_4\alpha).$$

Let

$$F_1(\alpha) = w_1(\lambda_1\alpha) v(\lambda_2\alpha, X_1) w(\lambda_3\alpha, Y) w(\lambda_4\alpha, Y),$$

where w_1 is given by (4.10) below, and define

$$J_2(\eta) = \int_{\mathbb{R}} F_1(\alpha) K_\tau(\alpha) e(-\alpha\eta)\, d\alpha. \tag{4.9}$$

Keeping (formally) the definition of F_2 the same as in Section 4.1, we see that it suffices to establish (4.3).

Consider the integral over the major arc $\mathfrak{M}$. As before, we can replace f and h by v and w, so it remains to be shown that

$$\int_{\mathfrak{M}} \big|G(\alpha) - F_2(\alpha)\big|^2 K_\tau(\alpha)\, d\alpha \ll \tau X Y^4 \exp\big(-(\log N)^{1/4}\big),$$

where

$$G(\alpha) = g(\lambda_1\alpha) v(\lambda_2\alpha, X_1) w(\lambda_3\alpha, Y) w(\lambda_4\alpha, Y).$$

Using (4.6), we can prune $\mathfrak{M}$ to

$$\mathfrak{M}_0 = \Big\{\alpha :\ |\alpha| < X^{-3} \exp\big((\log N)^{1/4}\big)\Big\}.$$

For $\alpha \in \mathfrak{M}_0$ and $r = 4, 5, 6$, the prime number theorem and partial summation yield the approximations

$$g_r(\lambda_1\alpha) = \int_X^{2X} c_r(t) e(\lambda_1\alpha t^3)\, dt + O\big(X \exp\big(-(\log N)^{1/4}\big)\big),$$

where

$$c_r(t) = \mathop{\sum\cdots\sum}_{\substack{z \le p_1 \le \cdots \le p_{r-1} \\ p_1 \cdots p_{r-2} p_{r-1}^2 \le t}} \frac{1}{p_1 \cdots p_{r-1}\big(\log t - \log(p_1 \cdots p_{r-1})\big)}.$$

Hence, for $\alpha \in \mathfrak{M}_0$,

$$g(\lambda_1\alpha) = \sum_{r=4}^{6} g_r(\lambda_1\alpha) = w_1(\lambda_1\alpha) + O\big(X \exp\big(-(\log N)^{1/4}\big)\big),$$

where

$$w_1(\beta) = \int_X^{2X} \big(c_4(t) + c_5(t) + c_6(t)\big) e(\beta t^3)\, dt. \tag{4.10}$$

So, we can also replace g by w_1 and, thus, finish the proof of (4.5).

Inequalities (4.4), (4.7), and (4.8) also stay true. The proofs are identical to the ones gives before, the only exception being the estimation of g on the minor arc $\mathfrak{m}$, which is, in fact, easier. One just needs to observe that the summation variables in the sums g_r can always be combined as to produce a ‘type II sum’, which can then be estimated via [8, Lemma 4]. Thus, (4.3) follows. ■

4.3 The integrals $J_1(\eta)$ and $J_2(\eta)$

Finally, we will show that J_1 and J_2 have the properties mentioned in Section 2. For example, Fourier inversion and some routine calculations show that

$$J_1(\eta) = \frac{1}{81} \underset{\substack{X^3<t_1\le 8X^3 \\ X_1^3<t_2\le 8X_1^3 \\ Y^3<t_3,t_4\le 8Y^3}}{\int\cdots\int} \frac{(t_1\cdots t_4)^{-2/3}}{\log t_2\cdots\log t_4} \widehat{K}\left(\frac{\lambda_1 t_1+\cdots+\lambda_4 t_4-\eta}{\tau}\right) dt_1\cdots dt_4.$$

In view of (2.2), it then follows that there exist constants $c_2 > c_1 > 0$, depending only on the λ_i's, for which

$$c_1\tau X^{-1}Y^2L^{-3} \le J_1(\eta) \le c_2\tau X^{-1}Y^2L^{-3}.$$

Futhermore, since K. Kawada [7, p. 18] has shown that

$$w_1(\beta) = Cw(\beta, X) + O\left(XL^{-2}\right),$$

with an absolute constant $C \in (0, 0.182)$, we can easily derive similar esimates for J_2 as well as prove (2.10).

This completes the proof of Theorem 1. ■

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