MORLEY RANK IN HOMOGENEOUS MODELS

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Abstract. We define an appropriate analog of the Morley rank in a totally transcendental homogeneous model with type diagram D. We show that if $RM[p] = \alpha$, then for some $1 \leq n < \omega$ the type p has n, but not $n + 1$, distinct D-extensions of rank α. This is surprising, because the proof of the statement in the first-order case depends heavily on compactness. We also show that types over $(D, \aleph_0)$-homogeneous models have multiplicity (Morley degree) 1.

Introduction

In the context of uncountably categorical theories two ranks play an important role: Morley's rank $RM[p]$ and Shelah's 2-rank. It is known that they are both bounded by ω_1 if and only if the theory is totally transcendental, and either rank can be used in the proof of Morley's theorem for countable first order theories, see for example [5] and [2]. The use of the R-rank gives a slightly shorter proof; and a key advantage of the Morley rank is that its value corresponds to classical dimension.

By contrast, the studies of categoricity in non-first order frameworks have used only the 2-rank R. Shelah developed a 2-rank in the context of models of an $L_{\omega_1,\omega}$ sentence in [13], Lessmann defined an analog of that rank for the homogeneous case in [10].

The main goal of this paper is to introduce a Morley-like rank for totally transcendental homogeneous models. We see this as a first step towards using ranks to measure complexity of (type-)definable sets in non-first order contexts in a meaningful way.

In Section 1, we introduce the context and notations. We make an effort to keep the presentation self-contained, but we assume some familiarity with the basics of homogeneous model theory. A good treatment of these can be found in [4] and [8] (these papers use the homogeneous model terminology) as well as in [10] (the term *finite diagram* appearing in that paper is not used now; instead one refers to a large

Date: September 20, 2005.

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strongly homogeneous model that realizes only the types in the "diagram" $D \subset S(\emptyset)$ as to the *homogeneous model*).

Section 2 is the main part of the paper. There we define an analog of Morley rank for the homogeneous context and show that the rank RM is bounded if and only if the homogeneous model is totally transcendental. We also establish that if $RM[p] = \alpha < \infty$, then p has a maximal finite number of contradictory D-extensions of RM-rank α, i.e., one can meaningfully define multiplicity of a type in a homogeneous totally transcendental model.

This is accomplished with the help of a different 2-rank R^* that we also introduce in Section 2. The main property of R^* is that $R^*[p] \geq \alpha + n$ if p has 2^n contradictory D-extensions of R^*-rank α. This rank is closer related to the first order 2-rank than the rank R introduced by Lessmann. In particular, for *any* algebraic type p, the value $R[p]$ is at most 1; while R^* gives the expected answer. The definitions for the ranks R^* and RM are similar in flavor to the definition of the rank in [8].

Along the way, we establish the ultrametric property for the rank RM, using what we call a *weak ultrametric property* for rank R^*; and find a formula that ties the values of the ranks R^* and RM. The formula is the same as the one obtained by Baldwin in [1] for the first order case; but the argument is substantially different, since we are not allowed to use compactness.

The last section connects stationarity, which is defined using a 2-rank in the homogeneous case, with multiplicity 1 in the sense of Morley rank. In essence, stationarity says that a type p over a $(D, \aleph_0)$-homogeneous model M with $R^*[p] = \alpha$ has a unique extension of R^*-rank α to any superset of M (the actual statement is stronger). By our results from Section 2, showing multiplicity 1 in the sense of Morley rank is equivalent to proving that the ordinal α cannot be a successor. This is the main result in Section 3.

In this paper, the main use of homogeneity is via Fact 1.3. We ask whether or not it is possible to extend the results to the case of totally transcendental classes of atomic models, perhaps under the additional assumption of excellence.

The authors are grateful to Andreas Blass for helpful conversations during their work on the paper; and to Rami Grossberg and John Baldwin for comments on the drafts of this paper.

1. Preliminaries

Fix a first order theory T and a model $M \models T$.

Definition 1.1. For a set $A \subset M$, the set of types $D(A) := \{\mathrm{tp}(\bar{a}/\emptyset) \mid \bar{a} \in A\}$ is called the *diagram of* A. The diagram of T is $D(T) := S(\emptyset)$.

For a fixed $D \subset D(T)$, we call A a D-set if $D(A) \subset D$. If $M \models T$ and $D(M) \subset D$, we call M a *D-model*.

The object of our study is the class of D-models, with an additional assumption.

Definition 1.2. Denote by $S^n_D(A)$ the collection of all complete types in n variables such that for all $\bar{c} \models p$ the set $A \cup \bar{c}$ is a D-set. Accordingly, $S_D(A) := \bigcup_{n<\omega} S^n_D(A)$.

Following [10], a D-model M is (D, λ)*-homogeneous* if M realizes all the types $\{p \in S_D(A) \mid A \subset M, |A| < \lambda\}$.

Compactness theorem no longer holds in this context. In particular, it is not clear if it is possible to realize the D-types over sets in some D-model containing the set without any additional assumptions on the class of all the D-structures. By the context of homogeneous models we mean a class of D-models under the assumption that there exists a monster D-model, i.e., (D, χ)-homogeneous model for some very large χ. We denote such model $\mathfrak{C}$, and call it simply a homogeneous model. An alternative approach, which does not make a big difference, is to demand $\mathfrak{C}$ to be strongly χ-homogeneous for some large χ, see for example [4].

A key property that holds in the homogeneous context is the "weak compactness"; it was used originally by Shelah in [12], and was explicitly stated in [7], and later in [4]. We use it freely throughout the paper.

Fact 1.3. *A type $p \in S(A)$ is realized in $\mathfrak{C}$ if and only if for every finite $\bar{a} \in A$ the type $p \restriction \bar{a}$ is realized in $\mathfrak{C}$.*

To study totally transcendental homogeneous models, a rank was introduced by Lessmann in [10].

Definition 1.4. Let p be a type over a finite $B \subset |\mathfrak{C}|$.

(1) $R[p] \geq 0$ if p is realized in $\mathfrak{C}$.
(2) for α limit ordinal, $R[p] \geq \alpha$ if $R[p] \geq \beta$ for all $\beta < \alpha$.
(3) $R[p] \geq \alpha + 1$ if
  (a) there are $\varphi(\bar{x}, \bar{y})$ and $\bar{a} \in \mathfrak{C}$ such that
  $$R[p \cup \varphi(\bar{x}, \bar{a})] \geq \alpha \quad \text{and} \quad R[p \cup \neg\varphi(\bar{x}, \bar{a})] \geq \alpha;$$
  (b) for every $\bar{b} \in |\mathfrak{C}|$ there is a complete type $q(\bar{x}, \bar{y}) \in D$ such that $R[p \cup q(\bar{x}, \bar{b})] \geq \alpha$.

As usual,
$R[p] = -1$ if $R[p] \not\geq 0$;
$R[p] = \alpha$ if $R[p] \geq \alpha$ and $R[p] \not\geq \alpha + 1$;
$R[p] = \infty$ if $R[p] \geq \alpha$ for all $\alpha \in \mathrm{On}$;
if q is a type over a subset of $\mathfrak{C}$ which is not necessarily finite, we let

$$R[q] := \mathrm{Min}\{R[p] \mid p \subseteq q, \mathrm{dom}(p) \text{ finite}\}.$$

Fact 1.5 (Properties of the rank)**.**

(1) Invariance: *if* $f \in \mathrm{Aut}(\mathfrak{C})$*, then* $R[p] = R[f(p)]$.
(2) Monotonicity: *If* $p \vdash q$*, then* $R[p] \leq R[q]$.
(3) Finite character: *For any type* q*, there is* $p \subset q$*,* $\mathrm{dom}(p)$ *finite, such that* $R[p] = R[q]$.

Definition 1.6. A homogeneous model $\mathfrak{C}$ is *totally transcendental* if $R[p] < \infty$ for all D-types p.

Fact 1.7 ([10])**.** *If the homogeneous model* $\mathfrak{C}$ *is* λ*-stable for some* $\aleph_0 \leq \lambda < 2^{\aleph_0}$*, then it is totally transcendental.*

While the rank R serves well in the proof of the uncountable categoricity result, its behavior is quite exotic when it comes to measuring complexity of definable sets. Let us show this on a simple example.

Example 1.8. Let us deal with the simplest first order case, so $D = S(\emptyset)$. We claim that for any algebraic type p with more than one realization we have $R[p] = 1$. (The classical intuition is of course that the value of a 2-rank should be $\lfloor \log_2 n \rfloor$, $n = |p(\mathfrak{C})|$.)

Let $p(\mathfrak{C}) = \{a_i | i < n\}$. Consider the tuple $\bar{a} = a_0 \dots a_{n-1}$. Clearly, any complete D-type $q(\bar{x}, \bar{a})$ consistent with p will have just one realization; thus for this particular $\bar{a}$ we have $R[p \cup q(\bar{x}, \bar{a})] = 0$. By the clause (3,(ii)) in the definition of R, necessarily $R[p] \leq 1$.

As we already mentioned, the purpose of the clause (3,(ii)) is to make sure that unboundedness of the rank does lead to the existence of many D-types; this is a key point in the proof of Fact 1.7. In the following section, we define a variant of the 2-rank, R^*, that achieves the same goal, and is better-behaved.

2. The ranks RM and R^*

We start by defining an analog of Morley rank for the homogeneous model case. It is easy to see that our definition agrees with the classical one in the first order case.

Definition 2.1. Let $p(\bar{x})$ be a type over a finite $B \subset |\mathfrak{C}|$. Define $RM[p] \geq \alpha$ by induction.

(1) $RM[p] \geq 0$ if p is realized in $\mathfrak{C}$.
(2) $RM[p] \geq \alpha$, where α is a limit ordinal if $RM[p] \geq \beta$ for every $\beta < \alpha$.
(3) $RM[p] \geq \alpha + 1$, if there exist pairwise contradictory $\{\psi_i(\bar{x}, \bar{a}_i) \mid i < \omega\}$ such that for each $i < \omega$ and each $\bar{b} \in \mathfrak{C}$ there is $q_i(\bar{x}, \bar{b})$ with

$$RM[p(\bar{x}) \cup \{\psi_i(\bar{x}, \bar{a}_i)\} \cup q_i(\bar{x}, \bar{b})] \geq \alpha.$$

The agreements for the notation $RM[p] = -1$, $RM[p] = \alpha$, and $RM[p] = \infty$ are standard; for a type q over a subset of $\mathfrak{C}$ which is not necessarily finite,

$$RM[q] := \text{Min}\{RM[p] \mid p \subseteq q, \text{dom}(p) \text{ finite}\}.$$

Remark 2.2. Whenever $\psi(\bar{x}, \bar{a})$ is such that for each $\bar{b} \in \mathfrak{C}$ there is $q(\bar{x}, \bar{b})$ with $RM[p(\bar{x}) \cup \{\psi(\bar{x}, \bar{a})\} \cup q(\bar{x}, \bar{b})] \geq \alpha$, we say that $\psi(\bar{x}, \bar{a})$ *is a D-extension of p of RM-rank at least α*. This is a slight abuse of the terminology because, technically, $p(\bar{x}) \cup \{\psi(\bar{x}, \bar{a})\}$ is the extension.

Using this terminology, $RM[p] \geq \alpha + 1$ if and only if p has ω-many pairwise contradictory D-extensions of RM-rank at least α.

The usual properties, such as invariance, monotonicity and finite character are easy to establish for RM. We now work towards proving

Theorem (2.9). *Suppose that $RM[p] = \alpha$. Then there is a finite number n such that p has n, but not $n+1$, distinct D-extensions of RM-rank α.*

In other words, multiplicity for Morley rank makes sense in the homogeneous context as well, and if there are arbitrarily large finite number of pairwise contradictory D-extensions of RM-rank α, then we indeed can find an infinite number of pairwise contradictory D-extensions of RM-rank α. We also prove

Theorem (2.11). *The homogeneous model $\mathfrak{C}$ is totally transcendental if and only if $RM[p] < \infty$ for all D-types p.*

In particular, if $\mathfrak{C}$ is $\aleph_0$-stable, the rank RM is bounded.

To prove both results we introduce a version of the 2-rank that we call R^*. The defining property of R^* is that $R^*[p] \geq \alpha + n$ implies that p has at least 2^n contradictory D-extensions of R^*-rank at least α. We show that R^* is bounded if and only if R is; and that $R^*[p] \geq \omega \cdot \alpha$ if and only if $RM[p] \geq \alpha$ for all $\alpha \geq 0$. The two theorems then follow easily.

As a byproduct, we obtain a formula connecting Morley rank and multiplicity with the values of the rank R^*; it is the same formula that ties Morley rank and the 2-rank in the first order case.

We start with the definition of the rank R^*.

Definition 2.3. For a partial type $p(\bar{x}, \bar{b})$, where $\bar{b}$ is finite, define $R^*[p] \geq \alpha$ by induction.

(1) $R^*[p] \geq 0$ if p is realized in $\mathfrak{C}$.
(2) $R^*[p] \geq \alpha$, where α is a limit ordinal, if $R^*[p] \geq \beta$ for every $\beta < \alpha$.
(3) $R^*[p] \geq \alpha + n$, where α is a limit ordinal or 0, if there exist pairwise contradictory $\{\psi_i(\bar{x}, \bar{a}_i) \mid i < 2^n\}$ such that for each $i < 2^n$ and each $\bar{b} \in \mathfrak{C}$ there is a D-type $q_i(\bar{x}, \bar{b})$ with

$$R^*[p(\bar{x}) \cup \{\psi_i(\bar{x}, \bar{a}_i)\} \cup q_i(\bar{x}, \bar{b})] \geq \alpha.$$

As before, for an arbitrary type q we let $R^*[q] := \text{Min}\{R^*[p] \mid p \subseteq q, \text{dom}(p) \text{ finite}\}$.

It is immediate that R^* has the invariance, monotonicity, and finite character properties. The following claim shows that for each of the ranks R, RM, and R^* there is a "critical value", i.e., an ordinal such that the rank is unbounded if and only if it is bigger than the ordinal.

Claim 2.4. *Let $\mathcal{R}$ be any ordinal-valued rank defined on types that satisfies invariance and finite character in the sense of Fact 1.5. Then there is an ordinal $\alpha_{\mathcal{R}}$ such that $\mathcal{R}[p] \geq \alpha$ if and only if $\mathcal{R}[p] = \infty$ for any type p.*

Proof. By invariance, there are at most $|D| + |T|$ possible values for $\mathcal{R}[q]$, $|\text{dom}(q)| < \aleph_0$. So we can let $\alpha_{\mathcal{R}} := \sup\{\mathcal{R}[q] + 1 \mid |\text{dom}(q)| < \aleph_0,\ \mathcal{R}[q] < \infty\}$. Now for any type q over a finite set $\mathcal{R}[q] \geq \alpha_{\mathcal{R}}$ implies $\mathcal{R}[q] = \infty$.

For an arbitrary type p, using finite character find $q \subset p$ with finite domain such that $\mathcal{R}[p] = \mathcal{R}[q]$. Thus $\mathcal{R}[p] \geq \alpha_{\mathcal{R}}$ implies $\mathcal{R}[q] \geq \alpha_R$, and so $\mathcal{R}[p] = \mathcal{R}[q] = \infty$. ⊣

Now we are ready to show that $\mathfrak{C}$ is totally transcendental if and only if R^* is bounded. A direct comparison between the ranks R and R^* (inequality or almost inequality) does not seem feasible.

Lemma 2.5. *For any type p, $R[p] = \infty$ if and only if $R^*[p] = \infty$.*

Proof. The hard direction: we show that for any type p, if $R[p] = \infty$, then $R^*[p] \geq \alpha$ for all $\alpha \in \text{On}$. If $\alpha = 0$, the statement is obvious; for a limit α the implication follows from the induction hypothesis.

Suppose now that the statement is true for some $\alpha \in \mathrm{On}$, and that $R[p] = \infty$. Since R has invariance and finite character properties, there is an ordinal α_R as in Claim 2.4. In particular for any $n < \omega$ we have $R[p] \geq \alpha_R + n + 1$.

Subclaim 2.6. *If $R[p] \geq \alpha_R + n + 1$, then there are 2^n pairwise contradictory formulas $\{\psi_i(\bar{x}, \bar{a}_i) \mid i < 2^n\}$ such that for each $i < 2^n$ and for any $\bar{b} \in \mathfrak{C}$ there is a complete D-type $q_i(\bar{x}, \bar{b})$ with $R[p \cup q_i(\bar{x}, \bar{b}) \cup \{\psi_i(\bar{x}, \bar{a}_i)\}] \geq \alpha_R$.*

Proof of the Subclaim. Use induction on $n < \omega$. If $n = 0$, the statement follows from clause (3) in the definition of R; ψ_0 can be taken as $\bar{x} = \bar{x}$.

For the induction step, if $R[p] \geq \alpha_R + n + 2$, we can find a formula $\varphi(\bar{x}, \bar{a})$ with $R[p \cup \{\pm\varphi(\bar{x}, \bar{a})\}] \geq \alpha_R + n + 1$. By induction hypothesis, there are two sets of 2^n pairwise contradictory formulas $\{\psi_i^\ell(\bar{x}, \bar{a}_i^\ell) \mid i < 2^n\}$, $\ell = 0, 1$, such that for $\ell = 0, 1$ for each $i < 2^n$ and for all $\bar{b} \in \mathfrak{C}$ there is a complete D-type $q_i^\ell(\bar{x}, \bar{b})$ with

$$R[p \cup \{\varphi(\bar{x}, \bar{a})\} \cup q_i^0(\bar{x}, \bar{b}) \cup \{\psi_i^0(\bar{x}, \bar{a}_i^0)\}] \geq \alpha_R,$$
$$R[p \cup \{\neg\varphi(\bar{x}, \bar{a})\} \cup q_i^1(\bar{x}, \bar{b}) \cup \{\psi_i^1(\bar{x}, \bar{a}_i^1)\}] \geq \alpha_R,$$

Now the 2^{n+1} formulas

$$\{\varphi(\bar{x}, \bar{a}) \wedge \psi_i^0(\bar{x}, \bar{a}_i^0), \neg\varphi(\bar{x}, \bar{a}) \wedge \psi_i^1(\bar{x}, \bar{a}_i^1) \mid i < 2^n\}$$

are as needed. $\dashv$

By the choice of α_R and the induction hypothesis, the Subclaim gives that for any $n < \omega$ there are 2^n pairwise contradictory formulas $\{\psi_i(\bar{x}, \bar{a}) \mid i < 2^n\}$ such that for any $\bar{b} \in \mathfrak{C}$ there is a complete D-type $q(\bar{x}, \bar{b})$ with $R^*[p \cup q_i(\bar{x}, \bar{b}) \cup \{\psi_i\}] \geq \alpha$. Thus $R^*[p] \geq \alpha + n$ for all $n < \omega$.

For the converse, suppose $R^*[p] = \infty$, so $R^*[p] \geq \alpha_{R^*} + 1$. An easy inductive argument shows $R[p] \geq \alpha$ for all α. $\dashv$

We now isolate an important property of the rank R^*.

Proposition 2.7 (Weak ultrametric property)**.** *Let α be a limit ordinal or 0. Then $R^*[p] \geq \alpha$ if and only if for any $\varphi(\bar{x}; \bar{a})$ either $R^*[p \cup \{\varphi(\bar{x}; \bar{a})\}] \geq \alpha$ or $R^*[p \cup \{\neg\varphi(\bar{x}; \bar{a})\}] \geq \alpha$.*

Proof. One direction is clear by monotonicity of the rank R^*. For the other, we use induction on α.

The base case $\alpha = 0$ is clear. If α is a limit of limit ordinals, $\alpha = \sup\{\beta \mid \beta < \alpha,\ \beta \text{ is a limit ordinal}\}$, then the statement follows easily from induction hypothesis and the pigeonhole principle.

So let $\alpha = \beta + \omega$, where β is a limit ordinal. It suffices to show that for each $n < \omega$ either $R^*[p \cup \varphi] \geq \beta + n$ or $R^*[p \cup \neg\varphi] \geq \beta + n$.

Fix $n < \omega$. Since $R^*[p] \geq \beta + \omega$, we can choose 2^{n+1} pairwise contradictory extensions of p, each of rank at least β. More precisely, there are pairwise contradictory $\{\psi_i(\bar{x}, \bar{a}_i) \mid i < 2^{n+1}\}$ such that for each $i < 2^{n+1}$ and each $\bar{b} \in \mathfrak{C}$ there is a D-type $q_i^{\bar{b}}(\bar{x}, \bar{b})$ with

$$R^*[p(\bar{x}) \cup \{\psi_i(\bar{x}, \bar{a}_i)\} \cup q_i^{\bar{b}}(\bar{x}, \bar{b})] \geq \beta.$$

By induction hypothesis applied to the type $p(\bar{x}) \cup \{\psi_i(\bar{x}, \bar{a}_i)\} \cup q_i^{\bar{b}}(\bar{x}, \bar{b})$ and formula $\varphi(\bar{x}, \bar{a})$, for each $i < 2^{n+1}$

$$\text{either } R^*[p(\bar{x}) \cup \{\varphi(\bar{x}; \bar{a})\} \cup \{\psi_i(\bar{x}, \bar{a}_i)\} \cup q_i^{\bar{b}}(\bar{x}, \bar{b})] \geq \beta$$
$$\text{or } R^*[p(\bar{x}) \cup \{\neg\varphi(\bar{x}; \bar{a})\} \cup \{\psi_i(\bar{x}, \bar{a}_i)\} \cup q_i^{\bar{b}}(\bar{x}, \bar{b})] \geq \beta.$$

By pigeonhole principle, for each $\bar{b}$ either $p(\bar{x}) \cup \{\varphi(\bar{x}; \bar{a})\}$ or $p(\bar{x}) \cup \{\neg\varphi(\bar{x}; \bar{a})\}$ have at least 2^n pairwise contradictory extensions of rank β, each containing a complete D-type over $\bar{b}$.

Now note that it is impossible to have $\bar{b}_1, \bar{b}_2$ such that $p(\bar{x}) \cup \{\varphi(\bar{x}; \bar{a})\}$ has less than 2^n contradictory extensions of rank β containing a complete D-type over $\bar{b}_1$, while $p(\bar{x}) \cup \{\neg\varphi(\bar{x}; \bar{a})\}$ has less than 2^n contradictory extensions of rank β containing a complete D-type over $\bar{b}_2$. Indeed, letting $\bar{b} := \bar{b}_1 \hat{\ } \bar{b}_2$ we see that either φ or its negation will work for both $\bar{b}_1$ and $\bar{b}_2$.

So we conclude $R^*[p \cup \varphi] \geq \beta + n$ or $R^*[p \cup \neg\varphi] \geq \beta + n$. ⊣

We are now ready to prove the key lemma. The argument for the successor step in the proof of the harder direction shows why (and how) we can find an infinite number of pairwise contradictory D-extensions of RM-rank α given that there are arbitrarily large finite number of pairwise contradictory D-extensions of RM-rank α.

Lemma 2.8. *For any type p we have $R^*[p] \geq \omega \cdot \alpha$ if and only if $RM[p] \geq \alpha$.*

Proof. By induction on α. If $\alpha = 0$ or α is a limit ordinal, the statement is clear.

Suppose now $R^*[q] \geq \omega \cdot \alpha$ if and only if $RM[q] \geq \alpha$ for all q.

($\Rightarrow$) Suppose $R^*[p] \geq \omega \cdot \alpha + \omega$. It is enough to construct $\{\psi_i(\bar{x}, \bar{a}_i) \mid i < \omega\}$ such that

(1) for all $i < \omega$ we have $\psi_i(\bar{x}, \bar{a}_i) \vdash \bigwedge_{j<i} \neg\psi_j(\bar{x}, \bar{a}_j)$;
(2) for each $i < \omega$ and every $\bar{b} \in \mathfrak{C}$ there is a complete D-type $q_i(\bar{x}, \bar{b})$ such that $R^*[p(\bar{x}) \cup \{\psi_i(\bar{x}, \bar{a}_i)\} \cup q_i(\bar{x}, \bar{b})] \geq \omega \cdot \alpha$.
(3) $R^*[p(\bar{x}) \cup \{\bigwedge_{j \leq i} \neg\psi_j(\bar{x}, \bar{a}_j)\}] \geq \omega \cdot \alpha + \omega$ for all $i < \omega$.

Clearly, (1) implies that the formulas are pairwise contradictory, and (2) together with the induction hypothesis give $RM[p] \geq \alpha + 1$.

Now the construction. Since in particular $R^*[p] \geq \omega \cdot \alpha + 1$, there are $\varphi_0(\bar{x}, \bar{c}_0)$ and $\varphi_1(\bar{x}, \bar{c}_1)$ with $\varphi_1(\bar{x}, \bar{c}_1) \vdash \neg\varphi_0(\bar{x}, \bar{c}_0)$ and such that for every $\bar{b} \in \mathfrak{C}$ there is a complete D-type $q_\ell(\bar{x}, \bar{b})$, $\ell = 0, 1$, such that $R^*[p(\bar{x}) \cup \{\varphi_\ell(\bar{x}, \bar{c}_\ell)\} \cup q_\ell(\bar{x}, \bar{b})] \geq \omega \cdot \alpha$. By monotonicity we may assume that $\varphi_1(\bar{x}, \bar{c}_1) = \neg\varphi_0(\bar{x}, \bar{c}_0)$.

Since $\omega \cdot \alpha + \omega$ is a limit ordinal, by the weak ultrametric property we may assume that $R^*[p(\bar{x}) \cup \{\neg\varphi_0(\bar{x}, \bar{c}_0)\}] \geq \omega \cdot \alpha + \omega$. Let $\psi_0(\bar{x}, \bar{a}_0) := \varphi_0(\bar{x}, \bar{c}_0)$.

Now iterate: given $\{\psi_j(\bar{x}, \bar{a}_j) \mid j < i\}$ satisfying (1)–(3), we have $R^*[p(\bar{x}) \cup \{\bigwedge_{j<i} \neg\psi_j(\bar{x}, \bar{a}_j)\}] \geq \omega \cdot \alpha + \omega$, so in particular $R^*[p(\bar{x}) \cup \{\bigwedge_{j<i} \neg\psi_j(\bar{x}, \bar{a}_j)\}] \geq \omega \cdot \alpha + 1$. As before, this means that the set defined by $p(\bar{x}) \cup \{\bigwedge_{j<i} \neg\psi_j(\bar{x}, \bar{a}_j)\}$ can be split by some formula $\varphi_i(\bar{x}, \bar{c}_i)$ in such a way that

- for all $\bar{b} \in \mathfrak{C}$ there is $q(\bar{x}, \bar{b})$ with
$$R^*[p(\bar{x}) \cup \{\bigwedge_{j<i} \neg\psi_j(\bar{x}, \bar{a}_j)\} \cup \{\varphi_i(\bar{x}, \bar{c}_i)\} \cup q(\bar{x}, \bar{b})] \geq \omega \cdot \alpha$$
- and $R^*[p(\bar{x}) \cup \{\bigwedge_{j<i} \neg\psi_j(\bar{x}, \bar{a}_j)\} \cup \{\neg\varphi_i(\bar{x}, \bar{c}_i)\}] \geq \omega \cdot \alpha + \omega$.

Now we let $\psi(\bar{x}, \bar{a}_i) := \varphi_i(\bar{x}, \bar{c}_i) \wedge \bigwedge_{j<i} \neg\psi_j(\bar{x}, \bar{a}_j)$. This meets the conditions (1)–(3).

($\Leftarrow$) Suppose $RM[p] \geq \alpha + 1$. Then there are contradictory $\{\psi_i(\bar{x}, \bar{a}_i) \mid i < \omega\}$ such that for each $i < \omega$ and each $\bar{b} \in \mathfrak{C}$ there is $q_i(\bar{x}, \bar{b})$ with

$$RM[p(\bar{x}) \cup \{\psi_i(\bar{x}, \bar{a}_i)\} \cup q_i(\bar{x}, \bar{b})] \geq \alpha.$$

By induction hypothesis we get for each $i < \omega$

$$R^*[p(\bar{x}) \cup \{\psi_i(\bar{x}, \bar{a}_i)\} \cup q_i(\bar{x}, \bar{b})] \geq \omega \cdot \alpha,$$

so $R^*[p] \geq \omega \cdot \alpha + n$ for all $n < \omega$ and thus $R^*[p] \geq \omega \cdot \alpha + \omega$. ⊣

Now the main results follow:

Theorem 2.9. *Suppose that $RM[p] = \alpha$. Then there is a finite number n such that p has n, but not $n+1$, pairwise contradictory D-extensions of RM-rank α.*

Proof. Suppose not. Then for each $n < \omega$ we have $\{\psi_i^n(\bar{x}, \bar{a}_i^n) \mid i < 2^n\}$ pairwise contradictory D-extensions of RM-rank α. By Lemma 2.8, these are D-extensions of p of R^*-rank $\omega \cdot \alpha$. Thus $R^*[p] \geq \alpha \cdot \omega + n$ for all $n < \omega$. Therefore $R^*[p] \geq \omega \cdot (\alpha + 1)$, and so applying Lemma 2.8 again we get $RM[p] \geq \alpha + 1$, contradiction. ⊣

Notation 2.10. If $RM[p] = \alpha$, we denote the maximal number n of pairwise contradictory D-extensions of RM-rank α given by Theorem 2.9 by $dM[p]$.

Theorem 2.11. *The homogeneous model $\mathfrak{C}$ is totally transcendental if and only if $RM[p] < \infty$ for all D-types p.*

Proof. By Lemma 2.8, RM is bounded if and only if R^* is. So we are done by Lemma 2.5. ⊣

By [10], we immediately get

Corollary 2.12. *If $\mathfrak{C}$ is λ-stable for some $\aleph_0 \leq \lambda < 2^{\aleph_0}$, then RM is bounded.*

We also obtain the ultrametric property for RM.

Corollary 2.13. *Let $\{\psi_i(\bar{x}, \bar{a}_i) \mid i < l\}$ be such that $p \vdash \bigvee_{i<l} \psi_i(\bar{x}, \bar{a}_i)$. Then $RM[p] \geq \alpha$ if and only if for some $i < l$ we have $RM[p \cup \{\psi_i(\bar{x}, \bar{a}_i)\}] \geq \alpha$.*

Proof. If $RM[p \cup \{\psi_i(\bar{x}, \bar{a}_i)\}] \geq \alpha$ for some $i < l$, then $RM[p] \geq \alpha$ by monotonicity.

For the other direction, without loss of generality $l = 2$. Since $p \cup \{\neg\psi_0(\bar{x}, \bar{a}_0)\} \vdash \psi_1(\bar{x}, \bar{a}_1)$, by monotonicity of RM it is enough to prove the statement with $\psi_1(\bar{x}, \bar{a}_1) = \neg\psi_0(\bar{x}, \bar{a}_0)$. By Lemma 2.8, $R^*[p] \geq \omega \cdot \alpha$, so Proposition 2.7 gives $R^*[p \cup \{\psi_i(\bar{x}, \bar{a}_i)\}] \geq \omega \cdot \alpha$ for $i = 0$ or $i = 1$. Applying Lemma 2.8 again, we get $RM[p \cup \{\psi_i(\bar{x}, \bar{a}_i)\}] \geq \alpha$ for $i = 0$ or $i = 1$. ⊣

We finish the section with a formula tying R^* and RM.

Corollary 2.14. *Let $\mathfrak{C}$ be a homogeneous model. Then for any type p we have*

1. *$RM[p] = \infty$ if and only if $R^*[p] = \infty$;*
2. *$RM[p] = \alpha < \infty$ and $2^n \leq dM[p] < 2^{n+1}$ if and only if $R^*[p] = \omega \cdot \alpha + n$.*

Proof. (1) is immediate by Lemma 2.8, and (2) follows from the definitions, Lemma 2.8, and Theorem 2.9. ⊣

3. Relating stationarity and multiplicity

We address stationarity. Throughout this section, $\mathfrak{C}$ is a totally transcendental homogeneous model.

In the homogeneous context, as well as in the case of classes of atomic models, the notion of stationarity is defined through the 2-rank. We use our rank R^*, but the definition comes from [10] and [14].

Definition 3.1. A type p is called *stationary* if for every $A \supset \mathrm{dom}(p)$ there is a unique complete D-type $p_A \supset p$, which is a D-extension of p of rank $\alpha := R^*[p]$.

We start by showing that the types over $(D, \aleph_0)$-homogeneous models are stationary; the proof is along the same lines as the corresponding argument in [10].

Proposition 3.2. *Let $\mathfrak{C}$ be totally transcendental. Let $M \prec \mathfrak{C}$ be a $(D, \aleph_0)$-homogeneous model, $p_M \in S_D(M)$, and $\bar{b} \in M$ such that $R^*[p_{\bar{b}}] = R^*[p_M] = \alpha$, where $p_{\bar{b}} = p \restriction \bar{b}$. Then for every $A \subset \mathfrak{C}$ that contains $\bar{b}$ there is a unique D-extension $p_A \supset p_{\bar{b}}$ of rank α.*

In addition, the extension p_A does not split over $\bar{b}$.

Proof. Let $p_M \in S_D(M)$, let $\bar{d} \models p$, $\bar{d} \in \mathfrak{C}$. By finite character choose $\bar{b} \in M$ such that $R^*[p_M] = R^*[p_{\bar{b}}] = \alpha$.

Subclaim 3.3. *If $\bar{b} \in M$ is a finite subset such that $R^*[p_M] = R^*[p_{\bar{b}}] = \alpha$, then the type p_M does not split over $\bar{b}$.*

Proof. Suppose, for contradiction, that there are $\bar{c}_1, \bar{c}_2 \in M$ such that $\mathrm{tp}(\bar{c}_1/\bar{b}) = \mathrm{tp}(\bar{c}_2/\bar{b})$ and for some formula φ p contains both $\varphi(\bar{x}, \bar{c}_1)$ and $\neg\varphi(\bar{x}, \bar{c}_2)$. To get a contradiction, it is enough to show that $\varphi(\bar{x}, \bar{c}_1)$ and $\neg\varphi(\bar{x}, \bar{c}_1)$ are D-extensions of $p_{\bar{b}}$ of R^*-rank at least α.

Let $\bar{a} \in \mathfrak{C}$. We now find $q_\ell(\bar{x}, \bar{a})$, $\ell = 1, 2$ such that $R^*[p_{\bar{b}}(\bar{x}) \cup \{\varphi(\bar{x}, \bar{c}_1)\} \cup q_1(\bar{x}, \bar{a})] \geq \alpha$ and $R^*[p_{\bar{b}}(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{c}_1)\} \cup q_2(\bar{x}, \bar{a})] \geq \alpha$. By $(D, \aleph_0)$-homogeneity, there is $\bar{a}_1 \in M$ such that $\mathrm{tp}(\bar{a}\bar{b}\bar{c}_1) = \mathrm{tp}(\bar{a}_1\bar{b}\bar{c}_1)$. Let $q_1(\bar{x}, \bar{a}_1) := p_M \restriction \bar{a}_1$. Since $p_M \supset p_{\bar{b}}(\bar{x}) \cup \{\varphi(\bar{x}, \bar{c}_1)\} \cup q_1(\bar{x}, \bar{a}_1)$, by monotonicity and invariance,

$$R^*[p_{\bar{b}}(\bar{x}) \cup \{\varphi(\bar{x}, \bar{c}_1)\} \cup q_1(\bar{x}, \bar{a}_1)] \geq \alpha.$$

To find q_2, let $f \in \mathrm{Aut}_{\bar{b}}(\mathfrak{C})$ be such that $f(\bar{c}_2) = \bar{c}_1$. Let $M' := f(M)$. Then M' is $(D, \aleph_0)$-homogeneous, and $R^*[f(p)] = R^*[p_{\bar{b}}]$ by invariance. Take $\bar{a}_2 \in M'$ such that $\mathrm{tp}(\bar{a}\bar{b}\bar{c}_1) = \mathrm{tp}(\bar{a}_2\bar{b}\bar{c}_1)$. Let $q_2(\bar{x}, \bar{a}_2) := f(p_M) \restriction \bar{a}_2$. Since $f(p_M) \supset p_{\bar{b}}(\bar{x}) \cup \{\neg\varphi(\bar{x}, \bar{c}_1)\} \cup q_2(\bar{x}, \bar{a}_2)$, by monotonicity and invariance we get $R^*[p_{\bar{b}}(\bar{x}) \cup \{\varphi(\bar{x}, \bar{c}_1)\} \cup q_2(\bar{x}, \bar{a}_2)] \geq \alpha$. $\dashv$

Now we show existence. Given $A \subset \mathfrak{C}$, define

$$p_A := \bigcup_{\bar{a} \in A} \{p(\bar{x}, \bar{a}) \mid \exists\, \bar{a}' \in M \text{ such that } \mathrm{tp}(\bar{a}/\bar{b}) = \mathrm{tp}(\bar{a}'/\bar{b}) \text{ and } p(\bar{x}, \bar{a}') \subset p_M(\bar{x})\}.$$

Since p_M does not split over $\bar{b}$, the type p_A is a well-defined D-type over A. By finite character and invariance of R^*, we have $R^*[p_A] =$

$R^*[p_M] = \alpha$. Since the set A is arbitrary, the extension p_A is a D-extension of $p_{\bar{b}}$ of rank α. By construction, the type p_A does not split over $\bar{b}$.

Finally, the uniqueness. If $p_\ell \in S_D(A)$, $\ell = 1, 2$ are distinct D-extensions of $p_{\bar{b}}$ of R^*-rank α, then we have $R^*[p_{\bar{b}}] \geq \alpha + 1$, a contradiction. ⊣

We finish by connecting stationarity in the sense of Definition 3.1 with multiplicity 1 for types over $(D, \aleph_0)$-homogeneous models. We already showed that stationary types have unique D-extensions of the maximal R^*-rank, but this is not enough. We want to rule out the possibility of, for example, $R^*[p] = \omega + 2$, $p \in S_D(M)$. In that case, the type p would be stationary of R^*-rank $\omega + 2$ (by Proposition 3.2); but will have between 4 and 7 D-extensions of RM-rank 1.

To show that this is not possible, we establish

Proposition 3.4. *Suppose M is a $(D, \aleph_0)$-homogeneous model, $p \in S_D(M)$. Then for no $\alpha \geq 0$ we have $R^*[p] = \alpha + 1$.*

Proof. Suppose not. Let M be a $(D, \aleph_0)$-homogeneous model and let $p \in S_D(M)$ be such that $R^*[p] = \alpha + 1$ for some α. Then p has at least two (and no more than three) D-extensions $\varphi_1(\bar{x}, c_1)$, $\neg\varphi_1(\bar{x}, \bar{c}_1)$ of R^*-rank α (as before, we may assume that one D-extension is the negation of the other by monotonicity). By Proposition 3.2, since $R^*[p] = \alpha + 1$, there is a complete type $p_1 \in S_D(M\bar{c}_1)$ which is a D-extension of p of R^*-rank $\alpha + 1$. We may assume that $\varphi_1(\bar{x}, \bar{c}_1) \in p_1(\bar{x})$.

By definition of R^* we can find two more D-extensions of R^*-rank α of the type p_1, $\varphi_2(\bar{x}, \bar{c}_2)$ and $\neg\varphi_2(\bar{x}, \bar{c}_2)$. Repeating the steps above, we get a complete type $p_2(x)$ that contains $\varphi_2(\bar{x}, \bar{c}_2)$ and which is the unique D-extension of p, and therefore p_1, of R^*-rank $\alpha + 1$.

Finally, we now find two D-extensions, $\varphi_3(\bar{x}, \bar{c}_3)$ and $\neg\varphi_3(\bar{x}, \bar{c}_3)$, of R^*-rank α of the type p_2. So now we have constructed four contradictory D-extensions of the type p, each of R^*-rank α. Namely, we have

$$\neg\varphi_1(\bar{x}, \bar{c}_1), \quad \varphi_1(\bar{x}, \bar{c}_1) \wedge \neg\varphi_2(\bar{x}, \bar{c}_2),$$
$$\varphi_1(\bar{x}, \bar{c}_1) \wedge \varphi_2(\bar{x}, \bar{c}_2) \wedge \neg\varphi_3(\bar{x}, \bar{c}_3), \quad \text{and} \quad \bigwedge_{i=1}^{3} \varphi_i(\bar{x}, \bar{c}_i).$$

Thus, $R^*[p] \geq \alpha + 2$, contradiction. ⊣

Thus, for a type p over a $(D, \aleph_0)$-homogeneous model, the value $R^*[p]$ has to be either 0 or a limit ordinal. So we obtain

Corollary 3.5. *Let $\mathfrak{C}$ be a totally transcendental homogeneous model, let $M \prec \mathfrak{C}$ be $(D, \aleph_0)$-homogeneous. For any $p \in S(M)$, the multiplicity $dM[p]$ is equal to 1.*

Proof. Immediate by Proposition 3.4 and Corollary 2.14. $\dashv$

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