Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences
Tome 49, N° 1, 1996

MATHEMATIQUES
Théorie des nombres

ON A SYSTEM OF TWO DIOPHANTINE EQUATIONS WITH PRIME NUMBERS[1]

A. Kumchev

(Submitted by Corresponding Member S. Troyanski on September 12, 1995)

In 1937 VINOGRADOV [1] solved Goldbach's ternary problem. He proved that if N is a sufficiently large odd integer then the equation

$$p_1 + p_2 + p_3 = N \tag{1}$$

has solutions in prime numbers p_1, p_2, p_3.

In 1952 PIATETSKI–SHAPIRO [2] considered the inequality

$$|p_1^c + \cdots + p_r^c - N| < \varepsilon \tag{2}$$

where $c > 1$ is not an integer and $\varepsilon > 0$ is arbitrary small. He proved that if r is sufficiently large in terms of c then (2) has solutions in prime numbers $p_1, \ldots, p_r$ for all sufficiently large N. He also proved the solvability of (2) in the case of $1 < c < \frac{3}{2}$ and $r = 5$.

Vinogradov's theorem on (1) gives an argument to expect that if c is close to one then (2) has solutions when $r = 3$. TOLEV [3] proved this for $1 < c < \frac{15}{14}$.

A natural analogue of Waring's problem with noninteger degrees is the equation

$$[x_1^c] + \cdots + [x_r^c] = N$$

(here and later $[x]$ denote the integer part of x). It was considered by SEGAL [4], DESHOUILLERS [5] and ARHIPOV and ZHITKOV [6]. An equation of this type with $r = 3$, $1 < c < \frac{17}{16}$ and prime unknowns was considered by TOLEV and LAPORTA [7].

In [8] TOLEV considered a system of two inequalities of type (2) with $r = 5$ and degrees close to one.

In this paper we study for solvability in prime numbers $p_1, \ldots, p_5$ the system

$$\begin{aligned} [p_1^c] + \cdots + [p_5^c] &= N_1; \\ [p_1^d] + \cdots + [p_5^d] &= N_2 \end{aligned} \tag{3}$$

where c, d are close to one and N_1, N_2 are sufficiently large integers satisfying some natural conditions.

[1]This work was supported by the National Foundation of the Bulgarian Ministry of Education, Science and Technology (grant number MM-430/94).

We define

$$R = \sum \ln p_1 \ldots \ln p_5,$$

where the summation is over primes $p_1, \ldots, p_5$ satisfying (3).

We use the notation:

c, d, α, β are fixed real number satisfying

$$(4) \qquad \begin{aligned} &1 < d < c < \tfrac{39}{38}; \\ &1 < \alpha < \beta < 5^{1-\frac{d}{c}}, \end{aligned}$$

$x, y, t, t_1, \ldots, t_5$ – real numbers; $p, p_1, \ldots, p_5$ – primes; n, j – integers; ε – arbitrary small positive number not necessary the same in all appearances; $\Lambda(n)$ – von Mangoldt's function; $e(x) = e^{2\pi i x}$; $X = N_1^{\frac{1}{c}}$; $\delta = N_2/N_1^{\frac{d}{c}}$;

$$S(x, y) = \sum_{p \le X} \ln p \cdot e(x[p^c] + y[p^d])$$

$$I(x, y) = \int_0^X e(xt^c + yt^d)dt$$

$$(5) \qquad \gamma = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_0^1 e(xt^c + yt^d)dt \right)^5 \cdot e(-x - \delta y)dxdy.$$

The constants in the O-terms and in the $\ll$-symbols depend on c, d, α, β and ε.

We prove the next

Theorem. Suppose that c, d, α, β are real numbers which satisfy (4) and N_1, N_2 are sufficiently large integers such that

$$(6) \qquad \alpha \le N_2/N_1^{\frac{d}{c}} \le \beta.$$

Then

$$(7) \qquad R = \gamma \cdot X^{5-c-d} + O\left(X^{5-c-d} \cdot \exp\left(-(\ln X)^{\frac{1}{3}-\varepsilon}\right)\right).$$

The integral (5) is convergent and its value is a positive real number $\gg 1$.

Corollary. The system (3) has a solution in prime numbers when the conditions (4) and (6) hold and N_1, N_2 are sufficiently large.

An outline of the proof. Let $\eta > 0$ be a sufficiently small number depending on c, d; $\tau_1 = X^{\frac{3}{4}-c-\eta}$, $\tau_2 = X^{\frac{3}{4}-d-\eta}$ and let Ω_1, Ω_2 be the sets

$$\Omega_1 = \{(x, y) : \ |x| < \tau_1, |y| < \tau_2\};$$
$$\Omega_2 = \{(x, y) : \ |x| < 0.5, |y| < 0.5\} \setminus \Omega_1.$$

Then

$$(8) \qquad R = \int_{-0.5}^{0.5} \int_{-0.5}^{0.5} S^5(x, y) \cdot e(-xN_1 - yN_2)dxdy = R_1 + R_2,$$

where R_j denotes the contribution of Ω_j to the integral.

If $(x, y) \in \Omega_1$ then we obtain

$$S(x, y) = \sum_{j \le j_0} \sum_{X/2^j < n \le X/2^{j-1}} \Lambda(n) e(xn^c + yn^d) + O\left(X \cdot \exp(-\ln X)^{\frac{1}{2}}\right),$$

where $j_0 = [(\ln X)^{\frac{1}{2}} / \ln 2]$.

If $X \exp\left(-(\ln X)^{\frac{1}{2}}\right) \le Y \le X$, then similarly to Lemma 14 of [3] we have

$$\sum_{Y<n\le 2Y} \Lambda(n) \cdot e(xn^c + yn^d) = \int_Y^{2Y} e(xt^c + yt^d)dt + O\left(X \exp\left(-(\ln X)^{\frac{1}{3}-\varepsilon}\right)\right).$$

We obtain from the last two equalities that

$$S(x,y) = I(x,y) + O\left(X \cdot \exp\left(-(\ln X)^{\frac{1}{3}-\varepsilon}\right)\right). \tag{9}$$

Similarly to Lemmas 8 and 9 of [8] we have

$$\iint_{\Omega_1} |S(x,y)|^4 dxdy \ll X^{4-c-d} \cdot \ln^{12} X \tag{10}$$

$$\iint_{\Omega_1} |I(x,y)|^4 dxdy \ll X^{4-c-d} \cdot \ln^{8} X \tag{11}$$

It is easy to prove (see [9], p. 36-39) that

$$|I(x,y)| \ll \left(\frac{X}{|x|+|y|}\right)^{\frac{1}{d+1}}.$$

The last estimate combined with (9)-(11) shows that

$$R_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I^5(x,y) \cdot e(-xN_1 - yN_2)dxdy + O\left(X^{5-c-d} \exp\left(-(\ln X)^{\frac{1}{3}-\varepsilon}\right)\right).$$

Hence

$$R_1 = \gamma \cdot X^{5-c-d} + O\left(X^{5-c-d} \exp\left(-(\ln X)^{\frac{1}{3}-\varepsilon}\right)\right). \tag{12}$$

If $(x,y) \in \Omega_2$ then similarly to Lemmas 12-14 of [10] we get

$$|S(x,y)| \ll \ln^2 X \cdot \max_{(x,y)\in\Gamma} \left|\sum_{n\le X} \Lambda(n) e(xn^c + yn^d)\right| + X^{\frac{15+2c}{18}} \ln^3 X, \tag{13}$$

where

$$\Gamma = \left\{(x,y):\ |x| < X^{\frac{3-2c}{18}} + 1,\ |y| < X^{\frac{3-2d}{18}} + 1\right\} \setminus \Omega_1.$$

Using VAUGHAN's identity (see [11]) and the simplest van der Corput's estimates of exponential sums we obtain

$$\left|\sum_{n\le X} \Lambda(n) \cdot e(xn^c + yn^d)\right| \ll X^{\frac{15+2c}{18}} \ln^7 X \tag{14}$$

uniformly with respect to $(x,y) \in \Gamma$.

The estimates (13) and (14) imply

$$\max_{(x,y)\in\Omega_2} |S(x,y)| \ll X^{\frac{15+2c}{18}} \ln^9 X$$

and similarly to Lemma 14 of [8] we have

$$\int_{-0.5}^{0.5}\int_{-0.5}^{0.5} |S(x,y)|^4 dxdy \ll X^2 \ln^{10} X.$$

From the last two estimates we obtain

$$|R_2| \ll X^{\frac{51+2c}{18}} \ln^{19} X$$

which gives in the case of $1 < c < \frac{39}{38}$ that

$$|R_2| \ll X^{5-c-d-\rho} \tag{15}$$

where $\rho = \rho(c,d) > 0$.

Now the asymptotic formula (7) follows from (8), (12) and (15).

Using (4) and (6) we see that the system

$$\begin{aligned} t_1^c + \cdots + t_5^c &= 1 \\ t_1^d + \cdots + t_5^d &= \delta \end{aligned}$$

has a solution $t_1, \ldots, t_5$ such that

$$\begin{aligned} &0 < t_1 < t_2 < 1 \\ &0 \le t_3, t_4, t_5 \le 1. \end{aligned}$$

Similarly to Lemmas 11-13 in chapter 8 of [9] we obtain now that $\gamma \gg 1$.

This completes the proof of the Theorem.

Finally the author would like to thank D. I. Tolev for the regular attention and the helpful discussions.

REFERENCES

[1] Виноградов И. М. ДАН СССР, **16**, 1937, 291-294. [2] Пятецкий–Шапиро И. И. Мат. Сб., **30**, 1952, 105-120. [3] Tolev D. I. Acta Arith., **61**, 1992, 289-306. [4] Сегал Б. И. Теорема Варинга для степеней с дробными и иррациональными показателями, **5**, Тр. физ.-мат. ин-та им. В. А. Стеклова АН СССР, 1933, 73-86. [5] Deshouillers J.-M. Probleme de Waring avec exposants non entiers, Bull. Soc. math. France, fasc. 101, 1973, 285-295. [6] Архипов Г. И., А. Н. Житков. Изв. АН СССР, сер. матем., **48**, 1984, 1138-1150. [7] Лапорта М., Д. И. Толев. Об одном уравнении с простыми числами, Мат. Заметки, **57**, 1995, 926–929. [8] Tolev D. I. Acta Arith., **69**, 1995, 387-400. [9] Архипов Г. И., А. А. Карацуба, В. Н. Чубариков. Теория кратных тригонометрических сумм. Москва, Наука, 1987. [10] Буриев К. Аддитивные задачи с простыми числами, канд. дис. Москва, МГУ, 1989. [11] Vaughan R. C. Mathematica, **24**, 1977, 135-141.

Department of Mathematics
University of Plovdiv
24, Tzar Asen Str.
4000 Plovdiv, Bulgaria