GROUPOIDS, COVERS, AND 3-UNIQUENESS IN STABLE THEORIES

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Abstract. Building on Hrushovski's work in [5], we study definable groupoids in stable theories and their relationship with 3-uniqueness and finite internal covers. We introduce the notion of retractability of a definable groupoid (which is slightly stronger than Hrushovski's notion of eliminability), give some criteria for when groupoids are retractable, and show how retractability relates to both 3-uniqueness and the splitness of finite internal covers. One application we give is a new direct method of constructing non-eliminable groupoids from witnesses to the failure of 3-uniqueness. Another application is a proof that any finite internal cover of a stable theory with a centerless liaison groupoid is almost split.

Introduction

This paper explores connections between the uniqueness of a solution to a 3-amalgamation problem in a stable theory, finite internal covers, and groupoids definable in models of the theory.

A *groupoid* is a category in which every morphism is invertible. A good example to keep in mind is that of a fundamental groupoid of a topological space. In such groupoids, objects are points in the topological space and morphisms are homotopy classes of paths connecting the points. In particular, morphisms from a point to itself are homotopy classes of loops; they form a group under the operation of composition. If the space is path-connected, then the groups of loops are isomorphic. The groupoids constructed in this paper are very similar in nature, but with one important distinction: all the automorphism groups of objects are finite.

For a stable theory, the uniqueness of a solution to a 3-amalgamation problem (we abbreviate this as 3-*uniqueness*) means the following. Let $\{\overline{a}_0, \overline{a}_1, \overline{a}_2\}$ be algebraically closed sets, independent over a common subset A. Denote by $\overline{a}_{ij}$ the algebraic closure of $\overline{a}_i\overline{a}_j$, for $i < j < 3$, considered as an infinite tuple with some (arbitrary) well-ordering. Let σ_{ij} be an automorphism of the monster model that fixes $\overline{a}_i \cup \overline{a}_j$ pointwise (so σ_{ij} fixes A), but does not necessarily fix $\overline{a}_{ij}$ pointwise. We say that 3-uniqueness holds if for all such $\{\overline{a}_0, \overline{a}_1, \overline{a}_2\}$ and A we have

$$\text{tp}(\overline{a}_{01}\overline{a}_{12}\overline{a}_{02}) = \text{tp}(\sigma_{01}(\overline{a}_{01})\sigma_{12}(\overline{a}_{12})\sigma_{02}(\overline{a}_{02})).$$

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Stationarity implies that we may take all but one of the σ_{ij} to be the identity maps.

The property of 3-uniqueness is one of the properties in the family of generalized amalgamation properties: n-existence and n-uniqueness for $n \geq 2$. For first-order theories, the 2-existence property is the extension property of non-forking and 2-uniqueness is stationarity. So 2-uniqueness holds in stable theories. However, 3-uniqueness need not hold in a stable theory. In fact, we present a family of examples of almost strongly minimal theories that fail to have 3-uniqueness.

Our paper builds on the work of Hrushovski [5]. In that manuscript, Hrushovski shows, among many other things, that a stable theory T has the 3-uniqueness property if and only if every groupoid definable in the models of T is *eliminable*. The precise definition of eliminability is given in Definition 1.24; essentially it means that there is a definable full faithful functor from the groupoid to a (possibly, larger) groupoid that contains a definable object. Hrushovski shows that failure of 3-uniqueness implies that T has a finite internal cover which is not split. A separate construction connects any internal cover with its *liaison groupoid*; and it is shown that a liaison groupoid is equivalent to a group if and only if the corresponding cover is split. However, internality of the cover is shown by an indirect argument, so it is not clear how to obtain a definable groupoid from failure of 3-uniqueness.

In this paper, we present a direct construction of a non-eliminable groupoid witnessing failure of 3-uniqueness. Our idea is to construct these groupoids from "paths" in close analogy to fundamental groupoids from algebraic topology, and we show that any sufficiently symmetric witness to the failure of 3-uniqueness can be embedded into such a groupoid. We are optimistic that this construction may be generalizable to higher dimensions, so that failures of n-uniqueness would correspond to "non-eliminable $(n-2)$-groupoids" (which would be certain kinds of $(n-2)$-categories).

We introduce the notion of retractability for definable groupoids. The notion is closely related to, but is stronger than, eliminability. Essentially, a groupoid is retractable over a set A if there is an A-definable functor F from the groupoid to a groupoid with a single object. Alternatively, a groupoid is retractable if there exists an A-definable commuting system of morphisms between all the objects. One can think of the commuting system as the inverse image of the identity morphism under the functor F, and we show how to construct such a functor from a commuting system of morphisms in Lemma 1.25.

We find that retractability of a groupoid is somewhat easier to visualize than eliminability. Retractability of a finitary groupoid gives rise to a finite equivalence relation on the set of morphisms. This is a key property used in the proof of Theorem 3.11.

We obtain the following characterizations of eliminability and retractability that help to connect the notions. If $\mathcal{G}$ is a finitary groupoid defined over an algebraically closed set A then:

$$\mathcal{G} \text{ is retractable over } A \iff \forall a \underset{A}{\perp} b,\ \mathrm{Mor}(a,b) \cap \mathrm{dcl}_A(a,b) \neq \emptyset,$$

$$\mathcal{G} \text{ is eliminable over } A \iff \forall a \underset{A}{\perp} b,\ \mathrm{Mor}(a,b) \cap \mathrm{dcl}_A(\mathrm{acl}_A(a), \mathrm{acl}_A(b)) \neq \emptyset,$$

where a and b are objects of $\mathcal{G}$.

We show in Section 4 that not every eliminable groupoid is retractable; but the characterizations suggest that an eliminable groupoid can be "covered" by a family of retractable ones if we add a finite number of points to objects. This is the motivation behind our definition of a partial sentient cover of a groupoid (Definition 1.29).

Using our analysis of retractable groupoids in stable theories, we derive several new results. If $\mathcal{G}$ is a groupoid and a is an object of $\mathcal{G}$, let G_a be the automorphism group of a in $\mathcal{G}$. We show that if $\mathcal{G}$ is any connected type-definable groupoid such that for any object a, G_a is finite and contained in $\mathrm{dcl}(a)$, then the quotient of $\mathcal{G}$ by its center is almost retractable (i.e. retractable over algebraic parameters). An immediate corollary of this is that if $\mathcal{G}$ is a finitary groupoid as above such that G_a is centerless, then $\mathcal{G}$ itself is almost retractable (Corollary 1.21). In particular, the non-eliminable groupoids arising from a failure of 3-uniqueness have nontrivial centers. We also show that if M is a model of a stable theory and N is a finite internal cover of M such that the kernel $\mathrm{Aut}(N/M)$ is centerless, then N is an almost split cover of M (Theorem 3.11).

The paper is organized as follows. Section 1 is about the general theory of definable groupoids, and here we define key notions such as retractability, eliminability, and sentience. We give several criteria for retractability (especially in a stable theory), show how it is related to Hrushovski's eliminability, and prove that retractability can often be obtained by quotienting by the center.

In Section 2 we give an explicit construction of a non-eliminable groupoid that witnesses the failure of 3-uniqueness in a stable theory. This is done in several stages: first we show that the failure of 3-uniqueness property gives rise to a certain configuration of elements in the monster model of the theory called here *symmetric witness* to the failure. The symmetric witness is then used to define the set of objects and the notion of a path between objects. Certain equivalence classes of paths form the morphisms of the groupoid. We show that the groupoid is not eliminable; combining with results from Section 1, we see that the automorphism group of objects of such a groupoid has to have a non-trivial center.

The main result of Section 3 is Theorem 3.11 mentioned above. This theorem appears to be similar to Lemma 1.2 of [3]; we point out the differences between the results in Section 3. We state the definitions of a finite internal cover, notions of a split cover and a liaison groupoid of a cover. All these definitions are taken from Hrushovski's [5].

Finally, Section 4 gives some examples of groupoids. For instance, we show that eliminability is strictly weaker than retractability, and give a general scheme for building non-retractable groupoids. For any finite group G, we build a totally categorical almost strongly minimal theory with a connected definable groupoid $\mathcal{G}$ such that $G_a \cong G$ for any $a \in \mathrm{Ob}(\mathcal{G})$. Moreover, the groupoid $\mathcal{G}$ can be chosen so that $G_a \subseteq \mathrm{dcl}(a)$. We show that if G has nontrivial center, then $\mathcal{G}$ is non-retractable, establishing that Corollary 1.21 is optimal in a certain sense.

We use standard notation and terminology from model theory and stability theory; see [6] for background in stability theory. Throughout, "definable" means "definable over $\emptyset$," unless otherwise specified. All of our definability results have obvious generalizations to the case where "$\emptyset$" is replaced by "A" in both the hypothesis and the conclusion, where A is any small subset of the monster model $\mathfrak{C}$. We assume that we are dealing with a stable theory T (the few exceptions at the beginning of Section 1 are explicitly noted); we also assume that $T = T^{eq}$ throughout.

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1. Retractability of definable groupoids

In this section, we develop the general theory of retractable and eliminable groupoids in a stable theory. We prove some characterizations of retractability in Lemma 1.12, Proposition 1.14, and Proposition 1.16 and new characterizations of eliminability in Lemma 1.26 and Theorem 1.32. In particular, the last result shows that any almost eliminable groupoid can be modified cosmetically to make it almost retractable by taking a finite cover of its objects – this is the role of sentient cover described in Definition 1.30. Aside from these basic results, the most striking new result here is that any connected finitary sentient groupoid can be made almost retractable by quotienting by the center (Proposition 1.18).

We start with some basic definitions. The definitions make sense in an arbitrary first-order theory.

Definition 1.1 (Groupoid)**.** A *groupoid* $\mathcal{G} = (\mathrm{Ob}(\mathcal{G}), \mathrm{Mor}(\mathcal{G}))$ is a non-empty category in which every morphism is invertible. In addition to the set of objects $\mathrm{Ob}(\mathcal{G})$ and the set of morphisms $\mathrm{Mor}(\mathcal{G})$, groupoids come equipped with the following structure: the (partial) *composition* operation on morphisms, the *domain and range maps* (for each morphism, giving its domain and the range), and the *identity map* which, for each $a \in \mathrm{Ob}(\mathcal{G})$, gives the identity morphism $\mathrm{id}(a)\colon a \to a$.

Definition 1.2 (Subgroupoid)**.** A *subgroupoid* of a groupoid $\mathcal{G}$ is a subcategory which is also a groupoid. A *full subgroupoid* is a subgroupoid $\mathcal{G}^0$ such that for any $a, b \in \mathrm{Ob}(\mathcal{G}^0)$, $\mathrm{Mor}_{\mathcal{G}_0}(a, b) = \mathrm{Mor}_{\mathcal{G}}(a, b)$.

Definition 1.3 (Connected groupoid)**.** A groupoid is *connected* if for any two of its objects it contains a morphism that goes between those objects. (In other words, any two objects are isomorphic.)

If $\mathcal{G}$ is a groupoid and $a \in \mathrm{Ob}(\mathcal{G})$, then G_a denotes the group $\mathrm{Mor}_{\mathcal{G}}(a, a)$.

Definition 1.4 (Finitary groupoid)**.** A connected groupoid $\mathcal{G}$ is called *finitary* if every (equivalently, some) set G_a is finite.

Next we consider definability properties of groupoids. From this point on, we assume that we are working in some fixed complete theory T (so "definable" means "definable in T").

Definition 1.5 (Definable groupoid)**.**
(1) Let $\mathcal{G} = (\mathrm{Ob}(\mathcal{G}), \mathrm{Mor}(\mathcal{G}))$ be a groupoid. We say that $\mathcal{G}$ is *definable* if the sets $\mathrm{Ob}(\mathcal{G})$ and $\mathrm{Mor}(\mathcal{G})$ are definable, as well as the composition operation "$\circ$" and the domain, range and identity maps (respectively denoted by i_0, i_1, and id).
(2) More generally, $\mathcal{G}$ is type-definable if these sets and maps are type-definable.

Remark 1.6. In a type-definable groupoid $\mathcal{G}$, the operations $\circ$, $^{-1}$ (inversion of morphisms), id, i_0, and i_1 are relatively definable, by a simple compactness argument. If in addition the groupoid $\mathcal{G}$ is connected and finitary, then the morphisms are relatively definable. That is, there is a formula $\theta(z; x, y)$ such that for all $a, b \in \mathrm{Ob}(\mathcal{G})$, we have $f \in \mathrm{Mor}(a, b)$ if and only if $f \models \theta(z; a, b)$.

The following modification of Lemma 1.4 from [5] is often useful:

Fact 1.7. *Suppose that $\mathcal{G}^0$ is a connected finitary type-definable groupoid. Then there is a connected definable groupoid $\mathcal{G}$ such that $\mathcal{G}^0$ is a full subgroupoid of $\mathcal{G}$.*

1.1. **Retractable groupoids.** The results of this section hold in an arbitrary first order theory T.

Definition 1.8. If $\mathcal{G}$ is a type-definable connected groupoid, then $\mathcal{G}$ is *retractable* (*over* A) if there is a relatively (A-)definable set $\{f_{ab} : a, b \in \mathrm{Ob}(\mathcal{G})\}$ such that $f_{ab} \in \mathrm{Mor}_{\mathcal{G}}(a, b)$ and $f_{bc} \circ f_{ab} = f_{ac}$ for all $a, b, c \in \mathrm{Ob}(\mathcal{G})$.

We say that $\mathcal{G}$ is *almost retractable over A* if there is such a system definable over $\mathrm{acl}\,(A)$.

Remark 1.9. Connected groupoids seem to be the ones that are relevant to amalgamation problems, but one could define a groupoid with multiple isomorphism classes to be "locally retractable" if and only if each isomorphism class is retractable.

The condition $f_{bc} \circ f_{ab} = f_{ac}$ implies in particular that each morphism f_{aa} is the identity.

An easy modification of the argument in Lemma 1.4 from [5] shows that the property of being retractable can be passed from a type-definable groupoid to a definable groupoid:

Proposition 1.10. *Suppose that $\mathcal{G}^0$ is a connected finitary type-definable groupoid. Suppose in addition that $\mathcal{G}^0$ is retractable. Then there is a connected definable* retractable *groupoid $\mathcal{G}$ such that $\mathcal{G}^0$ is a full subgroupoid of $\mathcal{G}$.*

Proof. If $\mathcal{G}$ is retractable by the relatively definable system of maps f_{ab}, then we can pick S_0 in the proof of Lemma 1.4 in [5] so that $S_0(x_1) \wedge S_0(x_2)$ implies that $f_{x_1x_2}$ is well-defined and $\{f_{ab} : \models S_0(a) \wedge S_0(b)\}$ satisfies the three axioms of being a witness to retractability. □

Proposition 1.11. *Let $\mathcal{G}$ be a type-definable connected finitary groupoid. Then the following are equivalent:*

(1) *$\mathcal{G}$ is retractable;*
(2) *There is a definable groupoid $\mathcal{G}'$ with a single object and a* full*, faithful, relatively definable functor $F: \mathcal{G} \to \mathcal{G}'$;*
(3) *There is a definable groupoid $\mathcal{G}'$ with a single object and a faithful, relatively definable functor $F: \mathcal{G} \to \mathcal{G}'$.*

Proof. $1 \Rightarrow 2$: First, note that we can use Proposition 1.10 to find a definable groupoid $\mathcal{G}^1$ which is still retractable, and if we can obtain the desired F for $\mathcal{G}^1$, then $F \restriction \mathcal{G}$ is the relatively definable functor we want. So without loss of generality, $\mathcal{G}$ is definable.

Let

$$X = \{(a, f) : a \in \mathrm{Ob}(\mathcal{G}) \text{ and } f \in \mathrm{Mor}_{\mathcal{G}}(a, a)\},$$

and let E be the equivalence relation on X defined by

$$(a, f)E(b, g) \Leftrightarrow g = f_{ab} \circ f \circ f_{ba},$$

where f_{ab} and f_{ba} are in the commuting system of morphisms given by retractability.

Pick some definable point $*$ in a single-element sort (in T^{eq}). Then we define $\mathcal{G}'$ so that $\mathrm{Ob}(G') = \{*\}$ and $\mathrm{Mor}_{\mathcal{G}'}(*, *)$ is the set X/E with the natural notion of composition. The desired functor $F: \mathcal{G} \to \mathcal{G}'$ simply collapses all objects to $*$ and quotients arrows by E.

$2 \Rightarrow 3$: Trivial.

$3 \Rightarrow 1$: Given the functor F, faithfulness implies that for any $a, b \in \mathrm{Ob}(\mathcal{G})$, there is a unique arrow $f_{ab} \in \mathrm{Mor}_{\mathcal{G}}(a, b)$ such that $F(f_{ab})$ is the identity map; then the system $\{f_{ab} : a, b \in \mathrm{Ob}(\mathcal{G})\}$ witnesses retractability. □

The next lemma will be useful in proving retractability from "generic retractability."

Lemma 1.12. *Suppose the finitary connected groupoid $\mathcal{G}$ is type-definable and that $\mathcal{G}$ is covered by a family $\{\mathcal{G}_i : i \in I\}$ of type-definable retractable groupoids; that is,*

(1) *each $\mathcal{G}_i$ is type-definable and retractable;*
(2) *each $\mathcal{G}_i$ is a full subgroupoid of $\mathcal{G}$; and*
(3) $\mathrm{Ob}(\mathcal{G}) = \bigcup_{i \in I} \mathrm{Ob}(\mathcal{G}_i)$.

Then $\mathcal{G}$ is almost retractable.

Proof. For each $i \in I$, let $p_i(x)$ be a partial type (over $\emptyset$) that defines $\mathrm{Ob}(\mathcal{G}_i)$, and let $\varphi_i(y; x_0x_1)$ be a formula witnessing the retractability of $\mathcal{G}_i$, so that $\{\varphi_i(y; a, b) : a, b \models p_i\}$ defines a commuting family of morphisms in $\mathcal{G}_i$. By compactness and the fact that $\mathcal{G}$ is finitary, we may assume that the sets $\mathrm{Mor}_{\mathcal{G}}(a, b)$ are uniformly definable by some formula $\psi(y; ab)$. Furthermore, we can find formulas $\theta_i(x)$ implied by $p_i(x)$ such that:

(1) $\theta_i(x_0) \wedge \theta_i(x_1) \vdash \exists! y\, [\varphi_i(y; x_0x_1)]$;
(2) $\theta_i(x_0) \wedge \theta_i(x_1) \wedge \varphi_i(y; x_0x_1) \vdash \psi(y; x_0x_1)$;
(3) $\theta_i(x_0) \wedge \varphi_i(y; x_0x_0) \vdash y = \mathrm{id}(x_0)$; and
(4) $\theta_i(x_0) \wedge \theta_i(x_1) \wedge \theta_i(x_2) \wedge \varphi_i(y_0; x_0x_1) \wedge \varphi_i(y_1; x_1x_2) \vdash \varphi_i(y_1 \circ y_0; x_0x_2)$.

By compactness again, $p(\mathfrak{C})$ is equal to the union of a finite number of the sets $\theta_i(\mathfrak{C})$, say $\theta_0(\mathfrak{C}), \ldots, \theta_{n-1}(\mathfrak{C})$, and we may further assume that these sets are nonempty and pairwise disjoint.

Next, let $S = \{a_0, \ldots, a_{n-1}\}$ be a finite set of points in $\mathrm{dcl}^{eq}(\emptyset)$, and let $F_0 \colon \mathrm{Ob}(\mathcal{G}) \to S$ be the function that sends $a \in \theta_i(\mathfrak{C})$ to a_i. Then there is a definable groupoid $\mathcal{G}'$ such that $\mathrm{Ob}(\mathcal{G}') = S$ and F_0 is extendable to a faithful functor $F \colon \mathcal{G} \to \mathcal{G}'$, where F sends any c realizing $\varphi_i(y; ab)$ to id_{a_i}. Finally, note that since $\mathcal{G}'$ is definable and has only finitely many objects and morphisms, there is a faithful, $\mathrm{acl}(\emptyset)$-definable functor $F' \colon \mathcal{G}' \to \mathcal{G}''$ onto a groupoid $\mathcal{G}''$ with only one object. Thus $F' \circ F \colon \mathcal{G} \to \mathcal{G}''$ is faithful and $\mathrm{acl}(\emptyset)$-definable, and by Proposition 1.11, $\mathcal{G}$ is retractable over $\mathrm{acl}(\emptyset)$. □

1.2. **Characterization of retractability.** Throughout this section (and in fact everywhere from this point on), we assume that we are working in a stable theory T. If $p(x)$ is a complete, stationary type over A, by $p^{(2)}(x, y)$ we denote the type in $S(A)$ such that for all $(a, b) \models p^{(2)}(x, y)$ we have $a, b \models p$ and $a \mathop{\smile\!\!\!\!\!\!|}_A b$.

Definition 1.13 (Generic retractability)**.** A type-definable groupoid $\mathcal{G}$ is *generically retractable* if $\mathcal{G}$ is connected and for every non-algebraic type $p(\overline{x}) \in S(\mathrm{acl}(\emptyset))$ that extends the type defining $\mathrm{Ob}(\mathcal{G})$, there is a relatively definable system of morphisms $\left\{ f^p_{ab} \in \mathrm{Mor}_{\mathcal{G}}(a, b) : (a, b) \models p^{(2)} \right\}$ that "commutes generically:" that is, for any $(a, b, c) \models p^{(3)}$,

$$f^p_{bc} \circ f^p_{ab} = f^p_{ac}.$$

Proposition 1.14. *If $\mathcal{G}$ is type-definable, finitary, and generically retractable, then $\mathcal{G}$ is almost retractable.*

Proof. For any type $p(x) \in S(\mathrm{acl}(\emptyset))$ that extends the type defining $\mathrm{Ob}(\mathcal{G})$, let $\mathcal{G}_p$ be $\mathcal{G}$ restricted to the set of objects satisfying p. Note that if p is an algebraic type, then $\mathcal{G}_p$ has only finitely many objects and arrows and is therefore trivially retractable over $\mathrm{acl}(\emptyset)$. So by Lemma 1.12, we may assume that $\mathcal{G} = \mathcal{G}_p$ for some non-algebraic type p, and we will omit the superscript p from the f_{ab}'s.

Let $\phi_{ab}(x)$ be a formula that defines f_{ab} for $(a, b) \models p^{(2)}$.

Claim 1.15. *If $(a, b) \models p^{(2)}$, then $f_{ba} = f_{ab}^{-1}$.*

Proof. Pick c realizing $p|(a, b)$. Then since $\mathrm{tp}(c/a) = \mathrm{tp}(b/a)$,

$$f_{ba} \circ f_{ab} = f_{ca} \circ f_{ac}.$$

Therefore we can compute:

$$f_{ab} \circ f_{ba} = f_{ab} \circ f_{ca} \circ f_{bc} = f_{ab} \circ f_{ca} \circ f_{ac} \circ f_{ba}$$
$$= f_{ab} \circ f_{ba} \circ f_{ab} \circ f_{ba} = (f_{ab} \circ f_{ba})^2,$$

so $f_{ab} \circ f_{ba} = \mathrm{id}_a$. □

Now for arbitrary (not necessarily independent) $a, b \in \mathrm{Ob}(\mathcal{G})$, we define g_{ab} to be the element $f_{cb} \circ f_{ac}$, where c is some (any) realization of p independent from ab. Again, by definability of types, this is a well-defined, definable family of morphisms, and the identity $g_{aa} = \mathrm{id}_a$ follows from the previous claim, so all that remains is to check that $g_{bc} \circ g_{ab} = g_{ac}$.

So suppose that $\{a, b, c\}$ is an arbitrary set of realizations of p. Pick $d \models p|(abc)$ and $e \models p|(abcd)$. Then

$$g_{bc} \circ g_{ab} = g_{bc} \circ f_{db} \circ f_{ad} = f_{ec} \circ f_{be} \circ f_{db} \circ f_{ad}$$
$$= f_{ec} \circ f_{de} \circ f_{ad} = f_{ec} \circ f_{ae} = g_{ac},$$

as required.

□

As an application of the last proposition, we have:

Proposition 1.16. *Let $\mathcal{G}$ be a connected finitary type-definable groupoid. Then $\mathcal{G}$ is almost retractable if and only if the following holds:*

(†) *For any independent $a, b \in \mathrm{Ob}(\mathcal{G})$, $\mathrm{Mor}_{\mathcal{G}}(a, b) \cap \mathrm{dcl}(a, b) \neq \emptyset$.*

Proof. Suppose the condition (†) holds. We show that $\mathcal{G}$ is retractable. As in Proposition 1.14, we may assume that the objects of $\mathcal{G}$ satisfy the same complete type q over $\mathrm{acl}(\emptyset)$.

By the assumption of our proposition and stationarity of q, for any two independent $a, b \in \mathrm{Ob}(\mathcal{G})$, there are formulas $\{\phi_i(x, a, b) : i < n\}$ defining all the morphisms in the set $\mathrm{Mor}_{\mathcal{G}}(a, b) \cap \mathrm{dcl}(a, b)$, where n is the number of elements in the set.

Claim 1.17. *There is a number $i^* < n$ such that for all independent a, b, and c, if $f_{ab} \models \phi_{i^*}(x, a, b)$ and $f_{bc} \models \phi_{i^*}(x, b, c)$, then $f_{bc} \circ f_{ab} \models \phi_{i^*}(x, a, c)$.*

Proof. By definability of types, whenever $\{a, b, c\}$ is an independent set of objects of $\mathcal{G}$, if $f \in \mathrm{Mor}(a, b)$ is definable over ab and $g \in \mathrm{Mor}(b, c)$ is definable over bc, then $g \circ f$ is definable over ac.

Since the composition in $\mathcal{G}$ is (relatively) definable, the definability of morphisms gives a well-defined function $o\colon n \times n \to n$, where for $i, j < n$ the function $o(i, j)$ gives the index of the definable morphism given by the composition of the ith definable morphism from a to b with the jth definable morphism from b to c. For simplicity of notation, we will use multiplicative notation for the function o and write $i * j = k$ if $o(i, j) = k$.

Pick $i < n$. Since the composition in $\mathcal{G}$ is associative, the "power" of i (i.e., the $*$-product of i with itself) is well-defined. Since $\mathrm{Mor}(a, b)$ is finite for all a, b, there are natural numbers m and p such that $i^m = i^{m+p}$. Let k be such that $kp > m$. Then $(i^{kp})^2 = i^{kp+kp} = i^{kp}$. Thus, if $i^* := i^{kp}$, then we have that the composition of the morphisms given by the formula ϕ_{i^*} again satisfies the formula ϕ_{i^*}. □

So the formula $\phi := \phi_{i^*}$ witnesses the fact that $\mathcal{G}$ is generically retractable. By Proposition 1.14, $\mathcal{G}$ is retractable over $\mathrm{acl}(\emptyset)$.

Conversely, if $\mathcal{G}$ is retractable, then for all $a, b \in \mathrm{Ob}(\mathcal{G})$ we have $f_{ab} \in \mathrm{Mor}(a, b) \cap \mathrm{dcl}(a, b)$, so (†) holds. □

1.3. **Quotienting by the center to obtain a retractable groupoid.** Suppose $\mathcal{G}$ is a connected groupoid, G_a is the group $\mathrm{Mor}_{\mathcal{G}}(a, a)$. Since $\mathcal{G}$ is connected, for any two objects a, b of $\mathcal{G}$ the groups G_a and G_b are isomorphic. Let $Z(G_a)$ be the center of G_a. Then for any $f \in \mathrm{Mor}_{\mathcal{G}}(a, b)$, conjugation by f gives an isomorphism between $Z(G_a)$ and $Z(G_b)$. This allows us to obtain a quotient groupoid $\mathcal{G}/Z(\mathcal{G})$ (see the first section of [5] for more details).

Proposition 1.18. *Let $\mathcal{G}$ be a type-definable connected finitary groupoid such that for any $a \in \mathrm{Ob}(\mathcal{G})$, $G_a \subseteq \mathrm{dcl}(a)$. Then $\mathcal{G}/Z(\mathcal{G})$ is almost retractable.*

Proof. By Lemma 1.12, it suffices to prove the proposition in the case where any two $a, b \in \mathrm{Ob}(\mathcal{G})$ satisfy the same type $p \in S(\mathrm{acl}(\emptyset))$. Pick an arbitrary $a \in \mathrm{Ob}(\mathcal{G})$; let n be the size of G_a. Since $G_a \subseteq \mathrm{dcl}(a)$, there are formulas $\{\varphi_i^a(x, a) : i < n\}$ such that each $\varphi_i^a(x, a)$ has a unique realization c_i^a and $\{c_i^a : i < n\} = G_a$. Note that since any two objects a, b in $\mathcal{G}$ satisfy the same complete type, we may assume that $\varphi_i^a(x, y) = \varphi_i^b(x, y)$, and furthermore the map $\psi_{ab}\colon G_a \to G_b$ that sends c_i^a to c_i^b is a group isomorphism.

For any $f \in \mathrm{Mor}_{\mathcal{G}}(a, b)$, let $f_*\colon G_a \to G_b$ be the group isomorphism defined by $f_*(x) = f \circ x \circ f^{-1}$.

Claim 1.19. *For any two independent $a, b \in \mathrm{Ob}(\mathcal{G})$, there is an arrow $f \in \mathrm{Mor}_{\mathcal{G}}(a, b)$ such that $\psi_{ab} = f_*$.*

Proof. First note that we can quotient the system of G_a's by the maps ψ_{ab} to obtain a group G that is isomorphic to G_a by a system of maps $\psi_a\colon G_a \to G$, and $\psi_b \circ \psi_{ab} = \psi_a$. For any $f \in \mathrm{Mor}_{\mathcal{G}}(a, b)$, let $\widehat{f} = \psi_b \circ f_* \circ \psi_a^{-1}$. So $\widehat{f} \in \mathrm{Aut}(G)$, and a simple calculation shows that $\widehat{f \circ g} = \widehat{f} \circ \widehat{g}$.

Stationarity implies that for any $(c, d) \models p^{(2)}$ and any $f \in \mathrm{Mor}_{\mathcal{G}}(a, b)$, there is a $g \in \mathrm{Mor}_{\mathcal{G}}(c, d)$ such that $g_* cd \equiv f_* ab$, and hence $\widehat{g} = \widehat{f}$. So if we pick any

$g \in \mathrm{Mor}_{\mathcal{G}}(a,b)$ and let k be the order of $\widehat{g}$, we can pick morphisms $g_1, \ldots, g_k$ over generic objects such that $g_1 \circ \ldots \circ g_k \in \mathrm{Mor}_{\mathcal{G}}(a,b)$ and $\widehat{g_i} = \widehat{g}$. Then

$$g_1 \widehat{\circ \ldots \circ} g_k = \widehat{g_1} \circ \ldots \circ \widehat{g_k} = \widehat{g}^k = 1.$$

Let $f = g_1 \circ \ldots \circ g_k$. Then since $\psi_b \circ \psi_{ab} \circ \psi_a^{-1}$ and $\psi_b \circ f_* \circ \psi_a^{-1}$ are both the identity map, $f_* = \psi_{ab}$.

□

Fix $(a,b) \models p^{(2)}$. While the map ψ_{ab} is certainly definable over ab, the arrow $f \in \mathrm{Mor}_{\mathcal{G}}(a,b)$ given by the claim above might not be, since there may be $g \neq f$ such that $g_* = f_*$. But $g_* = f_*$ if and only if $g^{-1} \circ f \in Z_a(\mathcal{G})$, so $\mathcal{G}/Z(\mathcal{G})$ is generically retractable. By Proposition 1.14, $\mathcal{G}/Z(\mathcal{G})$ is almost retractable.

□

Remark 1.20. In the last proposition, the assumption that the theory is stable can be weakened to: all the tuples in $\mathrm{Ob}(\mathcal{G})$ come from a collection of stable, stably embedded sorts in T^{eq}.

Corollary 1.21. *Suppose that $\mathcal{G}$ is a type-definable connected finitary groupoid such that*

(1) *the automorphism groups G_a of objects are centerless, and*
(2) *$G_a \subset \mathrm{dcl}(a)$ for all $a \in \mathrm{Ob}(\mathcal{G})$.*

Then $\mathcal{G}$ is almost retractable.

Remark 1.22. In Section 4.2, we show how to construct a connected definable groupoid with $G_a \cong G$, where G is an arbitrary finite group. Moreover, the groupoid $\mathcal{G}$ is chosen so that $G_a \subseteq \mathrm{dcl}(a)$ for each $a \in \mathrm{Ob}(\mathcal{G})$. We show in Claim 4.3 that if G has nontrivial center, then $\mathcal{G}$ is not almost retractable. Thus, no weaker condition on the groups G_a implies that the groupoid is almost retractable.

1.4. **Eliminability and retractability.** Here we connect the notion of an eliminable groupoid, introduced by Hrushovski in [5], with the notion of a retractable groupoid. The next two definitions are from [5].

Definition 1.23. Two type-definable groupoids are *equivalent* if there is a type-definable groupoid $\mathcal{G}$ and relatively definable functors $F_i \colon \mathcal{G}_i \to \mathcal{G}$ $(i = 1, 2)$ such that F_i is full and faithful and its image intersects every isomorphism class of $\mathcal{G}$. (Without loss of generality, $\mathrm{Ob}(\mathcal{G})$ is a disjoint union of $F_1(\mathrm{Ob}(\mathcal{G}_1))$ and $F_2(\mathrm{Ob}(\mathcal{G}_2))$.)

Definition 1.24. The type-definable groupoid $\mathcal{G}$ is *eliminable* if it is equivalent to a groupoid with a single object. $\mathcal{G}$ is *almost eliminable* if it is eliminable over $\mathrm{acl}(\emptyset)$.

Note that not every connected eliminable groupoid is retractable (see the "simplest example" in Section 4.1), but we do have the converse:

Lemma 1.25. *If $\mathcal{G}$ is retractable, then it is eliminable.*

Proof. Suppose $\{f_{ab} : a, b \in \mathrm{Ob}(\mathcal{G})\}$ is a commuting system witnessing the retractability of $\mathcal{G}$. We construct a definable functor from $\mathcal{G}$ to a groupoid $\mathcal{G}^*$ that contains a single object. Let $\mathrm{Ob}(\mathcal{G}^*) = \{*\}$, where $*$ is some definable element in T^{eq}, and let G^*_* be $\mathrm{Mor}(\mathcal{G})$ modulo conjugation by the maps f_{ab}. The functor F sends every object of $\mathcal{G}$ to $*$ and every morphism of $\mathcal{G}$ to the equivalence class containing that morphism (in particular, F sends each f_{ab} to the identity element of $\mathcal{G}^*_*$). □

Eliminability of a groupoid is defined as a "global" concept, so it is useful to note that in the cases we care about, it has a nice "local" characterization:

Lemma 1.26. *If $\mathcal{G}$ is a type-definable connected finitary groupoid, then $\mathcal{G}$ is almost eliminable if and only if:*

($*$) *For any two independent $a, b \in \mathrm{Ob}(\mathcal{G})$, $\mathrm{Mor}_{\mathcal{G}}(a, b) \subset \mathrm{dcl}(\mathrm{acl}\,(a), \mathrm{acl}\,(b))$.*

Remark 1.27. For a finitary groupoid, the condition ($*$) is equivalent to the following: for any two independent $a, b \in \mathrm{Ob}(\mathcal{G})$, we have

$$\mathrm{Mor}_{\mathcal{G}}(a, b) \cap \mathrm{dcl}(\mathrm{acl}\,(a), \mathrm{acl}\,(b)) \neq \emptyset.$$

Proof. $\Rightarrow$: Say $\mathcal{G}^*$ is an extension of $\mathcal{G}$, type-definable over $\mathrm{acl}\,(\emptyset)$, with a single new object $*$ such that $\mathcal{G}$ is a full subgroupoid of $\mathcal{G}^*$. Since $\mathcal{G}^*$ is finitary, for any $a \in \mathrm{Ob}(\mathcal{G})$, $\mathrm{Mor}_{\mathcal{G}^*}(a, *)$ is a finite a-definable set (not just a type-definable set). By composing maps, it follows that *any* element of $\mathrm{Mor}_{\mathcal{G}}(a, b)$ is in $\mathrm{dcl}(\mathrm{acl}\,(a), \mathrm{acl}\,(b))$. (Note that this direction did not use the stability of T.)

$\Leftarrow$: Without loss of generality, $\mathrm{Ob}(\mathcal{G})$ is infinite, since otherwise every object and every morphism of $\mathcal{G}$ is definable over $\mathrm{acl}\,(\emptyset)$ and it is trivial to extend $\mathcal{G}$ by a formal object $*$.

Let p be a non-algebraic type in $S(\mathrm{acl}\,(\emptyset))$ that is realized in $\mathrm{Ob}(\mathcal{G})$. Let $q(\bar{x}) \supseteq p(x)$ be a type in $S(\mathrm{acl}\,(\emptyset))$ in a larger set of variables (i.e., $x \subset \bar{x}$) such that any realization $\bar{b}$ of q is in the algebraic closure of its subtuple $b \models p$, and for any independent $a \models p$ and $\bar{b} \models q$, both $\mathrm{Mor}_{\mathcal{G}}(a, b)$ and $\mathrm{Mor}_{\mathcal{G}}(b, a)$ are contained in $\mathrm{dcl}(\mathrm{acl}\,(a), \bar{b})$.

We will define a groupoid $\mathcal{G}^*$ extending $\mathcal{G}$ with a single new object $*$, which should be thought of as representing a generic realization of q. Fix any $a \models p$ and any $\bar{b} \models q|a$, where again $\bar{b}$ is in the algebraic closure of $b \models p$. There are elements $c_0, \ldots, c_{n-1} \in \mathrm{acl}\,(a)$ and formulas $\left\{\theta_i(x; c_i, \bar{b}) : i < n\right\}$ such that each $\theta_i(x; c_i, \bar{b})$ has a unique realization f_i, the f_i's are all distinct, and $\{f_i : i < n\} = \mathrm{Mor}_{\mathcal{G}}(a, b)$. Let $\psi(y_0, \ldots, y_{n-1}; a)$ isolate $\mathrm{tp}(c_0, \ldots, c_{n-1}/a)$. Since p is complete and stationary, for any a' satisfying p,

(1) $a' \in \mathrm{Ob}(\mathcal{G})$,
(2) $\psi(\overline{y}; a')$ isolates an algebraic type with the same number of realizations as $\psi(\overline{y}; a)$, and

(3) For any $\overline{b}' \models q|a'$ and for any $(c'_0, \dots, c'_{n-1})$ realizing $\psi(\overline{y}; a')$, the formulas $\theta_i(x; c'_i, \overline{b}')$ define all the elements of $\mathrm{Mor}_{\mathcal{G}}(a', b')$.

Pick a set of distinct elements $\{*_i : i < n\}$ from $\mathrm{acl}(\emptyset)$, and for any a' satisfying p, let

$$M_{a'} = \left\{ (c'_i, *_i) : i < n \text{ and } \exists\, c'_0, \dots, c'_{i-1}, c'_{i+1}, \dots, c'_{n-1} \text{ such that } \psi(\overline{c'}; a') \right\}.$$

If $(c, *_i), (d, *_j)$ are in $M_{a'}$, then we let $(c, *_i) \sim (d, *_j)$ just in case $\theta_i(x; c, \bar{b}')$ and $\theta_j(x; d, \bar{b}')$ define the same element for some (any) $\bar{b}' \models q|a'$. If a' realizes p, we define $\mathrm{Mor}_{\mathcal{G}^*}(a', *)$ to be $M_{a'}/\!\sim$. Note that since $\mathcal{G}$ is connected, there is only one way (up to interdefinability) to extend the definition of $\mathrm{Mor}_{\mathcal{G}^*}(a', *)$ to all $a' \in \mathrm{Ob}(\mathcal{G})$. We define $\mathrm{Mor}_{\mathcal{G}^*}(*, a')$ to be a set of formal inverses of elements of $\mathrm{Mor}_{\mathcal{G}^*}(a', *)$, and by definability of types it is straightforward to define composition in $\mathcal{G}^*$ so that $\mathrm{Mor}_{\mathcal{G}^*}(*, b') \circ \mathrm{Mor}_{\mathcal{G}^*}(a', *) = \mathrm{Mor}_{\mathcal{G}}(a', b')$. Finally, we define $\mathrm{Mor}_{\mathcal{G}^*}(*, *)$ to be the set

$$\bigcup_{a' \in \varphi(\mathfrak{C})} \mathrm{Mor}_{\mathcal{G}}(a', *) \times \mathrm{Mor}_{\mathcal{G}^*}(*, a')$$

modulo the equivalence relation defined by equality of composition. □

Question 1.28. Is the stability hypothesis necessary in the lemma above?

Propositions 1.16 and 1.26 show that the properties of retractability and eliminability of a groupoid $\mathcal{G}$ depend on whether or not the set $\mathrm{Mor}_{\mathcal{G}}(a, b)$ can be "recovered" from the objects a and b or the algebraic closures of the objects. The purpose of the following two definitions is to connect the notions of eliminability and retractability by showing how to expand the objects of an eliminable groupoid to obtain retractability.

Definition 1.29. Let $\mathcal{G}$ be a (type-)definable groupoid. We say that $\mathcal{G}$ is *sentient* if for all independent $a, b \in \mathrm{Ob}(\mathcal{G})$, we have $\mathrm{Mor}(a, b) \cap \mathrm{dcl}(\mathrm{acl}(a), \mathrm{acl}(b)) = \mathrm{Mor}(a, b) \cap \mathrm{dcl}(a, b)$.

Definition 1.30. Suppose that $\mathcal{G}$ is a type-definable finitary groupoid, and $p_1(x), \dots, p_n(x)$ are *strong* types such that $p_i(\mathfrak{C}) \subseteq \mathrm{Ob}(\mathcal{G})$. Then we can pick a finite tuple $x' \supseteq x$ and types $p'_i(x') \supseteq p_i(x)$ such that

(1) Any a'_i realizing p'_i is contained in the algebraic closure of the corresponding realization a_i of p_i;
(2) If $a \in p_i(\mathfrak{C})$, $b \in p_j(\mathfrak{C})$, and a and b are independent, then a and b can be extended to tuples a', b' realizing p'_i and p'_j respectively such that $\mathrm{Mor}_{\mathcal{G}}(a, b) \cap \mathrm{dcl}(\mathrm{acl}(a), \mathrm{acl}(b)) = \mathrm{Mor}_{\mathcal{G}}(a, b) \cap \mathrm{dcl}(a', b')$.

Given such types p'_i, a *sentient partial cover of* $\mathcal{G}$ (generated by the types p_i) is the groupoid $\mathcal{G}'$ such that $\mathrm{Ob}(\mathcal{G}') = \bigcup_{1 \le i \le n} p'_i$ and $\mathrm{Mor}_{\mathcal{G}'}(a', b') = \mathrm{Mor}_{\mathcal{G}}(a, b) \times \{(a', b')\}$. Note that $\mathcal{G}'$ is type-definable over $\mathrm{acl}(\emptyset)$ and $\mathcal{G}'$ is sentient by construction.

Remark 1.31. Since the strong types p_i, $1 \leq i \leq n$ are disjoint, the union $\bigcup_{1 \leq i \leq n} p'_i$ is a disjoint union.

A sentient partial cover of a $\emptyset$-definable groupoid $\mathcal{G}$ is defined over a larger, in general, set $\operatorname{acl}(\emptyset)$. This difference disappears when we make the assumption that $\operatorname{acl}(\emptyset) = \operatorname{dcl}(\emptyset)$.

Note that if $\mathcal{G}'$ is a sentient partial cover of $\mathcal{G}$, then we have a natural relatively definable functor $\pi \colon \mathcal{G}' \to \mathcal{G}$ mapping $a'_i \models p'_i$ to its corresponding realization a_i of p_i, and π is full and faithful.

Theorem 1.32. *In a stable theory T, if $\mathcal{G}$ is a type-definable finitary groupoid, then $\mathcal{G}$ is almost eliminable if and only if any sentient partial cover $\mathcal{G}'$ of $\mathcal{G}$ is almost retractable.*

Proof. If $\mathcal{G}$ is almost eliminable, then it satisfies $(*)$ of Lemma 1.26, which implies that any sentient cover $\mathcal{G}'$ satisfies $(\dagger)$ of Proposition 1.16. This, in turn, implies that $\mathcal{G}'$ is almost retractable. Conversely, suppose that every sentient cover of $\mathcal{G}$ is almost retractable. So for any independent pair of objects $a, b \in \operatorname{Ob}(\mathcal{G})$, the sentient cover $\mathcal{G}'_{a,b}$ generated by the types $p_1 = \operatorname{stp}(a)$ and $p_2 = \operatorname{stp}(b)$ is almost retractable, hence connected, and therefore $\mathcal{G}$ is connected as well. Furthermore, the almost-retractability of $\mathcal{G}'_{a,b}$ gives at least one morphism $f \in \operatorname{Mor}_{\mathcal{G}}(a, b) \cap \operatorname{dcl}(\operatorname{acl}(a), \operatorname{acl}(b))$, and since $\mathcal{G}$ is finitary, in fact $\operatorname{Mor}_{\mathcal{G}}(a, b) \subseteq \operatorname{dcl}(\operatorname{acl}(a), \operatorname{acl}(b))$. So by Lemma 1.26, $\mathcal{G}$ is almost eliminable. ☐

Theorem 1.33. *A connected finitary sentient type-definable groupoid is almost eliminable if and only if it is almost retractable.*

Proof. Any almost retractable groupoid is almost eliminable by Lemma 1.25. For the other direction, an eliminable groupoid $\mathcal{G}$ satisfies

$$\operatorname{Mor}(a, b) \subset \operatorname{dcl}(\operatorname{acl}(a), \operatorname{acl}(b))$$

by Lemma 1.26. Since $\mathcal{G}$ is sentient, we have $\operatorname{Mor}(a, b) \subset \operatorname{dcl}(a, b)$, so $\mathcal{G}$ is almost retractable by Proposition 1.16. ☐

Theorem 1.34. *If T is stable, then the following are equivalent:*

(1) *Every type-definable over* $\operatorname{acl}(\emptyset)$ *connected finitary sentient groupoid is almost retractable;*
(2) *Every definable over* $\operatorname{acl}(\emptyset)$ *connected finitary groupoid is almost eliminable.*

Proof. $1 \Rightarrow 2$: Suppose that $\mathcal{G}$ is an $\operatorname{acl}(\emptyset)$-definable connected finitary groupoid. For any $a, b \in \operatorname{Ob}(\mathcal{G})$, let $\mathcal{G}'_{a,b}$ be a sentient partial cover of $\mathcal{G}$ generated by $p_1 = \operatorname{stp}(a)$ and $p_2 = \operatorname{stp}(b)$. Then $\mathcal{G}'_{a,b}$ is almost retractable, so $\mathcal{G}$ satisfies condition $(*)$ of Lemma 1.26 for a and b; so by that lemma, $\mathcal{G}$ is almost eliminable.

$2 \Rightarrow 1$: Suppose $\mathcal{G}^0$ is a type-definable over $\operatorname{acl}(\emptyset)$ connected finitary sentient groupoid. By Fact 1.7, $\mathcal{G}^0$ is a full subgroupoid of some $\operatorname{acl}(\emptyset)$-definable connected finitary groupoid $\mathcal{G}$. By assumption, $\mathcal{G}$ is almost eliminable, so condition $(*)$ of Lemma 1.26 holds for $\mathcal{G}$. So by sentience of $\mathcal{G}^0$, for any independent

$a, b \in \mathrm{Ob}(\mathcal{G}^0)$, $\mathrm{Mor}_{\mathcal{G}^0}(a, b) \subseteq \mathrm{dcl}(a, b)$, and by Proposition 1.16, $\mathcal{G}^0$ is almost retractable. □

2. Retractable groupoids and 3-uniqueness

Existence of non-retractable groupoids in a stable theory that fails 3-uniqueness logically follows from Hrushovski's [5] and our results in Section 1. Hrushovski shows that failure of 3-uniqueness implies that T has a finite internal cover which is not split (the unsplit cover is then linked to a certain groupoid which has to be non-eliminable). However, the argument for internality of the cover is indirect; in particular, there is no explicit definable bijection between the new sort of the cover and the "old" sorts. So it is not clear what the non-eliminable groupoid looks like, and what relation it has to the amalgamation problem that fails to have a unique solution.

This section gives an explicit construction of a type-definable non-eliminable (hence, non-retractable) finitary sentient groupoid in a stable theory that fails 3-uniqueness. To do this, we establish two new facts about the failure of 3-uniqueness: we show that there is always a symmetric witness to such a failure, and that a symmetric witness can be embedded into a non-retractable groupoid. The morphisms of this groupoid will be finite paths modulo a homotopy-like equivalence relation, and the key lemma that every path is equivalent to a two-step path is essentially a form of the old result that germs of functions in a stable theory generate a group in two steps (see [4]).

To set some notation, given a sequence $\langle a_i : i \in I\rangle$, a parameter set A, and $S \subseteq I$, the symbol $\overline{a}_S$ denotes the algebraic closure $\mathrm{acl}\,(A \cup \{a_i : i \in S\})$. We regard $\overline{a}_S$ as a (usually infinite) tuple. Without loss of generality, we may take A to be algebraically closed. Thus, we assume $A = \mathrm{acl}\,(A)$ everywhere below in this section.

If in addition the sequence $\langle a_i : i \in I\rangle$ is indiscernible over the set A, then we always assume that we have chosen the enumerations of these tuples $\overline{a}_S$ so that if $|S| = |S'|$ then $\overline{a}_S \equiv_A \overline{a}_{S'}$. We write "$\overline{a}_i$" for $\overline{a}_{\{i\}}$ and "$\overline{a}_{ij}$" for $\overline{a}_{\{i,j\}}$.

Recall that in a stable theory 3-uniqueness fails if and only if for some (algebraically closed) set A, there are tuples a_1, a_2, and a_3 independent over A and an automorphism σ of $\mathfrak{C}$ that fixes $\overline{a}_1\overline{a}_2$ such that

$$\mathrm{tp}(\overline{a}_{12}\overline{a}_{23}\overline{a}_{31}) \neq \mathrm{tp}(\sigma(\overline{a}_{12})\overline{a}_{23}\overline{a}_{31})$$

(by our agreement $A \subset \overline{a}_i$, $i = 1, 2$, so σ fixes $\mathrm{acl}\,(Aa_1)$ and $\mathrm{acl}\,(Aa_2)$ but does not fix the algebraic closure of Aa_1a_2).

The following fact appears in [2] and in the proof of Lemma 3.2 in [5].

Fact 2.1. *Let T be a stable theory, and suppose that a_1, a_2, a_3 are tuples which form an independent set over an algebraically closed set A. Then the following are equivalent:*

(1) *For any $\sigma \in \mathrm{Aut}_{\overline{a}_1\overline{a}_2}(\mathfrak{C})$, $\mathrm{tp}(\overline{a}_{12}\overline{a}_{23}\overline{a}_{31}) = \mathrm{tp}(\sigma(\overline{a}_{12})\overline{a}_{23}\overline{a}_{31})$;*
(2) $\mathrm{dcl}(\overline{a}_1, \overline{a}_2) = \mathrm{dcl}(\overline{a}_{23}, \overline{a}_{31}) \cap \overline{a}_{12}$.

Throughout this section, T is assumed to be a stable theory.

2.1. **Witnesses to non-3-uniqueness.**

Definition 2.2. 1. A *witness to non-3-uniqueness* (in a stable theory) is a finite sequence $\{a_1, a_2, a_3\}$ and an algebraically closed set A together with elements f_{12}, f_{23}, and f_{31} such that:

(1) $f_{ij} \in \overline{a}_{ij}$;
(2) $f_{12} \notin \mathrm{dcl}(\overline{a}_1\overline{a}_2)$; and
(3) $f_{12} \in \mathrm{dcl}(f_{23}, f_{31})$.

A witness to non-3-uniqueness is *symmetric (over A)* if $a_1, a_2 \in \mathrm{dcl}(f_{12})$, a_1, a_2, a_3 is a Morley sequence over a common subset A, and:

(4) $a_1a_2f_{12} \equiv_A a_2a_3f_{23} \equiv_A a_3a_1f_{31}$;
(5) There is a formula $\theta(x, y, z)$ over A such that f_{12} is the unique realization of $\theta(x, f_{23}, f_{31})$, f_{23} is the unique realization of $\theta(f_{12}, y, f_{31})$, and f_{31} is the unique realization of $\theta(f_{12}, f_{23}, z)$.

Note that the symmetry condition (5) implies in particular that each f_{ij} is in the definable closure of the other two.

The purpose of this section is to construct a symmetric witness to non-3-uniqueness, given *any* T which does not have 3-uniqueness.

Lemma 2.3. *The following are equivalent:*

(1) *T does not have 3-uniqueness property;*
(2) *there is an algebraically closed set A and a Morley sequence $\{a_1, a_2, a_3\}$ over A which is a witness to non-3-uniqueness.*

Proof. The implication (2) $\Rightarrow$ (1) follows from Fact 2.1. So suppose 3-uniqueness fails and let $\{b_1, b_2, b_3\}$ be an independent set over A (but not necessarily a Morley sequence) witnessing the failure of 3-uniqueness: that is, there is $\sigma \in \mathrm{Aut}_{\overline{b}_1\overline{b}_2}(\mathfrak{C})$ such that

$$\mathrm{tp}(\overline{b}_{12}\overline{b}_{23}\overline{b}_{31}) \neq \mathrm{tp}(\sigma(\overline{b}_{12})\overline{b}_{23}\overline{b}_{31}).$$

Pick a Morley sequence $\{a_i : 1 \leq i \leq 3\}$ over A such that for $i = 1, 2, 3$, the element a_i is the triple $a_i[1]a_i[2]a_i[3]$, and

$$\mathrm{tp}(\mathrm{acl}\,(a_i[1]a_i[2]a_i[3])) = \mathrm{tp}(\overline{b}_{123}).$$

Without loss of generality, we may assume that $a_1[1] = b_1$, $a_2[2] = b_2$, and $a_3[3] = b_3$.

Stationarity guarantees that the map $\sigma \upharpoonright \overline{b}_{12}$ extends to an automorphism $\hat{\sigma} \in \mathrm{Aut}(\mathfrak{C})$ that fixes $\overline{a}_1 \cup \overline{a}_2$ pointwise. Then $\mathrm{tp}(\overline{a}_{12}\overline{a}_{23}\overline{a}_{31}) \neq \mathrm{tp}(\hat{\sigma}(\overline{a}_{12})\overline{a}_{23}\overline{a}_{31})$, so by Fact 2.1 and compactness, we obtain f_{ij}'s as in Definition 2.2.

$\square$

Theorem 2.4. *If T does not have 3-uniqueness, then there is a set A and a symmetric witness to non-3-uniqueness over A.*

Proof. By Lemma 2.3, we may assume that a_1, a_2, a_3 is a Morley sequence over A witnessing the failure of 3-uniqueness. Thus, the set $D_{12} := \overline{a}_{12} \cap \operatorname{dcl}(\overline{a}_{13}, \overline{a}_{23})) \setminus \operatorname{dcl}(\overline{a}_1, \overline{a}_2)$ is not empty. Adding the elements of the set A to the language if necessary, we may assume that $A = \emptyset$.

Let $c_{12} \in D_{12}$. Then there is a formula $\phi(x; a_1, a_2, a_3; d_{23}, e_{31})$ and formulas $\chi_c(x; a_1, a_2)$, $\chi_d(y; a_3, a_1)$, and $\chi_e(z; a_2, a_3)$ such that

(1) c_{12} is the unique solution of $\phi(x; a_1, a_2, a_3; d_{23}, e_{31})$;
(2) the formulas χ isolate algebraic types over their parameters;
(3) $\phi(x; a_1, a_2, a_3; d_{23}, e_{31}) \vdash \chi_c(x; a_1, a_2)$;
(4) $d_{23} \models \chi_d(y; a_1, a_3)$, and similarly for e_{31}.

Also, we assume for convenience that $a_1, a_2 \in \operatorname{dcl}(c_{12})$, $a_2, a_3 \in \operatorname{dcl}(d_{23})$, and $a_1, a_3 \in \operatorname{dcl}(e_{31})$.

Claim 2.5. *We may further assume that* $d_{23} \in \operatorname{dcl}(c_{12}, e_{31})$ *and that* $e_{31} \in \operatorname{dcl}(c_{12}, d_{23})$.

Proof. First, replace d_{23} by the finite set d'_{23} of all conjugates of d_{23} over $\{c_{12}, e_{31}\}$. Then since for any such conjugate $\tilde{d}_{23}$ of d_{23}, $c_{12} \in \operatorname{dcl}(\tilde{d}_{23}, e_{31})$, it follows that $c_{12} \in \operatorname{dcl}(d'_{23}, e_{31})$. Also it is clear that $d'_{23} \in \overline{a}_{23}$ and $d'_{23} \in \operatorname{dcl}(c_{12}, e_{31})$.

Next, replace e_{31} by the set e'_{31} of all conjugates of e_{31} over (c_{12}, d'_{23}). By the same arguments as before, the new triple $(c_{12}, d'_{23}, e'_{31})$ satisfies all the conditions we want. □

Now pick finite tuples $\tilde{c}_{12}, \tilde{d}_{12}$, and $\tilde{e}_{12}$ enumerating the solution sets of $\chi_c(\mathfrak{C}; a_1, a_2)$, $\chi_d(\mathfrak{C}; a_1, a_2)$, and $\chi_e(\mathfrak{C}; a_1, a_2)$ respectively, and let f_{12} be the tuple $(\tilde{c}_{12}, \tilde{d}_{12}, \tilde{e}_{12})$. Similarly, we define tuples f_{23} and f_{31} such that $a_1 a_2 f_{12} \equiv a_2 a_3 f_{23} \equiv a_3 a_1 f_{31}$.

It is not hard to check that $\{a_1, a_2, a_3\}$ with the three f_{ij}'s are a symmetric witness to failure of 3-uniqueness.

□

2.2. **Paths and the non-retractable groupoid.** The goal here is to prove:

Theorem 2.6. *If T is stable and does not have 3-uniqueness, then there is a set A and an A-type-definable finitary connected groupoid in T which is not eliminable over* $\operatorname{acl}(A)$.

For ease of notation, we will assume for the remainder of this subsection that $\operatorname{dcl}(\emptyset) = \operatorname{acl}(\emptyset)$, by the previous section we may also assume (after naming parameters):

Assumption 2.7. The set $\{a_1, a_2, a_3\}$ together with f_{12}, f_{23}, and f_{31} is a symmetric witness to failure of 3-uniqueness (over $\emptyset$).

Let $p = \operatorname{stp}(a_1)$.

Remark 2.8. If $(a, b, c) \models p^{(3)}$ and $(a, b, f) \equiv (a_1, a_2, f_{12}) \equiv (b, c, g)$, then stationarity implies that there is a unique h such that

$$(a, b, c, f, g, h) \equiv (a_1, a_2, a_3, f_{12}, f_{23}, f_{31}),$$

which we call "$g \circ f$." Similarly, for any f and g as above, there is also a unique h_1 such that $f \circ h_1 = g$, and a unique h_2 such that $h_2 \circ f = g$.

Definition 2.9. (1) If a realizes p, then an *n-step path from a* is a sequence $(b_0, g_1, b_1, g_2, \ldots, b_n)$ such that $b_0 = a$ and for every $i < n$, $b_i b_{i+1} g_{i+1} \equiv a_1 a_2 f_{12}$. The *endpoint* of this path is b_n.
(2) $P^n(a)$ denotes the set of all n-step paths starting from a, and $P^n(a, b)$ denotes the set of all n-step paths from a with endpoint b.

Definition 2.10. Suppose that $(b_0, g_1, b_1, g_2, \ldots, b_n)$ and $(d_0, h_1, d_1, h_2, \ldots, d_m)$ are two paths with the same starting point and the same endpoint. Let a^* be a realization of p that is independent from all the parameters listed and pick any g_0^* such that $a^* b_0 g_0^* \equiv a_1 a_2 f_{12}$. Then we can define a sequence $g_1^*, \ldots, g_n^*$ inductively so that for every i between 0 and n, $g_{i+1}^* = g_{i+1} \circ g_i^*$. From the second path, we can define a sequence $h_1^*, \ldots, h_m^*$ similarly. Then we say that the paths are *equivalent* if $g_n^* = h_m^*$, and we use $\sim$ to denote this equivalence relation.

Remark 2.11. (1) By the stationarity of p, the choice of a^* and g_0^* in the previous definition does not matter: if $g_n^* = h_m^*$ for one such choice, then they are equal for every such choice.
(2) By the definability of types, the relation of $\sim$ restricted to $P^n(a)$ is relatively definable.

Note that if the endpoint of the path α is the starting point of a path β, then we can define the concatenation $\alpha\beta$ of these paths, and if $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then $\alpha\beta \sim \alpha'\beta'$.

Lemma 2.12. *If α and β are two paths starting from a such that the endpoint of α is independent from the endpoint of β, then there is a 1-step path γ such that α concatenated with γ is equivalent to β.*

Proof. Let $\alpha = (b_0, g_1, b_1, g_2, \ldots, b_n)$ and $\beta = (d_0, h_1, d_1, h_2, \ldots, d_m)$. Pick some $a^* \models p|(a, b_n, d_m)$, and pick any f^* such that $a^* a f^* \equiv a_1 a_2 f_{12}$. Compute the "edge" elements g_n^* and h_m^* as in Definition 2.10. Then $a^* b_n g_n^* \equiv a_1 a_2 f_{12} \equiv a^* d_m h_m^*$, so there is a unique k such that $k \circ g_n^* = h_m^*$. Then the path $\gamma = (b_n, k, d_m)$ is what we want. ☐

Finally, we define a type-definable groupoid which we will show to be non-retractable and sentient, hence non-eliminable.

Definition 2.13. (1) If a and b realize p, then $M_{a,b}$ is the type-definable set $P^2(a, b)/\sim$.

(2) The operation $\circ$ on $M_{b,c} \times M_{a,b}$ is given by concatenation of paths. Specifically: if α and β are paths representing arrows in $M_{a,b}$ and $M_{b,c}$ respectively, then $\alpha\beta$ is a 4-step path from a to c, which by Lemma 2.12 is equivalent to a 2-step path from a to c, which we call $\beta \circ \alpha$.

Lemma 2.14. *There is an (a_1, a_2)-definable bijection between M_{a_1,a_2} and $X_{a_1a_2} := \{f : f \equiv_{a_1a_2} f_{12}\}$.*

Proof. The first step is to note that every $f \in X_{a_1a_2}$ definably encodes an element of M_{a_1,a_2}: namely, the $\sim$-class of all 2-step paths equivalent to the 1-step path (a_1, f_{12}, a_2) (which is nonempty by Lemma 2.12). The next step is to note that if f and f' are two distinct elements of $X_{a_1a_2}$, then they encode different elements of M_{a_1,a_2}: this is by the final clause in the definition of a symmetric failure to 3-uniqueness. Finally, the fact that every element of M_{a_1,a_2} is so coded follows by Lemma 2.12 again. □

Lemma 2.15. *The sets $p(\mathfrak{C})$ and $M_{a,b}(\mathfrak{C})$ form the objects and morphisms of a groupoid with $\circ$ as composition, which we will call $\mathcal{G}$.*

Proof. The associativity of $\circ$ is straightforward, so we explain how to construct the identity morphism from a to a. Pick any $b \models p|a$, any $a^* \models p|(a, b)$, and any elements f, f_0^*, f_1^* such that $abf \equiv a_1a_2f_{12} \equiv a^*af_0^*$ and $f_1^* = f \circ f_0^*$. Then by Remark 2.8, we can pick an element g such that $bag \equiv a_1a_2f_{12}$ and $f_0^* = g \circ f_1^*$. Then a routine calculation shows that the path (a, f, b, g, a) is an identity element for $M_{a,a}$ under $\circ$.

To show that $\mathcal{G}$ is a groupoid, suppose that $\alpha \in \mathrm{Mor}_{\mathcal{G}}(a, b)$ is represented by the 2-step path (a, f_1, c, f_2, b), and suppose that we have picked a realization a^* of $p|(a, b, c)$ and an element f_0^* such that $a^*af_0^* \equiv a_1a_2f_{12}$. Then (as in Definition 2.10) let $f_1^* = f_1 \circ f_0^*$ and let $f_2^* = f_2 \circ f_1^*$. By Remark 2.8, there are elements g_1 and g_2 such that $f_1^* = g_2 \circ f_2^*$ and $f_0^* = g_1 \circ f_1^*$. The path (b, g_2, c, g_1, a) represents an inverse to the path α. □

Claim 2.16. *The groupoid $\mathcal{G}$ is not eliminable.*

Proof. Otherwise, by Lemma 1.26, the set $\mathrm{Mor}_{\mathcal{G}}(a_1, a_2)$ would be contained in $\mathrm{dcl}(\overline{a}_1, \overline{a}_2)$, contradicting Lemma 2.14 and the choice of f_{12}. □

This completes the proof of Theorem 2.6. □

2.3. **Characterizing 3-uniqueness.**

Proposition 2.17. *Suppose that for all A-independent algebraically closed $\overline{a}$, $\overline{b}$, $\overline{c}$ we have $\mathrm{dcl}_A(\overline{a}\overline{b}) = \mathrm{dcl}_A(\mathrm{acl}\,(ac), \mathrm{acl}\,(bc)) \cap \mathrm{acl}\,_A(ab)$. Then every A-type-definable connected finitary sentient groupoid is retractable over $\mathrm{acl}\,(A)$.*

Proof. Without loss of generality, $A = \emptyset$. Let $\mathcal{G}$ be such a type-definable groupoid. By Lemma 1.12, we may assume that any two objects in $\mathcal{G}$ satisfy the same complete type p over $\mathrm{acl}\,(\emptyset)$. If p is an algebraic type, then every element of $\mathrm{Ob}(\mathcal{G})$ and $\mathrm{Mor}(\mathcal{G})$ is in $\mathrm{acl}\,(\emptyset)$, and $\mathcal{G}$ is trivially retractable over $\mathrm{acl}\,(\emptyset)$. So without loss of generality, $\mathrm{Ob}(\mathcal{G})$ is an infinite set. Pick $a, b, c \in \mathrm{Ob}(\mathcal{G})$ that form an independent set.

Since the sets of morphisms between any two objects are finite, the sets $\mathrm{Mor}(a, c)$ and $\mathrm{Mor}(b, c)$ are contained in $\mathrm{acl}\,(ac)$ and $\mathrm{acl}\,(bc)$ respectively. Using composition, we get that $\mathrm{Mor}(a, b) \subset \mathrm{dcl}(\mathrm{acl}\,(ac), \mathrm{acl}\,(bc)) \cap \mathrm{acl}\,(ab)$.

By the assumption of our proposition, we then have $\mathrm{Mor}(a, b) \subset \mathrm{dcl}(\overline{a}\overline{b})$. Since $\mathcal{G}$ is sentient, $\mathrm{Mor}(a, b) \subset \mathrm{dcl}(a, b)$. So by Proposition 1.16, $\mathcal{G}$ is almost retractable.

□

Putting together Theorem 1.34, Theorem 2.6, and Proposition 2.17, we get:

Theorem 2.18. *If T is stable, then the following are equivalent:*

1. *T has 3-uniqueness;*
2. *For every algebraically closed set A, every connected finitary A-type-definable sentient groupoid is retractable over A;*
3. *For every algebraically closed set A, every connected finitary A-definable groupoid is eliminable over A.*

Note that we cannot weaken the second condition above to "Every connected finitary A-type-definable sentient groupoid is retractable over A" for sets A that fail to be algebraically closed. For instance, if there is a 2-element definable set P in T whose elements have the same type, then we can construct a finite groupoid as in section 4.2 which sentient but is not retractable over $\emptyset$.

3. Finite internal covers

In this section, we describe definable groupoids associated to finite internal covers in a stable theory. Most of the definitions and basic results appear in [5], but we found it useful to spell some things out more explicitly and precisely. We use these results to translate a fact about retractable groupoids from Section 1 into the following fact about finite internal covers: if N is a finite internal cover of M such that $\mathrm{Aut}(N/M)$ is centerless, then the cover is almost split (Theorem 3.11 below). This fact resembles Lemma 1.2 of [3]: that lemma implies, in the terminology of our paper, that if $\mathrm{Aut}(N/M)$ is centerless and $\mathrm{dcl}_N(\emptyset) = \mathrm{acl}_N(\emptyset)$, then the group $\mathrm{Aut}(N/M)$ is trivial. However, our result is different since we make no assumption that $\mathrm{dcl}_N(\emptyset) = \mathrm{acl}_N(\emptyset)$.

We assume that all structures in this section are stable. We will further assume that all structures M in this section have the property $M \cong M^{eq}$. Recall that the *eq*-construction preserves everything that we care about: stability, what sets are definable in the sorts of the original language, and the automorphism group of the structure. If $L' \supseteq L$ is an expansion of L with a larger set of sorts and N is an L'-structure, then "$N \restriction L$" denotes the natural L-structure induced by N; namely, for every L-sort S, $S(N \restriction L) = S(N)$, and every basic relation and function in L has the same interpretation in $S \restriction L$ as in N.

Definition 3.1. ([5], Definition 2.2) If M and N are multisorted structures in languages L_0 and L_1 respectively with $L_0 \subseteq L_1$, then N is a *finite internal cover* over M if:

(1) $M = N \upharpoonright L_0$;
(2) Every subset of M which is definable over $\emptyset$ in L_1 is definable over $\emptyset$ in L_0;
(3) For any formula $\varphi(x, y)$ in L_1, where x is in a sort of L_0, there is a formula $\psi(x, z)$ in L_1, where z is in a sort of L_0, with the following property: for every $a \in N$, there is a $b \in M$ such that

$$\forall x \in M\, [\varphi(x, a) \leftrightarrow \psi(x, b)]\,;$$

(4) There is a *finite* tuple $\bar{b} \subseteq \operatorname{acl}_{L_1}(M)$ such that $N \subseteq \operatorname{dcl}_{L_1}(M\bar{b})$.

Remark 3.2. Item (3) in the previous definition is essentially the property that is usually called "stable embeddedness," but adapted to the context of two structures in different languages. This is essentially the same definition as in the appendix of [1], adapted to a more general context.

Because of condition (2), we can require that the formula $\psi(x, z)$ in condition (3) is an L_0-formula instead of merely an L_1-formula.

Remark 3.3. (1) Suppose that N is a finite internal cover of M, L_1 is the language of N, and L_0 is the language of M. Condition (4) implies that there is a finite set $\mathcal{S}$ of sorts in $L_1 \setminus L_0$ such that for any sort $S \in L_1 \setminus L_0$, N satisfies a sentence which implies that any element of S is in the definable closure of the L_0-sorts plus the sorts in $\mathcal{S}$. It follows that if N' is elementarily equivalent to N then N' is also a finite internal cover of $N' \upharpoonright L_0$.

(2) If N is a finite internal cover of M, then $\operatorname{Aut}(N/M)$ is finite. However, there are examples where N is an extension of M to new sorts, $\operatorname{Aut}(N/M) = 1$, and conditions (1) through (3) are satisfied, but not (4). For example, suppose that in M there is a definable group G equal to the direct product of ω-many copies of $\mathbb{Z}_2$, with each subgroup

$$G_i = \{g \in G : \text{the } i\text{th coordinate of } g \text{ is } 0\}$$

also M-definable. Suppose also that N is generated by one new sort S which is an affine copy of G, with definable subsets $S_i \subseteq S$ such that if $g \in G$ and $a \in S_i$, then $ga \in S_i$ if and only if $g \in G_i$. Then any two distinct points in $S(N)$ realize different types, but at the same time, *no* element of $S(N)$ is in $\operatorname{acl}_{L_1}(M)$. However, the models M and N are not saturated, and the binding group $\operatorname{Aut}(N'/M')$ is infinite in a saturated extension (N', M').

Definition 3.4. Suppose that N is a finite internal cover of M.

(1) N is an *almost split cover* if there is a finite $\emptyset$-definable set $C \subseteq N$ such that $N \subseteq \operatorname{dcl}(M \cup C)$.

(2) N is a *split cover* if C as above can be chosen so that any C-definable subset of M is $\emptyset$-definable.

Lemma 3.5. (1) *If N is a split [or almost split] finite internal cover of M and $(N'; M')$ is elementarily equivalent to $(N; M)$, then N' is a split [almost split] finite internal cover of M'.*

(2) *The finite internal cover $(N; M)$ is split if and only if it is almost split and for any saturated elementary extension $(N'; M')$ of $(N; M)$ with $|N| > |L_1| + \aleph_0$, the natural restriction map*

$$\mathrm{Aut}(N'/C) \to \mathrm{Aut}(M')$$

is surjective.

Proof. (1): Suppose that $C \subsetneq N$ is a finite, $\emptyset$-definable set witnessing that N is a split cover of M, and say $(N'; M') \equiv (N; M)$. Let $C' \subsetneq N'$ be the interpretation of the formula defining C in N'. Elementary equivalence implies that $N' \subseteq \mathrm{dcl}(M' \cup C')$ (which takes care of proving the preservation of almost-splitness). Suppose that $X \subseteq M'$ definable in N' by a formula $\varphi(x; \overline{b})$, where $\overline{b} \subseteq C'$, and let $\theta(y)$ be the formula defining C and C'. Then by the finiteness of C plus the fact that N is a split cover of M, there is a finite collection of formulas $\psi_1(x), \ldots, \psi_m(x)$ (over $\emptyset$) such that

$$N \vdash \forall y_1 \ldots y_k \left[\bigwedge_{1 \le i \le k} \theta(y_i) \to \left(\bigvee_{1 \le j \le m} \forall x \left[\varphi(x, \overline{y}) \leftrightarrow \psi_j(x) \right] \right) \right].$$

Therefore, N' satisfies this sentence as well, so X is $\emptyset$-definable.

(2): By the same argument as [1], Appendix, Lemma 1, equivalence of (5) and (6). (Note that the fact that N is a split cover of M implies that even after expanding the language of N to include constants for the elements of C, the sorts of M are, according to the terminology of that paper, stably embedded in N, and therefore the argument of Lemma 1 applies.) □

Remark 3.6. As observed in [5], N is an almost split cover of M if and only if it is split over $\mathrm{acl}(\emptyset)$ – that is, if we add constants to the language naming the elements of $\mathrm{acl}_N(\emptyset)$ and constants to M for the elements of $\mathrm{acl}_M(\emptyset)$, then the cover becomes split.

3.1. **Liaison groupoids.**

Definition 3.7. Suppose that N is a finite internal cover of M. Then a *liaison groupoid of N over M* is a connected groupoid $\mathcal{G}$ which is type-definable in M and has the properties:

(1) There is a connected finitary groupoid $\mathcal{G}^*$ such that $\mathcal{G}$ is a full subgroupoid of $\mathcal{G}^*$, the groupoid $\mathcal{G}^*$ is type-definable in N, and $\mathcal{G}^*$ contains a single new object $*$ (which is definable over $\emptyset$ in N);

(2) For any $a \in \mathrm{Ob}(\mathcal{G})$, all the elements of $\mathrm{Mor}_{\mathcal{G}^*}(a, *)$ have the same type over M;

(3) For any $a \in \mathrm{Ob}(\mathcal{G})$ and any $f \in \mathrm{Mor}_{\mathcal{G}^*}(a, *)$, $N \subseteq \mathrm{dcl}_N(Mf)$.

Theorem 3.8. *Any finite internal cover of M has a definable liaison groupoid.*

Proof. This is a strengthening of a special case of Proposition 1.5 of [5]. The proof is essentially the same, but we spell out some details to show how we can ensure that the liaison groupoid is definable and not just $\star$-definable.

Let N be a finite internal cover of M, and let L_1 and L_0 be the respective languages of N and M. Pick $c \in \mathrm{acl}_{L_1}(M)$ such that $N \subseteq \mathrm{dcl}_{L_1}(Mc)$, and pick $b \in M$ such that $\mathrm{tp}(c/b) \vdash \mathrm{tp}(c/M)$. In fact, we can pick an L_1-formula $\theta(x, y)$ such that $\theta(x, b) \vdash \mathrm{tp}(c/M)$.

Claim 3.9. *For any $c' \in N$ realizing $\mathrm{tp}(c/b)$, there is a $\sigma \in \mathrm{Aut}(N/M)$ such that $\sigma(c) = c'$.*

Proof. For any sort S in N, $S \subset \mathrm{dcl}(Mc)$, so there is a set $Q_c^S \subseteq M$ and a surjective function $f_c^S : Q_c^S \to S$ which are both definable in L_1 over c. Note that Q_c^S is definable in L_1 over b by the formula $\forall x \left[\theta(x, b) \to z \in Q_x^S\right]$, so from now on we will write this set as "Q_b^S."

Define σ on the sort S such that if $d \in S$ and $d = f_c(a)$ for some $a \in M$, then $\sigma(d) = f_{c'}(a)$. To check that this is well-defined, note that if $f_c(a) = f_c(a')$, then the fact that $\mathrm{tp}(c'/M) = \mathrm{tp}(c/M)$ implies that $f_{c'}(a) = f_{c'}(a')$. By the same argument, one can check that σ is surjective and that σ is an elementary map.

□

Suppose $\mathrm{Aut}(N/M) = k < \omega$. Note that by Claim 3.9, $\mathrm{tp}(c/b)$ has no more than k realizations in M, and the fact that $N \subseteq \mathrm{dcl}(Mc)$ means that it has *exactly* k realizations.

Let S be the sort in N to which c belongs. Then $N \subseteq \mathrm{dcl}(MS)$, and since $S \subseteq \mathrm{dcl}(Mc)$, there is a c-definable bijection $f_c : S \to Q$ between S and some subset Q of M which is definable in L_1, possibly over extra parameters from M. Without loss of generality (expanding the tuple b if necessary), we may assume that Q is definable over b, and we will write Q_b for Q below. By condition (2) of Definition 3.1, Q_b is L_0-definable.

Pick some formula $\varphi(x, y)$ in L_1 that is satisfied by (b, c) such that for any other $(b', c') \in N$ satisfying $\varphi(x, y)$,

1. $f_{c'}$ defines a bijection between S and $Q_{b'}$, and
2. $\varphi(b', y)$ has exactly k realizations.

By condition (2) of Definition 3.1, the formula $\exists y \varphi(x, y)$ is equivalent (modulo $\mathrm{Th}(N)$) to some formula $\theta(x)$ in the language of M.

Now we can let $\theta(x)$ define the set of objects of our liaison groupoid $\mathcal{G}$, and for any $a \in \mathrm{Ob}(\mathcal{G})$, the formula $\varphi(a, y)$ will define $\mathrm{Mor}_{\mathcal{G}^*}(a, *)$. Just as in the proof of Proposition 1.5 of [5], we can use this data to define all other morphisms in $\mathcal{G}^*$ as formal pairs of morphisms to $*$ modulo a definable equivalence relation.

Condition (3) of the definition of a liaison groupoid is straightforward to check, since any $f' \in \mathrm{Mor}_{\mathcal{G}^*}(a', *)$ is interdefinable over M with an element

$f \in \mathrm{Mor}_{\mathcal{G}^*}(a,*)$. Once we have this, it follows that for any $a' \in \mathrm{Ob}(\mathcal{G}^*)$ and any $f', f'' \in \mathrm{Mor}_{\mathcal{G}^*}(a',*)$, there is at most one $\sigma \in \mathrm{Aut}(M/N)$ such that $\sigma(f') = f''$; therefore by a counting argument, f' and f'' are in fact conjugate over M, and condition (2) follows.

□

Lemma 3.10. *If $\mathcal{G}$ is a liaison groupoid for N over M, then for any $a \in \mathrm{Ob}(\mathcal{G})$, $\mathrm{Mor}_{\mathcal{G}}(a,a) \cong \mathrm{Aut}(N/M)$.*

Proof. Pick any $f \in \mathrm{Mor}_{\mathcal{G}^*}(a,*)$. For any other $g \in \mathrm{Mor}_{\mathcal{G}^*}(a,*)$, by the argument in the last paragraph of the previous proof, there is a unique map $\tau_{f,g} \in \mathrm{Aut}(N/M)$ such that $\tau_{f,g}(f) = g$. Thus, $\mathrm{Aut}(N/M) = \{\tau_{f,g} : g \in \mathrm{Mor}_{\mathcal{G}^*}(a,*)\}$.

If $f' \in \mathrm{Mor}_{\mathcal{G}^*}(a,*)$, then

$$\tau_{f,g}(f') = \tau_{f,g}(f \circ (f^{-1} \circ f')) = g \circ (f^{-1} \circ f'),$$

so $\tau_{f,g}$ acts on $\mathrm{Mor}_{\mathcal{G}^*}(a,*)$ via left translation by $g \circ f^{-1}$, giving an isomorphism $\mathrm{Mor}_{\mathcal{G}}(a,a) \cong \mathrm{Aut}(N/M)$. □

Theorem 3.11. *If N is a finite internal cover of M and $\mathrm{Aut}(N/M)$ is centerless, then N is an almost split cover of M.*

Proof. We will add constants for $\mathrm{acl}_M(\emptyset)$ to the language of M and constants for $\mathrm{acl}_N(\emptyset)$ to the language of N, and then we will find a finite $\emptyset$-definable C such that $N \subseteq \mathrm{dcl}(M \cup C)$; this suffices, since any such C is contained in a finite $\emptyset$-definable set in the original language. Also, by Lemma 3.5, we may assume that N is saturated and $|N|$ is greater than the size of its language.

By Theorem 3.8, the cover has a definable liaison groupoid $\mathcal{G}$. Without loss of generality, for any $a \in \mathrm{Ob}(\mathcal{G})$, $G_a \subseteq \mathrm{dcl}_M(a)$. If not, we can pick any object a from $\mathrm{Ob}(\mathcal{G})$; then there is a formula $\varphi(x) \in \mathrm{tp}(a)$ and a formula $\psi(x,y)$ such that whenever $M \models \varphi(a')$, we have $a' \in \mathrm{Ob}(\mathcal{G})$ and $\psi(a',y)$ defines the set $G_{a'}$. Let $\mathcal{G}'$ be the groupoid whose objects are all the tuples $(a', \bar{b})$ such that $M \models \varphi(a')$ and $\bar{b}$ enumerates $\psi(a',y)$, and with the same morphisms as $\mathcal{G}$. Then $\mathcal{G}'$ is a definable liaison groupoid, and for any $c \in \mathrm{Ob}(\mathcal{G}')$, $G'_c \subseteq \mathrm{dcl}_M(c)$.

By Lemma 3.10, the automorphism groups of objects in $\mathcal{G}$ are finite and centerless, and so by Corollary 1.21, $\mathcal{G}$ is almost retractable in M, hence retractable in M (since $\mathrm{acl}_M(\emptyset) = \mathrm{dcl}_M(\emptyset)$).

Let $\{f_{ab} : a, b \in \mathrm{Ob}(\mathcal{G})\}$ be a system which is $\emptyset$-definable in M and witnesses the retractability of $\mathcal{G}$, and let $n = |G_a|$ for any $a \in \mathrm{Ob}(\mathcal{G})$.

Claim 3.12. *There is a full subgroupoid $\mathcal{G}^0$ of $\mathcal{G}$, definable in M over $\emptyset$, and a collection of formulas $\varphi_1(x,y), \ldots, \varphi_n(x,y)$ over $\emptyset$ such that*

1. *For each $a \in \mathrm{Ob}(\mathcal{G}^0)$ and each $i \leq n$, we have $M \models \exists! x\, \varphi_i(x,a)$;*
2. *For any $a \in \mathrm{Ob}(\mathcal{G}^0)$, $G^0_a = \bigcup_{1 \leq i \leq n} \varphi_i(M,a)$;*
3. *For any i and any $a, b \in \mathrm{Ob}(\mathcal{G}^0)$,*

$$\models \varphi_i(g,a) \iff \models \varphi_i(f_{ab} \circ g \circ f_{ba}, b).$$

Proof. First, pick any $a \in \mathrm{Ob}(\mathcal{G})$, and note that there are formulas $\varphi_i(x,y)$ for $1 \leq i \leq n$ satisfying the first two conditions just at the point a. Let

$$S = \Big\{a' \in \mathrm{Ob}(\mathcal{G}) : (\forall i \leq n)\ |\varphi_i(M,a')| = 1 \text{ and } G_{a'} = \bigcup_{1\leq i\leq n} \varphi_i(M,a')\Big\},$$

which is certainly a nonempty $\emptyset$-definable subset of $\mathrm{Ob}(\mathcal{G})$. Let $\widehat{E}$ be the equivalence relation on S given by

$$\widehat{E}(b,c) \iff \forall i \leq n\, \forall x \left[\varphi_i(x,b) \leftrightarrow \varphi_i(f_{bc} \circ x \circ f_{cb}, c)\right].$$

The relation $\widehat{E}$ is $\emptyset$-definable and has only finitely many classes, so there is some $\mathrm{acl}\,(\emptyset)$-definable set $S^0 \subseteq S$ consisting of one of its nonempty classes. Let $\mathrm{Ob}(\mathcal{G}^0) = S^0$, and property (3) follows immediately. □

Without loss of generality, $\mathcal{G} = \mathcal{G}^0$ as in the Claim above, since any definable full subgroupoid of a liaison groupoid is still a liaison groupoid.

Let $\mathcal{G}^*$ be the extension of $\mathcal{G}$ in N with a single new object as in the definition of a liaison groupoid. We define an equivalence relation E on the set $\bigcup_{a\in \mathrm{Ob}(\mathcal{G})} (\mathrm{Mor}_{\mathcal{G}^*}(a,*) \times \{a\})$ such that $E((g,a),(h,b))$ holds if and only if $h^{-1} \circ g = f_{ab}$. Note that E is definable (in N), and for each $a \in \mathrm{Ob}(\mathcal{G})$, $\mathrm{Mor}_{\mathcal{G}^*}(a,*)$ is in bijection with the set of all E-classes, so E has only finitely many classes. Let C be the set of E-classes.

Then C is a finite $\emptyset$-definable set in N (since we are assuming $N \cong N^{eq}$). It is easy to check that $N \subset \mathrm{dcl}_N(MC)$. Indeed, for any $a \in \mathrm{Ob}(\mathcal{G})$, any $f \in \mathrm{Mor}_{\mathcal{G}^*}(a,*)$ is definable from a, $*$, and the set C, and so all of N is definable from $M \cup C$ by the third condition on a liaison groupoid. □

4. Examples

4.1. **Simplest example.** The example presented in this section shows that it is possible to have a stable theory with 3-uniqueness property yet have a definable groupoid which is not retractable. There are two points of this example:

(1) the notion of retractability is strictly stronger than that of eliminability;
(2) while sentience of a finitary groupoid implies $G_a \subset \mathrm{dcl}(a)$ for every $a \in \mathrm{Ob}(\mathcal{G})$, the converse need not hold.

The structure M will have two sorts: I and G^*. The sort I is simply an infinite set, and G^* is a double-cover of I. That is, the elements of G^* have the form (a,δ), where $a \in I$ and $\delta \in \{0,1\}$. On G^*, we have a projection π onto the first coordinate.

The theory T of such a structure is totally categorical, and almost strongly minimal. The theory T has 3-uniqueness (and therefore also 4-existence). We explain how the relevant definable groupoid in T is *not* retractable.

Define a groupoid $\mathcal{G}$ in T as follows:

Let $\mathrm{Ob}(\mathcal{G}) = I$. The set of morphisms is defined by $\mathrm{Mor}(\mathcal{G}) = (G^* \times G^*)/\sim$, where $(x,y) \sim (x',y')$ just in case either

1. $(x, y) = (x', y')$, or
2. $\pi(x) = \pi(x')$, $\pi(y) = \pi(y')$, $x \neq x'$, and $y \neq y'$.

For $f \in \mathrm{Mor}(\mathcal{G})$, the domain and range maps are given by the coordinate projections.

We think of morphisms between a and b in $\mathrm{Ob}(\mathcal{G})$ as the bijections between the fibers $\pi^{-1}(a)$ and $\pi^{-1}(b)$. Namely, if $f = [(x, y)]_\sim$, then we think of f as the bijection that sends x to y, and sends the remaining element of $\pi^{-1}(a)$ to the remaining element of $\pi^{-1}(b)$. Composition in $\mathcal{G}$ is determined by the composition of bijections.

For convenience, we will keep using the groupoid language here, and refer to a morphism f from $a \in I$ to $b \in I$ as if f is a member of M. (It is a member of M^{eq}.) Also, we'll treat the composition $\circ$ as if it was a part of the language.

Note: there is a single isomorphism class in the groupoid; the automorphism groups of each of the objects are $\mathbb{Z}_2$; so there are two morphisms between any two objects. Thus both morphisms are in the algebraic closure of any pair of points in I.

Claim 4.1. *The groupoid $\mathcal{G}$ is not retractable.*

Proof. If a and b are any two distinct elements of $\mathrm{Ob}(\mathcal{G})$, then there is a $\sigma \in \mathrm{Aut}(\mathfrak{C})$ fixing a and b, fixing $\pi^{-1}(a)$ pointwise, and swapping the two elements of $\pi^{-1}(b)$. Then σ swaps the two elements of $\mathrm{Mor}_{\mathcal{G}}(a, b)$, and so $\mathrm{dcl}(a, b) \cap \mathrm{Mor}_{\mathcal{G}}(a, b) = \emptyset$. □

However:

Claim 4.2. *The groupoid $\mathcal{G}$ is eliminable.*

Proof. Pick any $* \in \mathrm{dcl}^{eq}(\emptyset)$. We define a groupoid $\mathcal{G}^*$ extending $\mathcal{G}$ such that $\mathrm{Ob}(\mathcal{G}^*) = \mathrm{Ob}(\mathcal{G}) \cup \{*\}$ and for any $a \in \mathrm{Ob}(\mathcal{G})$, $\mathrm{Mor}_{\mathcal{G}^*}(a, *) = \pi^{-1}(a) \times \{*\}$ and $\mathrm{Mor}_{\mathcal{G}^*}(*, a) = \{*\} \times \pi^{-1}(a)$. $\mathrm{Mor}_{\mathcal{G}^*}(*, *)$ consists of two different elements of $\mathrm{dcl}^{eq}(\emptyset)$. We think of elements of $\mathrm{Mor}_{\mathcal{G}^*}(a, *)$ and $\mathrm{Mor}_{\mathcal{G}^*}(*, a)$ as being enumerations of $\pi^{-1}(a)$, where $(x, *)$ represents the enumeration that begins with x. If $(x, *) \in \mathrm{Mor}_{\mathcal{G}^*}(a, *)$ and $(*, y) \in \mathrm{Mor}_{\mathcal{G}^*}(*, b)$, then we let $(*, y) \circ (x, *) = [(x, y)]_\sim$, which we think of as the bijection between $\pi^{-1}(a)$ and $\pi^{-1}(b)$ which sends x to y. The other compositions in $\mathcal{G}^*$ are defined similarly.

□

Finally, note that for any $a \in \mathrm{Ob}(\mathcal{G})$, both morphisms from a to itself are definable from a (one fixes $\pi^{-1}(a)$ and the other does not). So $G_a \subset \mathrm{dcl}(a)$ for all $a \in \mathrm{Ob}(\mathcal{G})$. However, $\mathcal{G}$ is not sentient since for $a \neq b \in \mathrm{Ob}(\mathcal{G})$ we have $\mathrm{Mor}(a, b) \subset \mathrm{dcl}(\mathrm{acl}\,(a), \mathrm{acl}\,(b))$, but $\mathrm{Mor}(a, b) \cap \mathrm{dcl}(a, b) = \emptyset$.

4.2. **Groupoids with arbitrary finite automorphism groups of objects.** In this subsection we address the question of which finite groups can be the automorphism groups of objects in a definable, non-retractable groupoid in a stable theory. We give a recipe for constructing examples of such groupoids whose object automorphism groups are arbitrary groups with nontrivial center. We construct the groupoids in such a way that $G_a \subset \operatorname{dcl}(a)$ for each object a.

Fix some finite group G. Our language consists of unary predicates I and P, a ternary predicate Q, and unary functions π_1 and π_2. In the standard model of our theory, I is an infinite set, $P = I^2 \times G$, $\pi_i \colon P \to I$ is the projection onto the ith coordinate, and Q is the relation on P^3 defined by

$$\models Q((i_1, i_2, x), (j_1, j_2, y), (k_1, k_2, z))$$

$$\Leftrightarrow i_2 = j_1 \ \& \ j_2 = k_2 \ \& \ i_1 = k_1 \ \& \ z = y \cdot x.$$

T is the complete theory of this structure.

We let $\pi \colon P \to I^2$ be the map $\pi_1 \times \pi_2$. For any $a \in I$, let G_a be the group $\pi^{-1}(a, a)$, and note that the identity element e_a of G_a is definable from a. We write "$h = g \circ f$" if h is the unique element such that $Q(f, g, h)$ holds. Also, for any $f \in \pi^{-1}(a, b)$, there is a unique element $g \in \pi^{-1}(b, a)$ such that $g \circ f = e_a$, and g is f-definable; we will call this g "f^{-1}."

For any finite set $A \subseteq \mathfrak{C}$ which is closed under taking projections, any permutation of $I(\mathfrak{C}) \setminus A$ can be extended to an automorphism of $\mathfrak{C}$. It follows that I is strongly minimal, so T is almost strongly minimal, hence stable.

Now we give a recipe for constructing automorphisms in $\operatorname{Aut}(\mathfrak{C}/I(\mathfrak{C}))$. Let $\{a_i : i < \kappa\}$ be some enumeration of $I(\mathfrak{C})$. Suppose we are also given one element g_i from each group G_{a_i}. From this data, we construct an automorphism $\sigma \in \operatorname{Aut}(\mathfrak{C}/I(\mathfrak{C}))$ as follows:

(1) $\sigma(a_i) = a_i$;
(2) If $i \neq 0$ and $x \in \pi^{-1}(a_0, a_i)$, then $\sigma(x) = g_i \circ x$;
(3) If $i \neq 0$ and $x \in \pi^{-1}(a_i, a_0)$, then $\sigma(x) = x \circ g_i^{-1}$;
(4) If $x \in \pi^{-1}(a_0, a_0)$, then $\sigma(x) = x$;
(5) If $i, j \neq 0$ and $x \in \pi^{-1}(a_i, a_j)$, then $\sigma(x) = g_j \circ x \circ g_i^{-1}$.

It is routine to check that the map σ preserves the Q relation, so σ is indeed an automorphism of $\mathfrak{C}$.

Finally we define the groupoid $\mathcal{G}$. Let $\operatorname{Ob}(\mathcal{G})$ be the set of all tuples of the form $(a, \overline{b})$ where $a \in I$ and $\overline{b}$ is an enumeration of G_a. Note that $\overline{b}$ has a definable group structure isomorphic to G. For any tuples $x = (a, \overline{b})$ and $y = (c, \overline{d})$ in $\operatorname{Ob}(\mathcal{G})$, $\operatorname{Mor}_{\mathcal{G}}(x, y)$ is the set $\pi^{-1}(a, c) \times \{(x, y)\}$. (The point of taking a Cartesian product with $\{(x, y)\}$ is to ensure that the domain and range maps are definable.) If $(f, (x, y))$ and $(g, (y, z))$ are elements of $\operatorname{Mor}(\mathcal{G})$, then their composition is defined to be $(h, (x, z))$, where h is the unique element such that $Q(f, g, h)$ holds.

Claim 4.3. *If G has a nontrivial center, then $\mathcal{G}$ is not retractable over* $\operatorname{acl}(\emptyset)$*, or even over any bounded set of parameters.*

Proof. Let A be any bounded subset of $\mathfrak{C}$. Note that any automorphism σ as above will fix A pointwise as long as a certain (small) set of the g_i's are identity elements. Pick such a σ such that every g_i is central but at least one of the g_i's is not the identity. Then (by Clause 1) σ fixes $\mathrm{Ob}(\mathcal{G})$ pointwise. But σ induces a permutation with no fixed points on at least one of the sets $\mathrm{Mor}_{\mathcal{G}}(a_0, a_i)$, so no element of this set is definable over $A \cup \{a_0, a_i\}$, and $\mathcal{G}$ cannot be retractable over A. □

4.3. 2**-element definable sets yield unsplit covers.** Here we explain a general method for constructing finite internal covers that are unsplit (but almost split) any time there is a 0-definable set P with two distinct elements with the same type. At the same time, we construct a non-retractable groupoid in such structures.

Let M be such a structure, and for concreteness let $P(M) = \{a_0, a_1\}$. We build a model of a finite internal cover N of M with one new sort $S(N)$ with four elements b_0, b_1, b_2, and b_3. $L(N)$ contains a new function symbol $\pi \colon S \to M$ and a new 4-ary predicate R. $\pi(b_i) = a_j$ if $i \equiv j \mod 2$, and $R(b_i, b_j, b_k, b_\ell)$ holds if and only if (i, j, k, ℓ) is a cyclic permutation of $(0, 1, 2, 3)$.

If $\sigma \in \mathrm{Aut}(N/M)$, then $\sigma(b_0)$ is b_0 or b_2, and because of the predicate R, the values of $\sigma(b_1), \sigma(b_2)$, and $\sigma(b_3)$ are determined by $\sigma(b_0)$; moreover, it is clear that there is a $\sigma \in \mathrm{Aut}(N/M)$ switching b_0 and b_2. Thus, $\mathrm{Aut}(N/M) \cong \mathbb{Z}_2$. Also, note that the cover (N/M) is trivially almost split, since S is finite and $\emptyset$-definable. On the other hand, if $C \subseteq N^{eq}$ is a finite, 0-definable set and $S(N) \subseteq \mathrm{dcl}(M \cup C)$, then it is straightforward to check that if $\sigma \in \mathrm{Aut}(M)$ is an order-2 automorphism mapping a_0 to a_1, then there is no corresponding $\sigma' \in \mathrm{Aut}(N/C)$ such that $\sigma' \restriction M = \sigma$. So the cover is not split.

The structure M also has a corresponding $\emptyset$-definable groupoid $\mathcal{G}$ which is not retractable: $\mathrm{Ob}(\mathcal{G}) = P$, every set $\mathrm{Mor}(a_i, a_j)$ is also equal to P (and for $i = 0, 1$, a_i represents the identity morphism from a_i to a_i, and the two morphisms from a_0 to a_1 and from a_1 to a_0 represented by a_i are inverses of one another). Then $\mathcal{G}$ is retractable over $\mathrm{acl}(\emptyset)$ (trivially, since it is finite) but not over $\emptyset$: if the morphism $f_{a_0 a_1}$ in a retraction is equal to a_0, for instance, then $f_{a_1 a_0}$ must equal a_1 (since $a_0 \equiv a_1$), but then $f_{a_1 a_0} \circ f_{a_0 a_1}$ is not the identity, a contradiction.

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